A uniform approach for Hermite Padé and simultaneous Padé Approximants and their Matrix-type generalizations

Bernhard Beckermann
Institut für Angewandte Mathematik
Universität Hannover, Welfengarten 1, W-3000 Hannover, Germany
and
George Labahn
Department of Computing Science
University of Waterloo, Waterloo, Ontario, Canada

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Abstract

This paper introduces the notion of a *power Hermite Padé approximant*, a generalization of the classical scalar Hermite Padé approximant. We show that this generalized form provides a uniform approach for different concepts of matrix-type Padé approximants. This includes descriptions of vector and matrix Padé approximants along with generalizations of simultaneous and Hermite Padé approximants.

A complete description of these new approximants, based on the characterization of a corresponding linear solution space, is given. A Padé-like table is introduced and the singular structure is studied. It is shown that the geometric structure of the singular blocks of this new table is made up of one or more combinations of triangles. In the special case of matrix Padé approximants the geometric structure of the combined singular areas consists of square blocks - exactly the same as in the classical scalar Padé case.

Key words: Vector Padé approximant, Hermite Padé approximant, simultaneous Padé approximant, matrix Padé approximant.

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1 Introduction

Throughout this paper we will assume that m is an integer with $m \geq 2$ and that $\mathbf{F} = (f_1, \ldots, f_m)^T$ is an m-tuple of formal power series with coefficients from a field $\mathbb{K}$ (typically a subfield of either the real or complex numbers). Moreover, for a space $\mathcal{S}$ with scalars from $\mathbb{K}$ (for instance $\mathcal{S} = \mathbb{K}^{p \times q}$, the space of $p \times q$ matrices over $\mathbb{K}$), $\mathcal{S}[z]$ will denote the set of polynomials in z with coefficients from $\mathcal{S}$ while $\mathcal{S}[[z]]$ represents the set of formal power series in z with coefficients from $\mathcal{S}$.

Hermite introduced two different types of generalizations of the ordinary Padé table. Given a multi-index $\mathbf{n} = (n_1, \ldots, n_m) \in (\mathbb{N}_0 \cup \{-1\})^m$, a *Hermite Padé approximant* of type $\mathbf{n}$ is a nontrivial tuple $\mathbf{P} = (P_1, \ldots, P_m) \in \mathbb{K}^{1 \times m}[z]$ of polynomials P_l having degrees bounded by the n_l such that

$$\mathbf{P}(z) \cdot \mathbf{F}(z) = P_1(z) f_1(z) + \ldots + P_m(z) f_m(z) = z^{\|\mathbf{n}\|-1} \cdot R(z) \text{ with } R \in \mathbb{K}[[z]], \tag{1}$$

where the norm of the multi-index $\mathbf{n}$ is defined by $\|\mathbf{n}\| := (n_1 + 1) + \ldots + (n_m + 1)$. In contrast, a *simultaneous Padé approximant* $\mathbf{Q} = (Q_1, \ldots, Q_m) \in \mathbb{K}^{1 \times m}[z]$ of type $\mathbf{n}$ consists of polynomials Q_l having degrees bounded by $\|\mathbf{n}\| - m - n_l$ such that for all $l, \lambda \in \{1, \ldots, m\}$

$$Q_l(z) \cdot f_\lambda(z) - Q_\lambda(z) \cdot f_l(z) = z^{\|\mathbf{n}\|-m+1} \cdot R_{l,\lambda}(z) \text{ with } R_{l,\lambda} \in \mathbb{K}[[z]]. \tag{2}$$

Obviously, if for example $f_m(0) \neq 0$, then for (2) it remains to consider the indices $\lambda = m, l \in \{1, 2, \ldots, m-1\}$. [1]

The Hermite Padé approximation problem includes many classical approximation problems such as Padé approximation ($m = 2$, $\mathbf{F} = (1, -f)^T$), algebraic approximants and G^3J approximants. We refer the reader to [1, Part II,pp.32-40] for further examples and [2, 3] or [12] for a bibliography.

As pointed out in [5, 6, 7, 8, 10], a Hermite Padé approximant $\mathbf{P}$ of type $(n_1 - 1, \ldots, n_{l-1} - 1, n_l, n_{l+1} - 1, \ldots, n_m - 1)$ and a simultaneous Padé approximant $\mathbf{Q}$ of type $(n_1, \ldots, n_{\lambda-1}, n_\lambda - 1, n_{\lambda+1}, \ldots, n_m)$ are connected via the duality relation

$$\mathbf{P}(z) \cdot \mathbf{Q}^T(z) = 0 \text{ if } l \neq \lambda \text{ and } \mathbf{P}(z) \cdot \mathbf{Q}^T(z) = c \cdot z^{\|\mathbf{n}\|-m} \text{ with } c \in \mathbb{K}, \text{ if } l = \lambda. \tag{3}$$

This duality relation has been used, for example, to derive algorithms where both types of approximants are computed simultaneously. Also, by this formula one could determine the structure of the singular simultaneous Padé solution table since the structure for Hermite Padé approximation is well-known [2]. The aim of this paper is to give a new uniform approach for both approximation problems instead of applying duality arguments. This approach not only includes Hermite Padé and simultaneous Padé approximants but also their matrix-type generalizations as introduced by several authors in the last years.

The paper is organized as follows: in Section 2 we introduce the power Hermite Padé approximant - a scalar concept that is a natural generalization of a matrix Padé approximant. These are also shown to provide a uniform description of both Hermite

[1]Following [6, 7, 10], $\mathbf{P}$ (and $\mathbf{Q}$) is also called the vector of "Latin" or "System I" polynomials ("German" or "System II" polynomials, respectively).

Padé and simultaneous Padé approximants. In Section 3 a linear system associated to these approximants is studied and a basis for this system is determined. In Section 4 we introduce the notion of a power Hermite Padé table and study its singular structure. A recursive algorithm to efficiently and reliably solve the power Hermite Padé approximation problem will be presented in a later paper [4].

2 Vector and Power Hermite Padé Approximants

The original motivation for our work comes from the study of matrix Padé approximants. These are defined as follows: let $p, q, r \in \mathbb{N}$, $M, N \in \mathbb{N}_0$ and $A \in \mathbb{K}^{p\times q}[[z]]$. Then a left-hand rectangular Matrix-Padé Form (P, Q) consists of $P \in \mathbb{K}^{r\times q}[z]$, $Q \in \mathbb{K}^{r\times p}[z]$, with *deg* $P \leq M$, *deg* $Q \leq N$ and the rows of Q being linearly independent over $\mathbb{K}$ such that $P(z) - Q(z)\cdot A(z) = z^{M+N+1}\cdot R(z)$, $R \in \mathbb{K}^{r\times q}[[z]]$. Of course one can also define a right-hand rectangular Matrix-Padé form in a similar manner. [2] We can rewrite the order condition for left-hand rectangular forms as $\mathbf{P}(z)\cdot\mathbf{G}(z) = z^{M+N+1}\cdot\tilde{R}(z)$ with $\mathbf{P}$ being a row of $(P, Q) \in \mathbb{K}^{r\times(p+q)}[z]$, $\mathbf{G} = \begin{pmatrix} \mathbf{I} \\ -A \end{pmatrix} \in \mathbb{K}^{(p+q)\times q}[[z]]$, $\mathbf{I}$ denoting an identity matrix of suitable size and $\tilde{R} \in \mathbb{K}^{1\times q}[[z]]$. This leads to the following canonical extension of the Hermite Padé definition to the vector case:

Definition 2.1. (Vector Hermite Padé Problem) *Let $s, \tau \in \mathbb{N}_0$, $s \geq 1$, $\mathbf{G} \in \mathbb{K}^{m\times s}[[z]]$ and let $\mathbf{n}$ be a multi-index. Find (at least $\|\mathbf{n}\| - s\cdot\tau$ many) linearly independent polynomial tuples $\mathbf{P} = (P_1, \ldots, P_m) \in \mathbb{K}^{1\times m}$ with* $\deg P_l \leq n_l$, $1 \leq l \leq m$ *such that $\mathbf{P}(z)\cdot\mathbf{G}(z) = z^\tau\cdot R(z)$, $R \in \mathbb{K}^{1\times s}[[z]]$.* □

Note that the problem of computing a simultaneous Padé approximant of type $(\rho_1, \rho_2, \ldots, \rho_m)$, $\rho = \rho_1 + \ldots + \rho_m$, also can be translated into the vector Hermite Padé formalism by setting $\mathbf{n} = (\rho - \rho_1, \ldots, \rho - \rho_m)$, $s = m-1$, $\tau = \rho+1$, (and hence $\|\mathbf{n}\| - s\cdot\tau = 1$) and

$$\mathbf{G}(z) = \begin{bmatrix} f_m(z) & 0 & \ldots & 0 & -f_1(z) \\ 0 & f_m(z) & \ddots & \vdots & -f_2(z) \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \ldots & 0 & f_m(z) & -f_{m-1}(z) \end{bmatrix}^T \in \mathbb{K}^{m\times(m-1)}[[z]]. \tag{4}$$

Further examples of vector Hermite Padé approximants are given in [4].

Since most of the results in this field are obtained for scalar approximation problems, it is of special interest to imbed the vector Padé approximation problem into a more general scalar concept. The method of accomplishing this is to apply the small 'trick'

$$\mathbf{F}(z) := \mathbf{G}(z^s)\cdot(1, z, \ldots, z^{s-1})^T \in \mathbb{K}^{m\times 1}[[z]]. \tag{5}$$

Definition 2.2. (Power Hermite Padé Approximant) *For a* $\mathbf{P} = (P_1, \ldots, P_m) \in \mathbb{K}^{1\times m}[z]$ *we define its* defect *(with respect to the multi-index* $\mathbf{n} = (n_1, \ldots, n_m)$*) and its*

[2] Rectangular-matrix types of Padé forms are used, for example, to compute the inverse of matrices partitioned into a rectangular-block Hankel or Toeplitz structure [9].

s- order *(with respect to* $s \in \mathbb{N}$ *) by*

$$\mathrm{dct}_{\mathbf{n}}\mathbf{P} := \min_{l}\{n_l + 1 - \deg\, P_l\}$$
$$\mathrm{ord}_s\mathbf{P} := \sup\{\sigma \in \mathbb{N}_0 \ : \mathbf{P}(z^s) \cdot \mathbf{F}(z) = z^\sigma \cdot R(z) \ \textit{with}\ R \in \mathbb{K}\,[[z]]\}.$$

where the zero polynomial has degree $-\infty$*. Then* $\mathbf{P} = (P_1, \ldots, P_m)$ *is a* Power Hermite Padé Approximant *(PHPA) of type* $(\mathbf{n}, \sigma, s)$*,* $\sigma \in \mathbb{N}_0$ *, if it satisfies the conditions*

$$\mathrm{ord}_s\mathbf{P} \geq \sigma \quad \textit{and} \quad \mathrm{dct}_{\mathbf{n}}\mathbf{P} > 0. \tag{6}$$

More generally, we define the finite-dimensional space $\mathcal{L}_\delta^\sigma$ *by*

$$\textit{for}\ \sigma \in \mathbb{N}_0\,,\ \delta \in \mathbb{Z} \cup \{+\infty\}\textit{:}\ \mathcal{L}_\delta^\sigma = \{\mathbf{P} \in \mathbb{K}^{1\times m}[z] : \mathrm{dct}_{\mathbf{n}}\mathbf{P} > -\delta, \mathrm{ord}_s\mathbf{P} \geq \sigma\}. \tag{7}$$

□

Note that the classical Hermite Padé approximation problem is included by setting $s = 1$ and $\sigma = \|\mathbf{n}\| - 1$. By equating coefficients, equation (6) results in a system of homogeneous linear equations. By comparing the number of unknowns to equations one can conclude that there exist at least $\|\mathbf{n}\| - \sigma$ PHPA's of type $(\mathbf{n}, \sigma, s)$ which are linearly independent over $\mathbb{K}$. Finally, we see from (5) that computing Vector Hermite Padé approximants of type $(\mathbf{n}, \tau)$ and dimension s is equivalent to the determination of PHPA's of type $(\mathbf{n}, \tau \cdot s, s)$, i.e. of the solution set $\mathcal{L}_0^{\tau \cdot s}$.

3 The PHPA solution set

Adapting the techniques of [2, Section 4], we obtain

Theorem 3.1. (Bases for the PHPA solution set) *For each* $\sigma \in \mathbb{N}_0$ *and for each multi-index* $\mathbf{n} = (n_1, \ldots, n_m)$ *there exist* $\mathbf{P}_1, \ldots, \mathbf{P}_m \in \mathbb{K}^{1\times m}[z]$ *such that for all* $\delta \in \mathbb{Z} \cup \{+\infty\}$

$$\textit{dim}\ \mathcal{L}_\delta^\sigma = \max\{\mathrm{dct}_{\mathbf{n}}\mathbf{P}_1 + \delta, 0\} + \ldots + \max\{\mathrm{dct}_{\mathbf{n}}\mathbf{P}_m + \delta, 0\}. \tag{8}$$
$$\mathcal{L}_\delta^\sigma = \{\alpha_1 \cdot \mathbf{P}_1 + \ldots + \alpha_m \cdot \mathbf{P}_m \,:\, \alpha_l \in \mathbb{K}\,[z], \deg\, \alpha_l < \mathrm{dct}_{\mathbf{n}}\mathbf{P}_l + \delta\} \tag{9}$$

Proof: For $\mathbf{P}, \mathbf{Q} \in \mathbb{K}^{1\times m}[z]$, $\alpha \in \mathbb{K}\,[z]$ we have

$$dct_{\mathbf{n}}(\mathbf{P} + \alpha \cdot \mathbf{Q}) \geq \min\{dct_{\mathbf{n}}\mathbf{P}, dct_{\mathbf{n}}\mathbf{Q} - deg\ \alpha\}.$$

Hence as in [2, p.14] we can construct $\mathbf{P}_1, \ldots, \mathbf{P}_m$, $\mathbf{P}_l = (P_{l,1}, \ldots, P_{l,m})$ by recurrence on $-\delta$ using the following rules: set $U_1 := \mathcal{L}_{+\infty}^\sigma$ and for $\lambda = 1, 2, \ldots, m$

$$\text{choose } \mathbf{P}_\lambda \in U_\lambda \text{ such that } dct_{\mathbf{n}}\mathbf{P}_\lambda = \max\{dct_n\mathbf{Q} : \mathbf{Q} \in U_\lambda\}\ , \tag{10}$$
$$\text{choose } l_\lambda \in \{1, \ldots, m\} \text{ such that } deg\ P_{\lambda, l_\lambda} = n_{l_\lambda} + 1 - dct_n P_\lambda \geq 0\ , \tag{11}$$
$$\text{define } U_{\lambda+1} := \{\mathbf{Q} = (Q_1, \ldots, Q_m) \in U_\lambda : deg\ Q_{l_\lambda} \leq n_{l_\lambda} - dct_n\mathbf{Q}\}\ . \tag{12}$$

□

Note that, since the components of a $\mathbf{P}_\lambda$ can only contain a common factor of the form z^j, the approximant P_λ is reducible if and only if $\mathbf{P}_\lambda(0) = 0$.

As an immediate consequence of Theorem 3.1, in general, the (left hand) square matrix Padé approximation problem as stated in the beginning of Section 2 ($p = q = r =: s$ and $m = 2s$) does not have a unique rational solution like in the scalar case. Moreover, there are three distinct and possible forms of a denominator matrix polynomial Q. First, the case occurs when $Q(z)$ is singular for all z [3] and hence no matrix rational form exists. This type of degeneracy is not found in the scalar case. Secondly, it is possible that $Q(0)$ is non-singular. Here we can form $Q(z)^{-1} \cdot P(z)$ and its matrix power series agrees with $A(z)$ to the full order condition. Finally, if $Q(z)$ is non-singular for some z but $Q(0)$ is singular, we can cancel P and Q by a common matrix polynomial factor on the right. Here, similar to the degenerate case found in scalar Padé approximation, the resulting matrix rational form $Q(z)^{-1} \cdot P(z)$ does not agree any more with $A(z)$ to the full order condition.

4 The PHPA Table

We have several degrees of freedom in defining a table of PHPA approximants. For example, we can consider the m-dimensional table of approximants of type $(\mathbf{n}, s, \|\mathbf{n}\| + t)$ with fixed m, s, t and parameter $\mathbf{n}$. This approach is of course influenced by the well-known results for Hermite Padé approximation [2, 11]. Rather than proceeding as mentioned above, we instead define a two-dimensional table of approximants by introducing the multi-indices

$$\text{for } M, N \in \mathbb{N}_0 \cup \{-1\}\text{: } \mathbf{n}(M,N) = \left(\underbrace{M, \ldots, M}_{\text{s}}, \underbrace{\left[\tfrac{sN+m-s-1}{m-s}\right], \ldots, \left[\tfrac{sN}{m-s}\right]}_{\text{m-s}} \right) \tag{13}$$

where $[\cdot]$ denotes the Gauß function. Then as an (M, N) entry of our PHPA table we take all PHPA's of type $(\mathbf{n}(M,N), s, s \cdot (M+N) + m - t)$ where $t := \min\{s, m-s\}$. Since $\|\mathbf{n}(M,N)\| = s \cdot (M+N) + m$, we always have at least t PHPA's that are linearly independent over $\mathbb{K}$.

Before discussing features of our PHPA table, let us have a closer look at special cases. Obviously, for $m = 2s = 2$, we obtain the classical (linearized) Padé table. For $s = 1$, $m > 2$, the PHPA table is a two-dimensional cut of the Hermite Padé table (see, e.g., [2]), more precisely, on position (M, N) we find (scalar) Hermite Padé approximants of type $(M, [\frac{N+m-2}{m-1}], \ldots, [\frac{N+1}{m-1}], [\frac{N}{m-1}])$. For $s = m-1$, $m > 2$, and $\mathbf{F}$ chosen as in (4),(5), the PHPA table contains simultaneous Padé approximants of type $(\rho_1, \ldots, \rho_{m-1}, \rho_m) = (N, \ldots, N, M + (2-m) \cdot N)$. Finally, for $2s = m > 2$ and hence $t = s > 1$, an (M, N) entry of the PHPA table can be used as a row of a left-hand square Matrix Padé form (P, Q) of dimension s with numerator degree M and denominator degree N. It is the latter example that motivates the approach that we have taken in defining the PHPA table.

[3] More precisely, its rows are linearly dependent over $\mathbb{K}[z]$.

Although having a different interpretation for different m, s, we are interested in singular blocks in the PHPA table. Here we distinguish between so-called *elementary* and *combined* singular blocks, the first being a set of coordinates (μ, ν) with a common PHPA entry $\mathbf{P}$ whereas for the second set we only demand that at position (μ, ν) we find a polynomial multiple of $\mathbf{P}$. [4] For Padé approximation ($m = 2$, $s = 1$) it is well-known that (i) elementary singular blocks are triangles and that (ii) maximal combined singular blocks are induced by irreducible approximants, (iii) look like squares, and (iv) never intersect (e.g., [1, Part I, pp.19-31]). In the next Theorem we show that (i) and (ii) also hold for PHPA tables for arbitrary $1 \le s < m$, (iii) still holds for arbitrary $2s = m \ge 2$ and in general (iv) is not valid for $m > 2$. For a PHPA $\mathbf{P}$, the following auxiliary integers are used: $M = M(\mathbf{P})$, $N = N(\mathbf{P})$ and $d = d(\mathbf{P})$, are uniquely defined by the relations $\mathit{deg}\, \mathbf{P} \le \mathbf{n}(M, N)$ (componentwise) but $\mathit{deg}\, \mathbf{P} \not\le \mathbf{n}(M-1, N)$, $\mathit{deg}\, \mathbf{P} \not\le \mathbf{n}(M, N-1)$, and $d \cdot s + m - t \le \mathit{ord}_s \mathbf{P} < (d+1) \cdot s + m - t$.

Theorem 4.1. (Singular blocks in the PHPA table) *Elementary singular blocks always have the form of a triangle. More precisely, a PHPA $\mathbf{P}$ is a (μ, ν) entry of the PHPA table if and only if*

$$M(\mathbf{P}) \le \mu \ , \ N(\mathbf{P}) \le \nu \ , \text{ and } \mu + \nu \le d(\mathbf{P}) \ . \tag{14}$$

The combined singular block induced by $\mathbf{P}$ contains exactly those coordinates (μ, ν) with

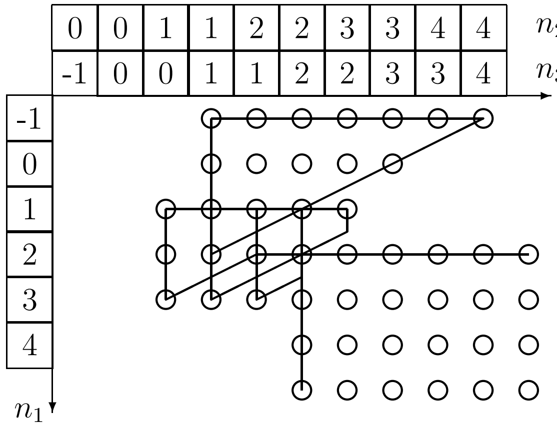
$$\begin{aligned} &M(\mathbf{P}) \le \mu \ , \ N(\mathbf{P}) \le \nu \le d(\mathbf{P}) - M(\mathbf{P}) \ , \text{ and} \\ &(m-s) \cdot \mu + (m-2s) \cdot \nu \le (m-s) \cdot d(\mathbf{P}) - s \cdot N(\mathbf{P}) + \kappa(\mathbf{P}) \end{aligned} \tag{15}$$

with a $\kappa(\mathbf{P}) \in \{0, 1, \ldots, s-1\}$. In addition, maximal combined singular blocks are induced by irreducible PHPA's.

Proof: The geometrical form (14) of an elementary singular block follows immediately from the definition of the PHPA table and of $M(\mathbf{P}), N(\mathbf{P}), d(\mathbf{P})$. For the second assertion it is sufficient to show that (15) describes the union of the elementary singular blocks induced by $z^j \cdot \mathbf{P}$, $j = 0, 1, 2, \ldots$. This is a direct consequence of the identities $M(z^j \cdot \mathbf{P}) = M(\mathbf{P}) + j$, $d(z^j \cdot \mathbf{P}) = d(\mathbf{P}) + j$ and $N(z^j \cdot \mathbf{P}) = N(\mathbf{P}) + 1 + [\frac{(m-s) \cdot j - \kappa - 1}{s}]$ with a suitable $\kappa = \kappa(\mathbf{P}) \in \{0, \ldots, s-1\}$ as above.
Finally, suppose that $\mathbf{P}$ is reducible, i.e. there exist $c \in \mathbb{K}$ and $\mathbf{Q} \in \mathbb{K}^{1 \times m}[z]$ with $\mathbf{P} = (z - c) \cdot \mathbf{Q}$. Then for each $j \in \mathbb{N}_0$ the elementary singular block induced by $z^j \cdot \mathbf{P}$ is a subset of that induced by $z^{j+1} \cdot \mathbf{Q}$. Hence the combined singular block induced by $\mathbf{P}$ is a subset of that induced by $\mathbf{Q}$ but not vice versa. $\square$

Figures 1-3 show possible maximal combined singular blocks for different special PHPA tables. The corresponding approximants are given in Tables 1-3. Notice that in each example there are intersecting combined singular blocks. [5]

[4] For the sake of simplicity, we will not discuss singular blocks at the 'border' of the PHPA table.
[5] The data required for the solution tables are obtained by the algorithm described in [4].

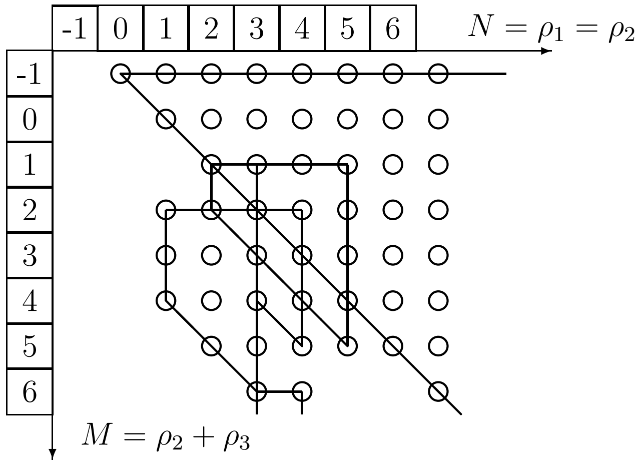


$f_1(z) = 2z^6 + \frac{1+z}{1-z}$
$f_2(z) = 1$
$f_3(z) = 2z^6 + \frac{1+z}{1-2z}$

FIG. 1: PHPA table for Hermite Padé approximation, $m = 3$, $s = 1$

Approximant	$M(\mathbf{P})$	$N(\mathbf{P})$	$d(\mathbf{P})$	combined block
$\mathbf{P}_1 = (-1+z, 1+z, 0)$	1	1	4	$\mathbf{P}_1$, $z\mathbf{P}_1$
$\mathbf{P}_2 = (0, 1+z, -1+2z)$	-1	2	4	$\mathbf{P}_2$, $z\mathbf{P}_2$, $z^2\mathbf{P}_2$, $z^3\mathbf{P}_2$
$\mathbf{P}_3 = (1-z, 0, -1+2z)$	1	2	5	$\mathbf{P}_3$, $z\mathbf{P}_3$
$\mathbf{P}_4 = (1-2z+z^2, z+z^2, -1+2z)$	2	3	6	$\mathbf{P}_4$
$\mathbf{P}_5 = (1-z, z+z^2, -1+z+2z^2)$	1	4	6	$\mathbf{P}_5$
$\mathbf{P}_6 = (1-3z+2z^2, z+z^2, -1+3z-2z^2)$	2	4	∞	$\mathbf{P}_6$

TABLE 1: Corresponding Hermite Padé approximants, $M \leq 4$, $N \leq 8$

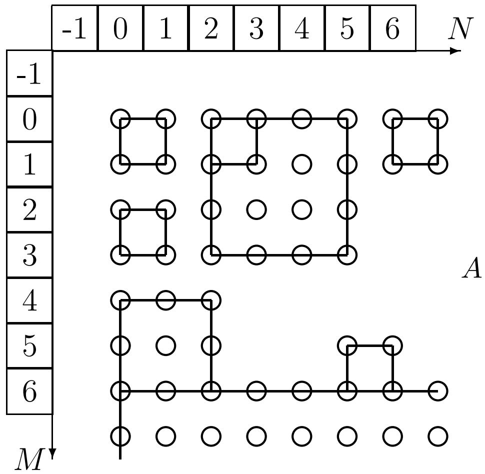


$$f_1(z) = 2z^6 + \frac{1+z}{1-z}$$
$$f_2(z) = 2z^6 + \frac{1+z}{1-2z}$$
$$f_3(z) = 1$$

FIG. 2: PHPA table for simultaneous Padé approximation, $m = 3$, $s = 2$

Approximant	$M(\mathbf{P})$	$N(\mathbf{P})$	$d(\mathbf{P})$	combined block
$\mathbf{P}_1 = (0, 0, 1)$	-1	0	-1	$z^j\mathbf{P}_1,\ j \in \mathbb{N}_0$
$(0, 0, z^{2j})$	-1	j	$2j-1$	
$(0, 0, z^{2j+1})$	-1	$j+1$	$2j$	
$\mathbf{P}_2 = (1-2z, 1-z,$ $1-4z+6z^2-6z^3+6z^4)$	1	2	4	$z^j\mathbf{P}_2,\ j = 0, 1, 2$
$\mathbf{P}_3 = (1-2z, 1-z,$ $1-4z+6z^2-6z^3+6z^4-6z^5+4z^6)$	1	3	6	$z^j\mathbf{P}_3,\ j = 0, 1, 2, 3, 4$
$\mathbf{P}_4 = (1-z-2z^2, 1-z^2,$ $1-3z+2z^2)$	2	1	5	$z^j\mathbf{P}_4,\ j = 0, 1, 2, 3, 4$
$\mathbf{P}_5 = (3-4z-4z^2, 3-z-2z^2,$ $3-10z+10z^2-6z^3+6z^4-6z^5)$	2	3	6	$z^j\mathbf{P}_5,\ j = 0, 1, 2, 3$
$\mathbf{P}_6 = (1+2z+2z^2+2z^3+2z^4-61z^5-60z^6,$ $1+3z+6z^2+12z^3+24z^4-15z^5-29z^6,$ $1-63z^5+62z^6)$	6	3	10	$z^j\mathbf{P}_6,\ j = 0, 1, 2$

TABLE 2: Corresponding simultaneous Padé approximants, $M \leq 6$, $N \leq 6$



$$A(z) = \begin{bmatrix} 1+z^2+2z^4-z^5+z^6 & -z^5 \\ z^7 & 1+z^2+z^4+z^7 \end{bmatrix}$$

Fig. 3: PHPA table for left-hand Matrix Padé approximation, $m = 4$, $s = 2$

Approximant	$M(\mathbf{P})$	$N(\mathbf{P})$	$d(\mathbf{P})$	combined block
$\mathbf{P}_1 = (1, 0, 1, 0)$	0	0	1	$z^j\mathbf{P}_1$, $j = 0, 1$
$\mathbf{P}_2 = (0, 1, 0, 1)$	0	0	1	$z^j\mathbf{P}_2$, $j = 0, 1$
$\mathbf{P}_3 = (1, 0, 1 - z^2, 0)$	0	2	3	$z^j\mathbf{P}_3$, $j = 0, 1$
$\mathbf{P}_4 = (0, 1, 0, 1 - z^2)$	0	2	5	$z^j\mathbf{P}_4$, $j = 0, 1, 2, 3$
$\mathbf{P}_5 = (1 + z^2, 0, 1, 0)$	2	0	3	$z^j\mathbf{P}_5$, $j = 0, 1$
$\mathbf{P}_6 = (0, 1 + z^2, 0, 1)$	2	0	3	$z^j\mathbf{P}_6$, $j = 0, 1$
$\mathbf{P}_7 = (1, -2, 1 - z^2 - z^4$ $+z^5 + 2z^6, -2 + 2z^2 + z^5 - 2z^6)$	0	6	7	$z^j\mathbf{P}_7$, $j = 0, 1$
$\mathbf{P}_8 = (0, 1 + z^2 + z^4, 0, 1)$	4	0	6	$z^j\mathbf{P}_8$, $j = 0, 1, 2$
$\mathbf{P}_9 = (1 + z + z^2 + 2z^4, 1 - z + 2z^2 - z^3$ $+2z^4 - 3z^5, 1 + z - z^3, 1 - z + z^2 - z^5)$	5	5	11	$z^j\mathbf{P}_9$, $j = 0, 1$
$\mathbf{P}_{10} = (1 + z^2 + 2z^4 - z^5 + z^6, -z^5, 1, 0)$	6	0	∞	$\mathbf{P}_{10}$

Table 3: Corresponding left-hand Matrix Padé approximants, $M \leq 6$, $N \leq 6$

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