Hyperexponential Solutions of Finite-rank Ideals in Orthogonal Ore Rings

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ABSTRACT

An orthogonal Ore ring is an abstraction of common properties of linear partial differential, shift and q-shift operators. Using orthogonal Ore rings, we present an algorithm for finding hyperexponential solutions of a system of linear differential, shift and q-shift operators, or any mixture thereof, whose solution space is finite-dimensional. The algorithm is applicable to factoring modules over an orthogonal Ore ring when the modules are also finite-dimensional vector spaces over the field of rational functions.

Categories and Subject Descriptors

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General Terms

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1. INTRODUCTION

A (partial) linear functional system consists of linear partial differential, shift, and q-shift operators. In this paper we consider those linear functional systems which have finite-dimensional solution spaces. In recent years there has been work on decomposing these systems into "subsystems" whose solution spaces are of lower dimension. This has been done following either ideal-theoretical [15, 16] or module-theoretical [2, 20] approaches. In both cases the methods require the computation of hyperexponential solutions of some linear functional systems obtained from either the associated equations or the integrable systems [19]. This observation motivates us to develop a general algorithm for hyperexponential solutions of linear functional systems.

Recall that a system of algebraic polynomials with a finite number of solutions can be solved as follows: compute a Gröbner basis G with respect to an elimination ordering; find the roots of the univariate polynomial in G, and finally perform back-substitution. Unfortunately, this approach does not seem to work well for linear functional systems, because a solution of an ordinary linear functional equation may contain several unspecified constants with respect to one operator. These constants are, however, not constants with respect to other operators. As such, the back-substitution introduces new unknowns in addition to introducing complicated irrational function coefficients (possibly involving integral signs). This means that we do not know what closed-form solutions of the new unknowns should be looked for in order to find hyperexponential solutions.

The method presented in this paper needs only operations on rational functions to combine univariate hyperexponential functions, although it has to solve several ordinary linear functional equations. However, there has been significant work done in recent years for computing hyperexponential solutions of linear ordinary functional equations. We refer the reader to [17, 7, 13, 14, 4] for algorithms and their implementations. In addition the algorithm in [16, §3.3] computes hyperexponential solutions for a linear partial differential system with finite-dimensional solution space. Our paper generalizes the ideas in [16, §2, 3 and 4] to linear difference and mixed systems.

Guided by the work of [9, 10] we introduce the notion of orthogonal Ore rings. These are general enough to include rings of partial differential, shift, and q-shift operators over the field of multivariate rational functions but also simple enough to analyze the nonlinear compatibility conditions of a rank-one ideal. Analysis of the compatibility conditions is both a key technique used in this paper and a basic tool for studying multivariate functions and sequences (see for example [5, 16]). Based on orthogonal Ore rings and the associated linear action of their elements, we can understand a hyperexponential function as a function whose partial "logarithmic derivatives" are rational. In differential algebra these are of course exponential functions while in difference algebra these are usually called hypergeometric terms. Finding hyperexponential solutions of a linear functional system is equivalent to finding effective rank-one ideals containing the ideal generated by the operators in the system.

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The main result of this paper is summarized in Theorem 7. Roughly speaking this says that in order to find hyperexponential solutions, we need to find univariate hyperexponential solutions of several ordinary linear functional equations, to compute rational solutions of some special nonlinear functional systems which can be decoupled by simple transformations and then finally to compute rational solutions of some special kind of parametric first-order ordinary linear functional systems. Due to page limitations, Theorem 7 is given for the bivariate case only. For the general case, a lengthy induction process is required and will be given in a forthcoming technical report.

The remainder of this paper is organized as follows: Section 2 specifies the problem to be solved and presents necessary preliminaries. Section 3 defines the notion of orthogonal Ore ring while the next section connects elements of an orthogonal Ore ring with some special pseudo-linear operators. Section 5 defines hyperexponential elements with the notion of similarity of such elements appearing in the next section. Section 7 briefly describes an algorithm for rational solutions of a linear functional system. Section 8 outlines the basic idea with the main algorithm given in Section 9.

2. PRELIMINARIES

Let C be a field of characteristic zero and $\mathbb{F}$ the field of rational functions in x_1, ..., x_n over C. Denote by $\mathbf{1}$ the identity mapping of $\mathbb{F}$, and by $\mathbf{0}$ the mapping that sends everything to zero. For $i = 1, \ldots, n$, let σ_i be an automorphism of $\mathbb{F}$ and δ_i be an additive mapping of $\mathbb{F}$ with the property that

$$\delta_i(ab) = \sigma_i(a)\delta_i(b) + \delta_i(a)b \quad \text{for all } a, b \in \mathbb{F}. \qquad (1)$$

Set

$$\Phi = \{(\sigma_1, \delta_1), \ldots, (\sigma_n, \delta_n)\}. \qquad (2)$$

The set Φ is said to be *commutative* if

$$\sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i, \; \sigma_i \circ \delta_j = \delta_j \circ \sigma_i, \; \delta_i \circ \delta_j = \delta_j \circ \delta_i,$$

where $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$. With the commutative set Φ, we can construct a well-defined Ore ring $\mathbb{F}[\partial_1, \ldots, \partial_n]$ over $\mathbb{F}$ whose multiplicative rules are

$$\partial_i \partial_j = \partial_j \partial_i \quad \text{and} \quad \partial_i a = \sigma_i(a)\partial_i + \delta_i(a), \qquad (3)$$

for all $i, j \in \{1, 2, \ldots, n\}$ and $a \in \mathbb{F}$ (see [10, §1.2]). Denote the noncommutative ring $\mathbb{F}[\partial_1, \ldots, \partial_n]$ by $\mathbb{A}$. Note that the first multiplicative rule hinges on the commutativity of Φ.

EXAMPLE 1. *Let* $\mathbb{F}=\mathbb{Q}(x,k)$ *and* $\Phi=\left\{\left(\mathbf{1}, \frac{\partial}{\partial x}\right), (\sigma_2, \mathbf{0})\right\}$, *where* σ_2 *sends* $a(x,k) \in \mathbb{F}$ *to* $a(x, k+1)$. *The set* Φ *is commutative. Then* $\mathbb{A} = \mathbb{F}(x,k)[\partial_x, E_k]$ *is an Ore polynomial ring with multiplicative rules* $\partial_x E_k = E_k \partial_x$, $\partial_x \, a = a \, \partial_x + \frac{\partial a}{\partial x}$ *and* $E_k \, a = a(x, k+1)E_k$. □

From now on, the set Φ is assumed to be commutative.

All ideals of $\mathbb{A}$ considered in this paper will be left ideals. Since $\mathbb{F}$ is a field, we can use a noncommutative version of Buchberger's algorithm to compute Gröbner bases (see Theorem 1.2 in [10]). An ideal I of $\mathbb{A}$ is of *finite rank* if the $\mathbb{F}$-linear space $\mathbb{A}/I$ is finite-dimensional with the *rank* of I then being defined to be the dimension of the vector space $\mathbb{A}/I$ over $\mathbb{F}$. Proposition 2.1 in [10] states that an ideal I is of finite rank if and only if it contains $g_i \in \mathbb{F}[\partial_i]$, for $i = 1, \ldots, n$. The set $\{g_1, \ldots, g_n\}$ is called a *rectangular system* of I. We are concerned with

Problem H. *Given a basis of an ideal I of finite rank, compute all ideals of rank one that contain I.*

We will see later in Section 5 that Problem H is slightly more general than that of finding hyperexponential solutions of I. Note, however, that the statement of Problem H does not involve any notion of "solutions" of I.

EXAMPLE 2. *Consider* $\mathbb{A}$ *from Example 1. Let*

$$L_1 = \partial_x^2 - \frac{2((k-x)^2+k)}{x(k-x)}\partial_x + \frac{((k-x)^3 - 3xk + 3k^2 + 2k)}{x^2(k-x)},$$

$$L_2 = E_k^2 - \frac{2x(x-k-1)}{x-k-2}E_k + \frac{x^2(x-k)}{x-k-2},$$

and I be the ideal generated by L_1 and L_2 in $\mathbb{A}$. One can verify that L_1 and L_2 is a Gröbner basis of I by, for example, the Maple packages `Ore_algebra` *and* `Groebner`*. Thus, $\mathbb{A}/I$ has rank four with an $\mathbb{F}$-linear basis $\{1, \partial_x, E_k, \partial_x E_k\}$. An rank-one ideal containing I is generated by*

$$\partial_x + \frac{x-k-1}{x} \quad \text{and} \quad E_k - x.$$

We show later in Example 10 that there are infinitely many rank-one ideals containing I. □

An ideal J of rank one in $\mathbb{A}$ can be generated by a basis

$$f_1 = \partial_1 - r_1, \ldots, f_n = \partial_n - r_n, \text{ with } r_i \in \mathbb{F}. \qquad (4)$$

Since the S-polynomial $g_{ij} = (\partial_i f_j - \partial_j f_i)$ belongs to J, reducing g_{ij} with respect to the basis (4) yields

$$(\sigma_j(r_i)r_j - \sigma_i(r_j)r_i + \delta_j(r_i) - \delta_i(r_j)) \in J.$$

As J is not the entire ring $\mathbb{A}$, we have

$$\sigma_j(r_i)r_j + \delta_j(r_i) = \sigma_i(r_j)r_i + \delta_i(r_j), \; (1 \le i < j \le n). \quad (5)$$

We call (5) the *compatibility conditions* of J. If both σ_i and σ_j are $\mathbf{1}$, (5) becomes $\delta_j(r_i) = \delta_i(r_j)$, which is the usual compatibility condition for first-order linear PDE's. If both δ_i and δ_j are $\mathbf{0}$, (5) becomes $\sigma_j(r_i)r_j = \sigma_i(r_j)r_i$ (see Definition 3 of [5]). If $\mathbb{A}$ is the ring in Example 1, then, (5) becomes $r_1(x, k+1)\, r_2(x,k) = r_2(x,k)\, r_1(x,k) + \frac{\partial r_2}{\partial x}$.

If we are given $r_1, \ldots, r_n$ satisfying (5), then (4) is a Gröbner basis and hence J is of rank one. A solution of Problem H is an algorithm to find all the elements $r_1, \ldots, r_n$ of $\mathbb{F}$ such that (5) holds, and such that every element of the given basis of I can be reduced to zero by the basis (4).

3. ORTHOGONAL ORE RINGS

In this section, we define the notion of orthogonal Ore rings, which are a special case of the Ore rings defined in [10]. Our motivation is from two observations: (i) the presence of arbitrary σ_i and δ_j would cause very complicated calculations; (ii) an operator in a linear functional system usually acts non-trivially on only one variable. Let $\mathbb{F}_i$ be the field $C(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, for $i = 1, \ldots, n$.

DEFINITION 1. *The set Φ in (2) is said to be* orthogonal *if, for $i = 1, \ldots, n$, the following conditions are satisfied: (i) $\sigma_i(x_i) \in C[x_i]$ and $\sigma_i(a) = a$ if and only if $a \in \mathbb{F}_i$; (ii) $\delta_i(x_i) \in C(x_i)$ and $\delta_i(a) = 0$ if and only if $a \in \mathbb{F}_i$; (iii) $\delta_i \neq \mathbf{0}$ if $\sigma_i = \mathbf{1}$. If Φ is orthogonal, the Ore ring $\mathbb{A}$ with multiplicative rules (3) is said to be* orthogonal.

If $\mathbb{A}$ is orthogonal, we have, for $i, j \in \{1, \ldots, n\}$ with $i \neq j$,

$$\partial_j x_i = x_i \partial_j, \ \partial_j \sigma_i(x_i) = \sigma_i(x_i)\partial_j, \ \partial_j \delta_i(x_i) = \delta_i(x_i)\partial_j, \quad (6)$$

The name "Orthogonal Ore rings" is due to the first equality in (6) and the fact $\partial_i x_i \neq x_i \partial_i$ by the third condition.

The goal of this paper is to provide a solution to Problem H when $\mathbb{A}$ is orthogonal.

Both the ring of usual partial differential operators and that of partial shift operators are instances of orthogonal Ore rings as is the ring in Example 1. One can observe that the pairs (σ_1, δ_1), ..., (σ_n, δ_n) in these examples have the property that either $\sigma_i = \mathbf{1}$ or $\delta_i = \mathbf{0}$. The next theorem shows that this observation is not a coincidence.

THEOREM 1. *Let $\mathbb{A}$ be an orthogonal Ore ring with Φ given in (2). Then $\mathbb{A}$ is isomorphic to another orthogonal Ore ring $\mathbb{B} = \mathbb{F}[\Delta_1, \ldots, \Delta_n]$ whose multiplicative rules are: (i) $\Delta_i \Delta_j = \Delta_j \Delta_i$, (ii) $\Delta_i a = \sigma_i(a)\Delta_i$ if $\sigma_i \neq \mathbf{1}$, and (iii) $\Delta_j a = a\Delta_j + \delta_j(a)$ if $\sigma_j = \mathbf{1}$, where $i, j \in \{1, \ldots, n\}$, $a \in \mathbb{F}$.*

Proof The Ore ring $\mathbb{B}$ is well-defined and orthogonal because the commutative set $\{(\sigma_1, \epsilon_1), \ldots, (\sigma_n, \epsilon_n)\}$, in which $\epsilon_i = \mathbf{0}$ if $\sigma_i \neq \mathbf{1}$, and $\epsilon_i = \delta_i$ if $\sigma_i = \mathbf{1}$, is orthogonal. Since $\sigma_i(x_i)$ differs from x_i if $\sigma_i \neq \mathbf{1}$, we can define

$$D_i = \Delta_i \text{ if } \sigma_i = \mathbf{1}, \quad D_i = \frac{\Delta_i + \delta_i(x_i)}{x_i - \sigma_i(x_i)} \text{ if } \sigma_i \neq \mathbf{1}.$$

A repeated use of (6) shows that $D_i D_j = D_j D_i$. Thus, the $\mathbb{F}$-linear mapping ϕ sending $\partial_1^{m_1} \cdots \partial_n^{m_n}$ to $D_1^{m_1} \cdots D_n^{m_n}$ is a well-defined bijection. By Theorem 3.1 in [11, §8.3] or Theorem 9 in [9], we have $\phi(\partial_i a) = D_i a$ for all $a \in \mathbb{F}$ and $i \in \{1, \ldots, n\}$. An easy induction then proves that ϕ is a ring homomorphism. □

From this point on we assume that $\mathbb{A}$ is an orthogonal Ore ring, in which $\sigma_i = \mathbf{1}$ if $\delta_i \neq \mathbf{0}$ and $\sigma_i \neq \mathbf{1}_i$ if $\delta_i = \mathbf{0}$, for $i = 1$, ..., n. The *differential index* $\rho(\mathbb{A})$ of $\mathbb{A}$ is defined to be the number of nonzero δ_i appearing in Φ. If $\rho(\mathbb{A})$ is equal to n (respectively, equal to 0), then $\mathbb{A}$ is the ring of partial differential (respectively, difference) operators. For a later convenience, the elements of Φ in (2) are so arranged that δ_j is unequal to $\mathbf{0}$ for $j = 1$, ..., $\rho(\mathbb{A})$, and δ_j is equal to $\mathbf{0}$ for $j = \rho(\mathbb{A}) + 1$, ..., n.

A rank-one ideal of $\mathbb{A}$ is said to be *effective* if it has a basis given in (4), in which r_i is nonzero for $i = \rho(\mathbb{A}) + 1$, ..., n. We will see in Section 5 that a rank-one ideal has a hyperexponential solution if and only if it is effective.

4. LINEAR ACTIONS AND SOLUTIONS

To solve Problem H for orthogonal Ore rings, we shall regard indeterminates ∂_1, ..., ∂_n as operators acting on some commutative rings containing $\mathbb{F}$ (referred as $\mathbb{F}$-algebras). This view enables us to connect ideals of rank one with certain "solutions" of Ore polynomials.

An $\mathbb{F}$-algebra $\mathbb{E}$ is said to be Φ-*compatible* if all the σ_i's and δ_j's can be extended on $\mathbb{E}$ in such a way that (i) all the σ_i's are injective endomorphisms of $\mathbb{E}$, (ii) equality (1) holds for all elements of $\mathbb{E}$, and (iii) Φ in (2) is commutative on $\mathbb{E}$. The reader is referred to [18, 19] for the existence of Φ-compatible $\mathbb{F}$-algebras in the ordinary case. The next example describes a Φ-compatible $\mathbb{F}$-algebra defined by an effective rank-one ideal.

EXAMPLE 3. *Let $\rho(\mathbb{A})$ be p and J be an effective rank-one ideal generated by elements in (4). We construct a Φ-compatible $\mathbb{F}$-algebra $\mathbb{E}$ by J. Let $\mathbb{E} = \mathbb{F}(z)$, where z is an indeterminate. We define $\delta_i(z) = r_i z$, $i = 1$, ..., p and $\sigma_j(z) = r_j z$, $j = p + 1$, ..., n. Then δ_i and σ_j can be naturally extended to $\mathbb{E}$ as in usual differential and difference algebra. Note that σ_i $(1 \leq i \leq p)$ and δ_j $(p+1 \leq j \leq n)$ are kept to be $\mathbf{1}$ and $\mathbf{0}$, respectively. Hence (1) holds for all elements of $\mathbb{E}$. The σ_j's are injective because $r_{p+1} \cdots r_n \neq 0$. The commutativity of Φ on $\mathbb{E}$ holds because of (5). Here we verify that $\delta_i \circ \sigma_j = \sigma_j \circ \delta_i$, $1 \leq i \leq p$ and $(p+1) \leq j \leq n$. We compute*

$$\delta_i \circ \sigma_j(z) = (\delta_i(r_j) + r_i r_j)z \text{ and } \sigma_j \circ \delta_i(z) = \sigma_j(r_i) r_j z. \quad (7)$$

Since $\sigma_i = \mathbf{1}$ and $\delta_j = \mathbf{0}$, (5) becomes $\sigma_j(r_i) r_j = \delta_i(r_j) + r_i r_j$. It then follows from (7) that $\delta_i \circ \sigma_j(z) = \sigma_j \circ \delta_i(z)$, which implies $\delta_i \circ \sigma_j(f) = \sigma_j \circ \delta_i(f)$ for all $f \in \mathbb{E}$, because δ_i acts on a product or a fraction as a usual partial differential operator and σ_j is a field homomorphism. The verification of $\sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i$ and $\delta_i \circ \delta_j = \delta_i \circ \delta_j$ is similar. □

An element c of $\mathbb{E}$ is called a *constant with respect to* σ_i and δ_i if $\sigma_i(c) = c$ and $\delta_i(c) = 0$. The element c is called a *constant* if it is a constant with respect to all σ_i and δ_i, for $i = 1$, ..., n. The reader is referred to [9] for a general way to define the action of Ore polynomials on $\mathbb{E}$. We need only a special kind of pseudo-linear operators.

DEFINITION 2. *Let $\mathbb{E}$ be a Φ-compatible $\mathbb{F}$-algebra. Define θ_i to be the mapping sending an element a of $\mathbb{E}$ to $\delta_i(a)$ for $i = 1, \ldots, \rho(\mathbb{A})$, and to $\sigma_i(a)$ if $i = \rho(\mathbb{A}) + 1, \ldots, n$.*

By Lemma 3 in [9] we conclude that θ_i is pseudo-linear with respect to σ_i and δ_i over $\mathbb{E}$, that is, θ_i is additive and

$$\theta_i(ab) = \sigma_i(a)\theta_i(b) + \delta_i(a)b, \ a, b \in \mathbb{E}. \quad (8)$$

Definition 2 and the commutativity of Φ imply $\theta_i \circ \theta_j = \theta_j \circ \theta_i$. This commutativity allows us to define the action of

$$P = \sum_{m_1, \ldots, m_n} p_{m_1, \ldots m_n} \partial_1^{m_1} \cdots \partial_n^{m_n} \quad (P \in \mathbb{A})$$

on $a \in \mathbb{E}$ as $P \bullet a = \sum_{m_1, \ldots, m_n} p_{m_1, \ldots m_n} \theta_1^{m_1} \circ \cdots \circ \theta_n^{m_n}(a)$. In this way the ∂_i's can be regarded as $\mathbb{E}$-pseudo-linear operators with respect to σ_i and δ_i on $\mathbb{E}$. If c is a constant with respect to σ_i and δ_i, then $\partial_i \bullet (ca) = c\partial_i \bullet a$ by (8). So a constant with respect to σ_i and δ_i is also simply called a ∂_i-constant. An element of $\mathbb{A}$ is a linear operator over the field of all constants in $\mathbb{E}$.

EXAMPLE 4. *In applications we are mainly interested in the following three operators: If σ_i is $\mathbf{1}$, then ∂_i is called a* differential operator. *If $\sigma_i(x_i) = x_i + 1$ and $\delta_i = 0$, then ∂_i is called a* shift operator. *If $\sigma_i(x_i) = q x_i$ and $\delta_i = 0$, where q is a constant in C but not a root of unity, then ∂_i is called a q-*shift operator. *If σ_i is not $\mathbf{1}$ and unspecified, then ∂_i is called a σ-shift operator. Consider the ring $\mathbb{A}$ in Example 1. By Definition 2, the application of ∂_x on a function $a(x, k)$ is $\frac{\partial a}{\partial x}$, and that of E_k is $a(x, k+1)$. Thus ∂_x and E_k can be viewed as differential and shift operators, respectively.* □

An element y of some Φ-compatible $\mathbb{F}$-algebra is said to be a *solution* of a subset S of $\mathbb{A}$ if the application of any element of S to y equals zero. Example 3 illustrates one way to introduce a Φ-compatible $\mathbb{F}$-algebra that contains a nonzero solution of an effective ideal of rank one. However, this way would introduce many unnecessary new constants. If we do not want any new constant, Φ-extensions may have zero divisors (see Example 0.1 in [18]). It is also clear that zero is the only solution of an ineffective ideal of rank one.

5. HYPEREXPONENTIAL ELEMENTS

A nonzero element h of some Φ-compatible $\mathbb{F}$-algebra is *hyperexponential* over $\mathbb{F}$ with respect to ∂_i if $\partial_i \bullet h = r_i h$, where $r_i \in \mathbb{F}$. The rational function r_i is called the ∂_i-*certificate* of h. Note that r_i is assumed to be nonzero in [9]. We can remove this restriction because the θ_i's in Definition 2 are very special. An element is *hyperexponential* over $\mathbb{F}$ (with respect to $\partial_1, \ldots, \partial_n$) if it is hyperexponential over $\mathbb{F}$ with respect to all the ∂_i's. All nonzero elements of $\mathbb{F}$ are hyperexponential and the product of two hyperexponential elements is hyperexponential.

Example 5. *Let $\mathbb{A}=C(m,k)[E_m, E_k]$, where E_m and E_k are shift operators with respect to m and k, respectively. A hyperexponential element over $C(m,k)$ is usually called a hypergeometric term. For example: $\binom{m}{k}$, $(m+k)!$, 3^{m+k} are all bivariate hypergeometric terms.* □

A relation between hyperexponential solutions and effective rank-one ideals is described in

Proposition 2. *An ideal I of $\mathbb{A}$ has a hyperexponential solution if and only if there exists an effective rank-one ideal containing I.*

Proof Let h be a hyperexponential solution of I and the ideal J be generated by $\left(\partial_1 - \frac{\partial_1 \bullet h}{h}\right), \ldots, \left(\partial_n - \frac{\partial_n \bullet h}{h}\right)$. The ideal J is of rank-one and effective because it has a nonzero solution h. Moreover, $I \subset J$ since h is a solution of I. Conversely, J has a hyperexponential solution z in the Φ-compatible field $\mathbb{F}(z)$ as described in Example 3. The element z is also a solution of I since $I \subset J$. □

Example 6. *The rank-one ideal in Example 2 has a hyperexponential solution* $\exp(-x)x^{k+1}$. *As pointed out in Example 2, there are infinitely many rank-one ideals containing I. Consequently, I has infinitely many different hyperexponential solutions, although two hyperexponential elements with constant ratio are usually considered to be equal.* □

In this paper, we merely treat hyperexponential elements as a convenient way to represent effective rank-one ideals. By "given (computing) a hyperexponential element", we mean that we are given (computing) its certificate(s). Note that, if h is hyperexponential with respect to all the ∂_i's, then its certificates $r_1, \ldots, r_n$ must satisfy (5).

6. SIMILARITY

In this section, we introduce the notion of similarity among hyperexponential elements. This notion helps us organize possibly infinitely many different hyperexponential solutions into finitely many equivalence classes. The notion of similarity is well known in the ordinary case [17] and has been extended to the partial differential case in [16]. In the following we first characterize the condition that an ideal of rank one can have a nonzero rational solution.

Proposition 3. *A rank-one ideal of $\mathbb{A}$ has a nonzero solution in $\mathbb{F}$ if and only if it has a basis $\left(\partial_1 - \frac{\partial_1 \bullet s_1}{s_1}\right), \ldots, \left(\partial_n - \frac{\partial_n \bullet s_n}{s_n}\right)$, where $s_1, \ldots s_n$ are nonzero elements of $\mathbb{F}$, and (5) is satisfied when $r_1, \ldots, r_n$ are replaced by $\frac{\partial_1 \bullet s_1}{s_1}, \ldots, \frac{\partial_n \bullet s_n}{s_n}$, respectively.*

Proof The necessity is trivial. Let p be the differential index $\rho(\mathbb{A})$. The proposition follows from Lemma 3.1 if p is equal to n. Thus we assume that $p < n$, and hence $\sigma_n \neq \mathbf{1}$.

We proceed by induction on n. If $n = 1$, then s_1 is the solution. Assume that the proposition holds for $(n-1)$. For all i with $(p+1) \leq i \leq (n-1)$, (5) becomes

$$\sigma_n\left(\frac{\sigma_i(s_i)}{s_i}\right)\frac{\sigma_n(s_n)}{s_n} = \sigma_i\left(\frac{\sigma_n(s_n)}{s_n}\right)\frac{\sigma_i(s_i)}{s_i}.$$

It follows that

$$\sigma_n\left(\frac{\sigma_i(t_i)}{t_i}\right) = \frac{\sigma_i(t_i)}{t_i}, \quad \text{where } t_i = \frac{s_i}{s_n} \tag{9}$$

so that $\frac{\sigma_i(t_i)}{t_i}$ is a ∂_n-constant. Since $\mathbb{A}$ is orthogonal, we have $\frac{\sigma_i(t_i)}{t_i} \in \mathbb{F}_n$. For all j with $1 \leq j \leq p$, (5) becomes

$$\sigma_n\left(\frac{\delta_j(s_j)}{s_j}\right)\frac{\sigma_n(s_n)}{s_n} = \frac{\sigma_n(s_n)}{s_n}\frac{\delta_j(s_j)}{s_j} + \delta_j\left(\frac{\sigma_n(s_n)}{s_n}\right). \tag{10}$$

Since $\sigma_j = \mathbf{1}$, expanding the last term of (10) yields

$$\delta_j\left(\frac{\sigma_n(s_n)}{s_n}\right) = \frac{s_n\delta_j \circ \sigma_n(s_n) - \sigma_n(s_n)\delta_j(s_n)}{s_n^2}. \tag{11}$$

Replacing the last term of (10) by the right hand side of (11), and moving $\frac{\sigma_n(s_n)}{s_n}$ from the left hand side of (10) to the right hand side, we get

$$\sigma_n\left(\frac{\delta_j(s_j)}{s_j}\right) = \frac{\delta_j(s_j)}{s_j} + \frac{\delta_j \circ \sigma_n(s_n)}{\sigma_n(s_n)} - \frac{\delta_j(s_n)}{s_n}.$$

Using the identity on logarithmic derivatives:

$$\frac{\delta_j(a)}{a} - \frac{\delta_j(b)}{b} = \frac{\delta_j(f)}{f}, \quad \text{where } f = \frac{a}{b}, \tag{12}$$

and $\sigma_n\left(\frac{\delta_j(s_j)}{s_j}\right) = \frac{\delta_j \circ \sigma_n(s_j)}{\sigma_n(s_j)}$, we see that

$$\sigma_n\left(\frac{\delta_j(t_j)}{t_j}\right) = \frac{\delta_j(t_j)}{t_j}, \quad \text{where } t_j = \frac{s_j}{s_n}. \tag{13}$$

Hence, $\frac{\delta_j(t_j)}{t_j}$ belongs to $\mathbb{F}_n$ by the same reason. A tedious but straightforward calculation verifies that, for $i, j = 1, \ldots, n-1$, (5) is satisfied when r_k is replaced by $\frac{\partial_k \bullet t_k}{t_k}$ for $k = 1, \ldots, n-1$ (One would see that such a verification is just a repetition of the calculation to get (9) or (13)). Hence, by induction, the rank-one ideal in $\mathbb{F}_n[\partial_1, \ldots, \partial_{n-1}]$ generated by $\left(\partial_1 - \frac{\partial_1 \bullet t_1}{t_1}\right), \ldots, \left(\partial_{n-1} - \frac{\partial_{n-1} \bullet t_{n-1}}{t_{n-1}}\right)$, has a nonzero solution r in $\mathbb{F}_n$. We claim that rs_n is a solution of I. The operator $\left(\partial_n - \frac{\partial_n \bullet s_n}{s_n}\right)$ annihilates rs_n, because r is a ∂_n-constant. If $i > p$, then $\frac{\sigma_i(rs_n)}{rs_n} = \frac{\sigma_i(s_i)}{s_i}$ since $\left(\partial_i - \frac{\partial_i \bullet t_i}{t_i}\right)$ annihilates r. It follows that $\left(\partial_i - \frac{\partial_i \bullet s_i}{s_i}\right)$ annihilates rs_n. If $j \leq p$, $\left(\frac{\delta_j(r)}{r} - \frac{\delta_j \bullet t_j}{t_j}\right) = 0$. By (12) $\frac{\delta_j(rs_n)}{rs_n} = \frac{\delta_j(s_j)}{s_j}$ and hence, $\left(\partial_j - \frac{\partial_j \bullet s_j}{s_j}\right)$ annihilates rs_n. □

Let h_1 and h_2 be hyperexponential over $\mathbb{F}$ with respect to ∂_i. We say that h_1 and h_2 are *similar* with respect to ∂_i (denoted by $h_1 \sim_i h_2$) if their ratio is the product of a ∂_i-constant and a rational function of $\mathbb{F}$. If h_1 and h_2 are hyperexponential over $\mathbb{F}$, then h_1 and h_2 are *similar* (denoted by $h_1 \sim h_2$) if their ratio is the product of a constant and a rational function of $\mathbb{F}$. Both $\sim_i$ and $\sim$ are equivalence relations. The ∂_i-constant in the definition of the similarity $\sim_i$ should be explicitly stated, because it may be not a ∂_j-constant when $j \neq i$, and, in general, it is not an element of $\mathbb{F}$. In the univariate case such a constant may be ignored. Clearly, $h_1 \sim_i h_2$ $(1 \leq i \leq n)$ if $h_1 \sim h_2$. The converse follows from the next lemma.

LEMMA 4. *Two hyperexponential elements h_1 and h_2 are similar if and only if $h_1 \sim_i h_2$ for all i with $1 \leq i \leq n$.*

Proof Assume that $h_1 \sim_i h_2$, for $i = 1, \ldots, n$ and denote by h the ratio of h_1 and h_2. Then $h = c_1 s_1, \ldots, h = c_n s_n$, where c_i is a ∂_i-constant and $s_i \in \mathbb{F}$. Thus $\partial_1 \bullet h = c_1 \partial_1 \bullet s_1$, $\ldots$, $\partial_n \bullet h = c_n \partial_n \bullet s_n$. These two equations imply

$$\frac{\partial_1 \bullet h}{h} = \frac{\partial_1 \bullet s_1}{s_1}, \ldots, \frac{\partial_n \bullet h}{h} = \frac{\partial_n \bullet s_n}{s_n}. \qquad (14)$$

It follows that $\frac{\partial_1 \bullet s_1}{s_1}, \ldots, \frac{\partial_n \bullet s_n}{s_n}$ satisfy (5) when r_i is replaced by $\frac{\partial_i \bullet s_i}{s_i}$. By Proposition 3, there exists $s \in \mathbb{F}$ such that $\frac{\partial_i \bullet s}{s} = \frac{\partial_i \bullet s_i}{s_i}$, for $i = 1, \ldots, n$. Equation (14) implies that both h and s are nonzero solutions of the ideal generated by $\left(\partial_1 - \frac{\partial_1 s_1}{s_1}\right), \ldots, \left(\partial_n - \frac{\partial_n s_n}{s_n}\right)$. Consequently, h is the product of s and a nonzero constant. □

The next lemma extends Theorem 5.1 in [17].

LEMMA 5. *Let $h_1, \ldots, h_m$ be hyperexponential elements with respect to ∂_k in a Φ-compatible field. If there exist nonzero ∂_k-constants $c_1, \ldots, c_m$, and nonzero rational functions $r_1, \ldots, r_m$ such that $(c_1 r_1 h_1 + \cdots + c_m r_m h_m) = 0$, then $h_i \sim_k h_j$ for some $1 \leq i < j \leq m$.*

Proof If ∂_k is a σ-shift operator, then the proof is the same as that of Theorem 5.1 in [17], which works verbatim in the differential case. □

Due to the existence of Wronskian and Casoratian determinants (see [12, p. 271]), each P in $\mathbb{F}[\partial_i]$ has at most $(\deg_{\partial_i} P)$ linear independent solutions over a field of constants with respect to ∂_i. This fact combined with Lemma 5 implies that

COROLLARY 6. *If $h_1, \ldots, h_m$ are pairwise dissimilar hyperexponential solutions of P in $\mathbb{F}[\partial_i]$ with respect to ∂_i, then m is no more than the degree of P in ∂_i.*

7. RATIONAL SOLUTIONS

In this section, we briefly describe an algorithm for rational solutions of a rectangular system $\mathcal{Q} = \{Q_1, \ldots, Q_n\}$, where $Q_i \in \mathbb{F}[\partial_i]$. Such an algorithm is needed in Section 9.

Assume that we are able to find rational solutions for each of the Q_i's. Let $q_{i1}, \ldots, q_{im_i}$ be a basis of the rational solutions of Q_i. Let g be the least common multiple of the denominators of all the q_{ij}'s, where $i = 1, \ldots, n$ and $j = 1, \ldots, m_i$. We claim that g is a common denominator of all rational solutions of $\mathcal{Q}$. For, otherwise, suppose that a rational solution r can be written as $p/(ug)$, where $p, u \in C[x_1, \ldots, x_n]$ and $\gcd(p, ug) = 1$. If $\deg_{x_i} u > 0$, then $r = \bar{p}/(\bar{u}g)$, where $\bar{p}$ and $\bar{u}$ are in $C[x_1, \ldots, x_n]$ and the degree of $\bar{u}$ in x_i equals zero, because r is a rational solution of Q_i, and so its denominator divides g over $\mathbb{F}_i[x_i]$. These two expressions of r imply that u divides $\bar{u}$, a contradiction. Thus, each rational solution of $\mathcal{Q}$ can be written as f/g, in which the polynomial f is yet to be determined. Regarding f as a new unknown and applying the Q_i's to f/g yields a rectangular system $\mathcal{P} = \{P_1, \ldots, P_n\}$ whose polynomial solutions are the numerators of the rational solutions of $\mathcal{Q}$.

Now, assume that we can find polynomial solutions for each of the P_i's. Let $p_{i1}, \ldots, p_{im_i}$ be a basis of the polynomial solutions of P_i. Then $\max(\deg_{x_i} p_{i1}, \ldots, \deg_{x_i} p_{im_i})$ is a degree bound for all the polynomial solutions of $\mathcal{P}$ in x_i $(1 \leq i \leq n)$. Using these bounds, we can then compute all polynomial solutions of $\mathcal{P}$ by solving a linear system over C.

EXAMPLE 7. *Let $\mathbb{A}$ be given in Example 1. We compute the rational solutions of the rectangular system:*

$$(x-k)\partial_x^2 + 2\partial_x, \text{ and}$$

$$(x-k-2)(k+2)E_k^2 - 2(x-k-1)(k+1)E_k + (x-k)k.$$

The first equation has rational solutions $(c_1 + c_2 x)/(x-k)$, where c_1 and c_2 are ∂_x-constants. The second equation has rational solutions $(c_3 + c_4 k)/((x-k)xk)$, where c_3 and c_4 are E_k-constants. Thus, rational solutions of the system have a common denominator $(x-k)xk$. Applying the two operators to $p/((x-k)xk)$, where p is an indeterminate, we see that p satisfies

$$x^2\partial_x^2 - 2x\partial_x + 2, \quad \text{and} \quad E_k^2 - 2E_k + 1.$$

The first operator has polynomial solutions $c_5 x + c_6 x^2$, where c_5 and c_6 are ∂_x-constants while the second has polynomial solutions $c_7 + c_8 k$, where c_7 and c_8 are E_k-constants. Thus the polynomial solutions of the second system are of the form

$$g = c_9 + c_{10} x + c_{11} k + c_{12} xk + c_{13} x^2,$$

where $c_9, \ldots c_{13}$ are constants. Substituting g into the second system gives $c_9 = 0$ and $c_{11} = 0$. The rational solutions of the first system are $c_{10} + c_{12} k + c_{13} x/(k(x-k))$. □

In our algorithm for computing hyperexponential solutions, rectangular systems are all consistent and of minimal degrees. Therefore, there is no need to compute Gröbner basis to check the consistency of the system. The reader is referred to [1, 6, 3, 8] for algorithms to compute polynomial and rational solutions of ordinary linear functional equations.

In addition to the assumptions that we can compute rational solutions of ordinary linear functional equations, we further assume that we can solve

Problem P. *Given an automorphism σ of $\mathbb{F}$ and a nonzero element a of $\mathbb{F}$, find a nonzero c in C such that the equation*

$$\sigma(z) = caz \qquad (15)$$

has a nonzero solution in $\mathbb{F}$.

If σ is the shift or q-shift operator, the algorithms in [1] can be easily adapted to solve Problem P, because the universal denominator of all rational solutions of (15) does not depend on the constant c. This assumption will be used in Section 9.

EXAMPLE 8. *Find a nonzero rational number c and a nonzero rational function z of $\mathbb{Q}(x,k)$ such that*

$$\sigma(z) = \frac{c(k-x)z}{k-x+1}$$

where σ is the automorphism of $\mathbb{Q}(x,k)$ sending x to x and k to $(k+1)$. A denominator of z is $(k-x)$ according to [1]. Hence, we need only to find c and a polynomial f such that $\sigma(f) = cf$. Hence, c has to be 1 and f can be chosen as 1 as well. The equation then has a solution $1/(k-x)$ for $c = 1$ and has no rational solutions for other values of c.

8. COMMON ASSOCIATES

The basic idea we use to solve Problem H is to compute hyperexponential solutions of an ideal I of finite rank, and then use Proposition 2 to get the effective rank-one ideals containing I. The ineffective ones can be obtained by an easy recursive process as such ideals must contain ∂_i for some i with $\rho(\mathbb{A}) < i \leq n$.

Let H be the set of hyperexponential solutions of I, and let $\{P_1, \ldots, P_n\}$ be a rectangular system contained in I, in which $P_i \in \mathbb{F}[\partial_i]$ for all i with $1 \leq i \leq n$. Denote by H_i the set of hyperexponential solutions of P_i with respect to ∂_i. Since $H \subset H_i$, H is empty if one of the H_i's is empty. Assume that all the H_i's are nonempty. Although each of the H_i's is infinite, the set $\bar{H}_i = H_i / \sim_i$ is finite by Corollary 6. Since an element h of H is similar to an element of H_i with respect to ∂_i, h must belong to an equivalence class in $\bar{H}_i$ for all i with $1 \leq i \leq n$. This observation motivates us to introduce

DEFINITION 3. *For $i = 1, \ldots, n$, let h_i be hyperexponential with respect to ∂_i. A hyperexponential element h is called a* common associate *of h_i if $h \sim_i h_i$, for $i = 1, \ldots, n$.*

Our approach involves two basic steps. First, we search for dissimilar common associates of elements in $H_1 \times \cdots \times H_n$, a finite number modulo $\sim$ since $\bar{H}_1 \times \cdots \times \bar{H}_n$ is finite. Let G be the set of dissimilar common associates. Then every element h of H is similar to one and only one $g \in G$ by Lemma 4 and so $h = crg$ for some rational function r and a constant c. The second step is to determine r by applying the P_j's to rg with r treated as an unknown. Since g is hyperexponential, the applications of the P_j's result in another rectangular system, and r is one of its rational solutions.

9. COMPUTING COMMON ASSOCIATES

In this section, we present an algorithm to construct common associates. Common associates reduce Problem H to finding rational solutions of some rectangular systems. Due to page limitations we now assume that $n = 2$ and leave the general result to a forthcoming technical report.

The next theorem, which is our main result, relates common associates to rational solutions of some nonlinear equations. These equations can be easily solved.

THEOREM 7. *Let $\mathbb{A}_2 = \mathbb{F}[\partial_1, \partial_2]$ be an orthogonal Ore ring, and let h_i be hyperexponential with respect to ∂_i with certificate u_i, $i = 1, 2$. Then h_1 and h_2 have a common associate if and only if one of the following conditions holds:*

1. Differential case. *If $\sigma_1 = \sigma_2 = \mathbf{1}$, then*
$$\delta_1\left(\frac{\delta_2(z)}{z}\right) = \delta_1 u_2 - \delta_2 u_1 \tag{16}$$
has a rational solution.

2. σ-Shift case. *If $\sigma_1 \neq \mathbf{1}$ and $\sigma_2 \neq \mathbf{1}$, then*
$$\sigma_1\left(\frac{\sigma_2(z)}{z}\right) = \frac{u_1\sigma_1(u_2)}{u_2\sigma_2(u_1)}\frac{\sigma_2(z)}{z} \tag{17}$$
has a rational solution.

3. Mixed case. *If $\sigma_1 = \mathbf{1}$ and $\sigma_2 \neq \mathbf{1}$, then*
$$\delta_1\left(\frac{\sigma_2(z)}{z}\right) = \frac{u_1u_2 + \delta_1(u_2) - \sigma_2(u_1)u_2}{u_2}\frac{\sigma_2(z)}{z} \tag{18}$$
has a rational solution.

Moreover, suppose there exists such a rational solution z. Let p_1 and p_2 be the numerator and denominator of z, respectively. Then
$$q_1 = \frac{\sigma_1(p_1)}{p_1}u_1 + \frac{\delta_1(p_1)}{p_1},\ q_2 = \frac{\sigma_2(p_2)}{p_2}u_2 + \frac{\delta_2(p_2)}{p_2} \tag{19}$$
are respective ∂_1 and ∂_2-certificates of a common associate of h_1 and h_2.

Proof First, we assume that h is a common associate of h_1 and h_2. Then
$$h = c_1r_1h_1 \quad \text{and} \quad h = c_2r_2h_2, \tag{20}$$
where $r_1, r_2 \in \mathbb{F}$ are nonzero, and c_1, c_2 are ∂_1 and ∂_2-constants, respectively. Applying operators ∂_1 to the first equation and ∂_2 to the second in (20) then gives
$$\begin{cases} \partial_1 \bullet h = c_1(\sigma_1(r_1)\partial_1 \bullet h_1 + \delta_1(r_1)h_1) \\ \partial_2 \bullet h = c_2(\sigma_2(r_2)\partial_2 \bullet h_2 + \delta_2(r_2)h_2). \end{cases} \tag{21}$$
Equations (20) and (21) imply
$$\partial_1 \bullet h = \underbrace{\left(\frac{\sigma_1(r_1)}{r_1}u_1 + \frac{\delta_1(r_1)}{r_1}\right)}_{w_1} h \tag{22}$$
and
$$\partial_2 \bullet h = \underbrace{\left(\frac{\sigma_2(r_2)}{r_2}u_2 + \frac{\delta_2(r_2)}{r_2}\right)}_{w_2} h. \tag{23}$$
Let J be the ideal generated by $(\partial_1 - w_1)$ and $(\partial_2 - w_2)$. Since h is a solution of J, the ideal is of rank one, and, hence, the compatibility condition (5) implies that
$$\sigma_2(w_1)w_2 + \delta_2(w_1) = \sigma_1(w_2)w_1 + \delta_1(w_2). \tag{24}$$
Set z to be the rational function r_1/r_2.

Differential case. Since both σ_1 and σ_2 are $\mathbf{1}$, (24) becomes
$$\delta_2\left(u_1 + \frac{\delta_1(r_1)}{r_1}\right) = \delta_1\left(u_2 + \frac{\delta_2(r_2)}{r_2}\right),$$
which implies (16) because δ_1 and δ_2 commute.

σ-Shift Case. Since both δ_1 and δ_2 are $\mathbf{0}$, (24) becomes
$$\frac{\sigma_2 \circ \sigma_1(r_1)}{\sigma_2(r_1)}\frac{\sigma_2(r_2)}{r_2}\sigma_2(u_1)u_2 = \frac{\sigma_1 \circ \sigma_2(r_2)}{\sigma_1(r_2)}\frac{\sigma_1(r_1)}{r_1}\sigma_1(u_2)u_1,$$
which implies (17) because σ_1 and σ_2 commute.

Mixed case. Since $\sigma_1 = \mathbf{1}$ and $\delta_2 = \mathbf{0}$, we have
$$w_1 = u_1 + \frac{\delta_1(r_1)}{r_1} \quad \text{and} \quad w_2 = \frac{\sigma_2(r_2)}{r_2}u_2.$$
Set $f = \frac{\sigma_2(r_2)}{r_2}$. Then the left hand side L of (24) becomes:
$$L = u_2\sigma_2(u_1)\,f + \underbrace{\frac{\sigma_2 \circ \delta_1(r_1)}{\sigma_2(r_1)}}_{a_1}\,fu_2,$$
and the right hand side R of (24) becomes:
$$u_1u_2f + \underbrace{\frac{\delta_1(r_1)}{r_1}}_{a_2}\,fu_2 + \delta_1(u_2)f + \underbrace{\frac{\delta_1 \circ \sigma_2(r_2)}{r_2}}_{a_3}\,u_2 - \underbrace{\frac{\delta_1(r_2)}{r_2}}_{a_4}\,fu_2.$$

Set $v = \frac{u_1 u_2 + \delta_1(u_2) - \sigma_2(u_1) u_2}{u_2}$. It follows from $L = R$ that

$$(a_4 - a_2) f + a_1 f - a_3 = v f.$$

Hence, $(a_4 - a_2) + \left(a_1 - \frac{a_3}{f}\right) = v$. Since $(a_4 - a_2) = -\frac{\delta_1(z)}{z}$ and $a_1 - \frac{a_3}{f} = \frac{\delta_1(\sigma_2(z))}{\sigma_2(z)}$, we find by logarithmic derivatives (see (12)) that (18) holds.

Conversely, assume that $z = p_1/p_2$ with $p_1, p_2 \in C[x_1, x_2]$ is a nonzero rational solution of (16) in the differential case, that of (17) in the σ-shift case, and that of (18) in the mixed case. Reversing the respective arguments to get (16), (17) and (18), we see that (24) holds in all three cases when r_1 and r_2 are replaced by p_1 and p_2, respectively. Thus, the ideal $\tilde{J}$ generated by

$$\partial_1 - \left(\frac{\sigma_1(p_1)}{p_1} u_1 + \frac{\delta_1(p_1)}{p_1}\right), \;\; \partial_2 - \left(\frac{\sigma_2(p_2)}{p_2} u_2 + \frac{\delta_2(p_2)}{p_2}\right)$$

is of rank one and clearly effective. Proposition 2 then implies that $\tilde{J}$ has a hyperexponential solution h. We shall verify that h is a common associate for the mixed case with differential and σ-shift cases handled in a similar manner. Recall $u_i = \frac{\partial_i \bullet h_i}{h_i}$, for $i = 1, 2$. Thus, applying the first generator to h yields

$$\frac{\delta_1(h)}{h} = \frac{\delta_1(h_1)}{h_1} + \frac{\delta_1(p_1)}{p_1} = \frac{\delta_1(p_1 h)}{p_1 h},$$

since $\sigma_1 = \mathbf{1}$. The ratio of h and $p_1 h_1$ is a ∂_1-constant, because of the logarithmic derivatives (see (12)) and $\partial_1 = \delta_1$. Similarly, applying the second operator to h yields

$$\frac{\sigma_2(h)}{h} = \frac{\sigma_2(p_2)}{p_2} \frac{\sigma_2(h_2)}{h_2},$$

since δ_2=0. Thus, $h/(p_2 h_2)$ is a ∂_2-constant since ∂_2=σ_2. □

An algorithm for computing rational solutions of (16) is presented in [16]. To compute rational solutions of (17) and (18), we need a technical lemma. For a nonzero rational function $r \in C(x_1, x_2)$ with numerator a and denominator b, let the contents of a and b with respect to x_1 be a_1 and b_1, respectively. Normalize $a_1, b_1 \in C[x_2]$ to be monic. We call a_1/b_1 the univariate part of r, denoted by $\mathrm{uv}_1(r)$, and $r/\mathrm{uv}_1(r)$ the bivariate part, denoted by $\mathrm{bv}_1(r)$.

LEMMA 8. *If $r \in C(x_1, x_2)$ is nonzero, then*

$$\sigma_2(\mathrm{uv}_1(r)) = c\,\mathrm{uv}_1(\sigma_2(r)) \text{ and } \sigma_2(\mathrm{bv}_1(r)) = \mathrm{bv}_1(\sigma_2(r))/c$$

for some $c \in C$. Moreover, if $s \in C(x_1, x_2)$ is nonzero, then $\mathrm{uv}_1(rs) = \mathrm{uv}_1(r)\mathrm{uv}_1(s)$ and $\mathrm{bv}_1(rs) = \mathrm{bv}_1(r)\mathrm{bv}_1(s)$.

Proof Since $\mathbb{A}_2$ is orthogonal, $\sigma_2(x_2)$ is in $C[x_2]$ with positive degree and $\sigma_2(x_1) = x_1$. So $\sigma_2(\mathrm{uv}_1(r))$ is in $C(x_2)$. Write the numerator of $\mathrm{bv}_1(r)$ as a polynomial in x_1 with coefficients, say $p_1, \ldots, p_m$ in $C[x_2]$. Since $p_1, \ldots, p_m$' are relatively prime, so are $\sigma_2(p_1), \ldots, \sigma_2(p_n)$. The same argument applies to the denominator of $\mathrm{bv}_1(r)$. Thus, $\sigma_2(\mathrm{uv}_1(r))$ and $\mathrm{uv}_1(\sigma_2(r)$ can only differ by a multiplicative constant. So we can assume that $\sigma_2(uv_1(r)) = c\,\mathrm{uv}_1(\sigma_2(r))$. It follows that $\sigma_2(\mathrm{bv}_1(r)) = \mathrm{bv}_1(\sigma_2(r))/c$. The last two equalities hold by Gauss's lemma. □

For the σ-shift case, we set $v = \frac{u_1 \sigma_1(u_2)}{u_2 \sigma_2(u_1)}$ and $y = \frac{\sigma_2(z)}{z}$. Equation (17) then becomes

$$\sigma_1(y) = vy \quad \text{and} \quad \sigma_2(z) = yz. \tag{25}$$

First, we compute a nonzero rational solution of the first equation in (25). If there are no such solutions, then (17) has no rational solution. Assume now that f is such a solution. It follows that all rational solutions of the first equation of (25) are in form af where a is an arbitrary nonzero element in $C(x_2)$. Our task is to decide for which a the second equation of (25) has a nonzero rational solution $g(x_1, x_2)$. Second, assume $\sigma_2(g) = afg$, so $\mathrm{bv}_1(\sigma_2(g))$=$\mathrm{bv}_1(afg)$. By Lemma 8, $c_1 \sigma_2(\mathrm{bv}_1(g)) = \mathrm{bv}_1(a)\mathrm{bv}_1(f)\mathrm{bv}_1(g)$. Since $\mathrm{bv}_2(a)$ is in C, we have $\sigma_2(\mathrm{bv}_1(g)) = c\,\mathrm{bv}_1(f)\mathrm{bv}_1(g)$. Consequently, there are nonzero $z \in \mathbb{F}$ and $c \in C$ such that

$$\sigma_2(z) = c\,\mathrm{bv}_1(f) z. \tag{26}$$

Note that such a z is a rational solution of (17), because the function $c\,\mathrm{bv}_1(f)$ is a solution of the first equation of (25). The problem of finding rational solutions of (17) is so reduced to Problem P described in Section 7.

For the mixed case, set v=$(u_1 u_2 + \delta_1(u_2) - \sigma_2(u_1) u_2)/u_2$ and $y = \sigma_2(z)/z$. Equation (18) then becomes

$$\delta_1(y) = vy \quad \text{and} \quad \sigma_2(z) = yz. \tag{27}$$

In the same vein, we compute a nonzero rational solution f of the first equation in (27), and then, find a nonzero rational function $z \in \mathbb{F}$ and a nonzero constant $c \in C$ that solve (26). Such a rational function z is a solution of (18).

EXAMPLE 9. *Let $\mathbb{A}$ be as in Example 1. We compute the common associate of h_1=$\exp(-x)x^{k+1}$ and h_2=$kx^k/(x-k)$. The system (27) becomes*

$$\frac{\partial y(x,k)}{\partial x} = vy(x,k) \text{ and } z(x,k+1) = c\mathrm{bv}_k(y) z(x,k),$$

where $v = \frac{-1}{(x-k)(x-k-1)}$ and $c \in C$. The first equation has a rational solution $y = (k-x)/(k-x+1)$. The second then has a solution $1/(k-x)$ with $c = 1$ (see Example 8). By Theorem 7, a common associate of h_1 and h_2 has ∂_x-certificate $(1+k-x)/x$ and E_k-certificate $(k+1)x/k$. The associate can be written as $k\exp(-x)x^{k+1}$. □

We can now outline an algorithm for hyperexponential solutions of an ideal I with finite rank. Assume that we are given a finite basis of I. First, we compute a Gröbner basis of I and then use linear algebra to construct a rectangular system $\mathcal{P} = \{P_1, P_2\}$ contained in I. In fact, the rectangular system will be of minimal degrees. We then find the set H_i of hyperexponential solutions of P_i with respect to ∂_i, for $i = 1, 2$. If one of the H_i's is empty, then I has no hyperexponential solution. Otherwise we choose dissimilar elements $h_{i1}, \ldots, h_{im_i}$ such that any element of H_i is similar to one and only one h_{ij_i}. For each pair (h_{1j_1}, h_{2j_2}), we then use Theorem 7 to compute a common associate, discarding pairs without common associates. If no common associate is found then we are done. Otherwise assume that the common associates are $h_1, \ldots, h_m$. Then, for each $k = 1, \ldots, m$, apply the P_i's to rh_k to obtain a new rectangular system $\mathcal{Q}_k$ over $\mathbb{F}$, since $\partial_i \bullet h_k = u_{ik} h_k$ with $u_{ik} \in \mathbb{F}$. Find rational solutions of $\mathcal{Q}_k$. If no rational solution is found, discard h_k. Write the general rational solution r_k of $\mathcal{Q}_k$ as a linear combination of rational functions, in which unspecified constants appear linearly. Apply each operator in the given basis to $r_k h_k$ to get a linear algebraic system in these constants and solve these linear systems.

EXAMPLE 10. *Let $\mathbb{A}$ be as in Example 1 and let us find hyperexponential solutions of the rank-four ideal given in Example 2. We use the Maple command* `DEtools[expsols]` *to find that all hyperexponential solutions of* L_1 *are similar to* $h_1 = \exp(-x)x^{k+1}$ *with respect to* ∂_x. *Likewise, the command* **LREtools[hypergeomsols]** *finds that all hyperexponential (hypergeometric) solutions of* L_2 *are similar to the term* $h_2 = \frac{kx^k}{x-k}$ *with respect to* E_k. *As shown in Example 9, a common associate of* h_1 *and* h_2 *is* $h = k\exp(-x)x^{k+1}$. *Hence, all hyperexponential solutions of* I *are similar to* h. *Apply* L_1 *and* L_2 *to* rh *to get a new rectangular system (in* r*), which is the first system in Example 7. Taking the rational solution of the system from Example 7, we conclude that the ideal* I *has hyperexponential solutions in the form*

$$\frac{c_1 + c_2 k + c_3 x}{x - k}\exp(-x)x^{k+1}. \qquad \square$$

EXAMPLE 11. *Consider the shift algebra given in Example 5. We compute the hyperexponential (hypergeometric) solutions of the ideal* I *generated by*

$$\begin{aligned} L_1 \quad = \quad & E_m^2 + \frac{18+(m-k)(9-(m+k)(m+k+1))}{(m-k)(m+k-3)-3}E_m \\ & + \frac{3(4k^2-10k+(m-k)(2k-5+(m+k)(m+k-1)))}{(m-k)(m+k-3)-3} \end{aligned}$$

and

$$\begin{aligned} L_2 \quad = \quad & E_k^2 - \frac{2+(m-k)(4(m+k)(m+k+1)-1)}{(m-k)(2m+2k-1)+1}E_k \\ & + \frac{(m-k)((m+k)(4m+4k-2)+-8k)-16k^2}{(m-k)(2m+2k-1)+1}. \end{aligned}$$

The operators L_1 *and* L_2 *form a Gröbner basis of* I. *The hyperexponential solutions of* L_1 *with respect to* E_m *are either similar to* $h_{11}{=}3^m$ *or to* $h_{12}{=}\Gamma(m+k)$*, while the hyperexponential solutions of* L_2 *are either similar to* $h_{21}{=}1$ *or to* $h_{22}{=}2^k\Gamma(m{+}k)$. *The pairs* (h_{11}, h_{22}) *and* (h_{21}, h_{22}) *have no common associates, because the system (25) for each of the pairs has no rational solution. On the other hand, the pair* (h_{11}, h_{21}) *has a common associate* $h_1 = 3^m$, *and* (h_{12}, h_{22}) *has a common associate* $h_2 = 2^k\Gamma(m+k)$. *Hence, the hyperexponential solutions of* I *are in the form of either* r_1h_1 *and* r_2h_2, *where* r_1, r_2 *are in* $C(m,k)$. *Using the algorithm given in Section 7, we get* $r_1{=}(m-k)$ *and* $r_2{=}1$. *The hyperexponential solutions of* I *are then* $c_1(m-k)3^k$ *and* $c_2 2^k\Gamma(m+k)$*, where* c_1 *and* c_2 *are constants.* $\square$

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