Inversion of Toeplitz Structured Matrices Using Only Standard Equations

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Abstract

Formulas for the inverse of layered or striped Toeplitz matrices in terms of solutions of standard equations are given. These results are also generalized, in the generic case, to mosaic Toeplitz matrices and also to Toeplitz plus Hankel matrices.

Key words: Toeplitz matrices, Hankel matrices, Mosaic Toeplitz matrices, Toeplitz plus Hankel matrices, matrix inversion.

Subject Classifications: AMS(MOS): 15A09, 15A57.

1 Introduction

Gohberg and Semencul [5] have shown that for the generic case of a Toeplitz matrix $A = [a_{i-j}]_{i,j=1}^{m}$, it is enough to solve the two equations

$$A \cdot X = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad A \cdot Y = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

in order to obtain A^{-1}. Note that there are $2m-1$ involved parameters in the definition of Toeplitz matrices and the requirement of solving two sets of linear equations is therefore minimal. Denote by $\{E^{(i)}\}_{i=1}^{m}$ the standard basis, and define an equation of the type $A \cdot X = E^{(i)}$ as a standard equation. In Gohberg and Krupnik [4] it is shown that generically one can use the standard equations $A \cdot X = E^{(1)}$, $A \cdot Z = E^{(2)}$ in order to obtain A^{-1}. Ben-Artzi and Shalom [1] have generalized these results showing that three standard equations, when properly chosen, will always be enough in order to construct the inverse of a Toeplitz matrix. Recently, the authors [12] proved that actually two standard equations are sufficient as the solution of the third equation can be obtained using the entries $[a_{i-j}]$ as well.

In this paper, we consider wider classes of matrices. Let $A = [A_i]_{i=1}^{k}$ be an m-by-m matrix in which $A_i = [a_{p-q}^{(i)}]_{p=1,q=1}^{m_i \ \ m}$ (with $\sum_{i=1}^{k} m_i = m$), namely, each A_i is an m_i-by-m Toeplitz structured matrix. Such a matrix is referred to as a layered Toeplitz matrix while its transpose is a striped Toeplitz matrix. Note that such a matrix involves $(k+1) \cdot m - k$ parameters. Lerer and Tismenetsky [13] have shown that solving $k+1$ systems of equations, not all being standard equations, is enough to reconstruct A^{-1}. We show that this can be done with $k+1$ standard equations in the following way:

Theorem 1.1. Let A be an m-by-m layered Toeplitz matrix with layers of size $m_1, \ldots, m_k$. Suppose there are solutions $X^{(M_p)} = col(x_j^{(M_p)})_{j=1}^{m}$ to the k standard equations

$$A \cdot X^{(M_p)} = E^{(M_p)} \quad p = 1, \ldots, k,$$

where, for each p, $M_p = 1 + \sum_{i<p} m_i$ denotes the first row of the p-th layer. For each p let h_p be the index of the largest non-zero component of $X^{(M_p)}$ and let j be any index such that $h_j = max\{h_p\}_{p=1,\ldots,k}$.

a) If $m_j < m - h_j + 1$ then A is not invertible.

b) If $m_j \geq m - h_j + 1$ and there is a solution to the standard equation

$$A \cdot X^{(M_j+m-h_j+1)} = E^{(M_j+m-h_j+1)}$$

then A is invertible with inverse

$$A^{-1} = row[row[(Q+P)^{i-1} \cdot X^{(M_p)}]_{i=1}^{m_p}]_{p=1}^{k}$$

where $Q = S_m - \sum_{p=1}^{k} X^{(M_p)} \cdot F^{(M_p)} \cdot A \cdot S_m$ (with S_m is the m-by-m lower shift matrix) and $P = (X^{(M_j+m-h_j+1)} - Q^{m-h_j+1} \cdot X^{(M_j)}) \cdot F^{(m)} / x_{h_j}^{(M_j)}$. $\square$

Layered and striped Toeplitz matrices are special cases of a larger family called mosaic Toeplitz matrices (see for example Heinig and Amdebrhan [8]). In Section 4, we show that in the generic case we can obtain the inverse through solutions of $k+\ell$ standard equations. Finally, in Section 5 we also consider the class of Toeplitz plus Hankel matrices (see [1], [9]) and show that in the generic case we need to solve four standard equations in order to determine the inverse matrix.

2 Preliminaries

In this section we prove some preliminary results necessary for our inversion formulae. For convenience the following notation is useful. For the rest of this paper $row(b_j)_{j=1}^k$ will denote the matrix $[b_1,\cdots,b_k]$, while $col(b_j)_{j=1}^k$ will denote the matrix $[b_1,\cdots,b_k]^T$. Note that this notation is valid if the b_j are scalars, vectors or even matrices of appropriate sizes. The following result from Gohberg and Shalom [6] will be used throughout this paper.

Lemma 2.1. Let A be an m-by-m matrix, and P, Q m-by-m matrices satisfying $P\cdot A = A\cdot Q$. Suppose $X^{(1)},\ldots,X^{(k)}$ are column vectors such that the matrix

$$R = row[row[P^{i-1}\cdot A\cdot X^{(p)}]_{i=1}^{m_p}]_{p=1}^{k}$$

is invertible (for a given set of integers $\{m_p\}$ with $m=\sum_{p=1}^k m_p$). Then A is invertible and has inverse given by

$$A^{-1} = row[row[Q^{i-1}\cdot X^{(p)}]_{i=1}^{m_p}]_{p=1}^{k}\cdot R^{-1}$$

□

Remark: The case $k=1$ of Lemma 2.1 first appeared in Ben-Artzi and Shalom [1].

Let $A = [A_{p,q}]_{p=1,q=1}^{k\ \ \ell}$ be an m-by-m matrix where $A_{p,q} = [a_{i-j}^{(p,q)}]_{i=1,j=1}^{m_p\ n_q}$ with $m = \sum_{p=1}^k m_p = \sum_{q=1}^{\ell} n_q$. Such a matrix is called a *mosaic Toeplitz matrix* having k layers and ℓ stripes. Define

$$S_{(M)} = \begin{bmatrix} S_{m_1} & & & \\ & S_{m_2} & & \\ & & \ddots & \\ & & & S_{m_k} \end{bmatrix} \quad \text{and} \quad S_{(N)} = \begin{bmatrix} S_{n_1} & & & \\ & S_{n_2} & & \\ & & \ddots & \\ & & & S_{n_q} \end{bmatrix}$$

where S_i is the i-by-i lower shift matrix $S_i = [\delta_{p,q+1}]_{p,q=1}^i$. In addition define integers M_p, $p=1,2,\ldots,k$ and N_q, $q=1,2,\ldots,\ell$ by $M_p = 1+\sum_{i<p} m_i$ and $N_q = \sum_{i\le q} n_i$. For a given integer p, M_p marks the first row of the p-th layer of A, while for a given q, N_q denotes the last column of the q-th stripe.

Lemma 2.2. Let A be a mosaic Toeplitz matrix. Then

$$S_{(M)} \cdot A - A \cdot S_{(N)} = \sum_{q=1}^{\ell} S_{(M)} \cdot A \cdot E^{(N_q)} \cdot F^{(N_q)} - \sum_{p=1}^{k} E^{(M_p)} \cdot F^{(M_p)} \cdot A \cdot S_{(N)}. \qquad (1)$$

where $E^{(i)}$ is the i-th standard column vector and $F^{(i)} = (E^{(i)})^T$ is the i-th standard row vector.

Proof: From the definitions of $S_{(M)}$ and $S_{(N)}$ along with the mosaic structure of A we see that

$$S_{(M)} \cdot A - A \cdot S_{(N)} = [S_{m_p} \cdot A_{p,q} - A_{p,q} \cdot S_{n_q}]_{p=1,q=1}^{k\ \ \ \ell}$$

Since each $A_{p,q}$ is a Toeplitz matrix we have

$$S_{m_p} \cdot A_{p,q} - A_{p,q} \cdot S_{n_q} = \begin{bmatrix} -a_{n-1}^{(p,q)} & \cdots & -a_{-(n_q-1)}^{(p,q)} & 0 \\ 0 & \cdots & 0 & a_{-(n_q-1)}^{(p,q)} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & a_{m_p-1-n_q}^{(p,q)} \end{bmatrix}$$

hence $S_{(M)} \cdot A - A \cdot S_{(N)}$ is given by

$$\begin{array}{cc}
 & \begin{array}{ccccccccc} & & & N_1 & & & & & N_\ell \\ & & & \downarrow & & & & & \downarrow \end{array} \\
\begin{array}{c} M_1 \rightarrow \\ \\ \\ \\ \\ M_k \rightarrow \\ \\ \\ \\ \end{array} &
\left[\begin{array}{ccccccccc}
-a_{-1}^{(1,1)} & \cdots & -a_{-(n_1-1)}^{(1,1)} & 0 & & -a_{-1}^{(1,\ell)} & \cdots & -a_{-(n_\ell-1)}^{(1,\ell)} & 0 \\
0 & \cdots & 0 & a_{-(n_1-1)}^{(1,1)} & \cdots & 0 & \cdots & 0 & a_{-(n_\ell-1)}^{(1,\ell)} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & \cdots & 0 & a_{m_1-1-n_1}^{(1,1)} & & 0 & \cdots & 0 & a_{m_1-1-n_\ell}^{(1,\ell)} \\
 & & \vdots & & & & & \vdots & \\
-a_{-1}^{(k,1)} & \cdots & -a_{-(n_1-1)}^{(k,1)} & 0 & & -a_{-1}^{(k,\ell)} & \cdots & -a_{-(n_\ell-1)}^{(k,\ell)} & 0 \\
0 & \cdots & 0 & a_{-(n_1-1)}^{(k,1)} & \cdots & 0 & \cdots & 0 & a_{-(n_\ell-1)}^{(k,\ell)} \\
\vdots & & \vdots & \vdots & & \vdots & & \vdots & \vdots \\
0 & \cdots & 0 & a_{m_k-1-n_1}^{(k,1)} & & 0 & \vdots & 0 & a_{m_k-1-n_\ell}^{(k,\ell)}
\end{array}\right]
\end{array}$$

The lemma is clear from the above decomposition. □

We remark that Lemma 2.2 is a natural generalization ot the well known rank two decomposition of $ST - TS$ for a shift matrix S and a Toeplitz matrix T.

Lemma 2.3. (see also Heinig and Rost [9]): Let $A = [A_{p,q}]_{p=1,q=1}^{k\ \ \ell}$ with $A_{p,q} = [a_{i-j}^{(p,q)}]_{i=1,j=1}^{m_p\ \ n_q}$ be a mosaic Toeplitz matrix. With the notation as above, the matrix A is invertible if and only if the following equations are soluble:

$$\begin{aligned} A \cdot X^{(M_p)} &= E^{(M_p)}, & p = 1, \ldots, k \qquad (2)\\ A \cdot Z^{(N_q)} &= S_{(M)} \cdot A \cdot E^{(N_q)}, & q = 1, \ldots, \ell. \qquad (3) \end{aligned}$$

In this case

$$A^{-1} = row[row[Q^{i-1} \cdot X^{(M_p)}]_{i=1}^{m_p}]_{p=1}^{k}$$

where

$$Q = S_{(N)} + \sum_{q=1}^{\ell} Z^{(N_q)} \cdot F^{(N_q)} - \sum_{p=1}^{k} X^{(M_p)} \cdot F^{(M_p)} \cdot A \cdot S_{(N)}.$$

Proof: We will apply Lemma 2.1 to A with $P = S_{(M)}$, Q given above and the columns $X^{(M_p)}$, $p = 1, \ldots, k$ determined by equation (3). Note that the structure of $S_{(M)}$ implies that

$$P^{i-1} \cdot A \cdot X^{(M_p)} = E^{(M_p+i-1)} \quad for \quad 1 \leq i \leq m_p, \quad p = 1, \ldots, k.$$

Therefore

$$R = row[row[P^{i-1} \cdot A \cdot X^{(M_p)}]_{i=1}^{m_p}]_{p=1}^{k}$$

is precisely the identity matrix. Consequently the formula for A^{-1} holds. □

3 Layered and Striped Toeplitz Matrices

In this section we prove the main Theorem stated in the introduction, along with some other related results. Therefore we now consider the special case where $\ell = 1$, that is, of a mosaic Toeplitz matrix having a single stripe. Such matrices are referred to as *layered Toeplitz matrices*. In this case, Lemma 2.3 implies that determining the invertibility of A (and also its inverse) is accomplished by solving the $k + 1$ linear equations

$$\begin{aligned} A \cdot X^{(M_p)} &= E^{(M_p)} & p = 1, \ldots, k, \qquad (4)\\ A \cdot Z &= S_{(M)} \cdot A \cdot E^{(m)}. & \qquad (5) \end{aligned}$$

The inverse in the case that such solutions have been found is then given by

$$A^{-1} = row[row[Q^{i-1} \cdot X^{(M_p)}]_{i=1}^{m_p}]_{p=1}^{k}$$

where $Q = S_m + Z \cdot F^{(m)} - \sum_{p=1}^{k} X^{(M_p)} \cdot F^{(M_p)} \cdot A \cdot S_m$.

Proof of Theorem 1.1.

If by chance $S_{(M)} \cdot A \cdot E^{(m)} = 0$, then the invertibility of A can be determined with only k standard equations. In the Toeplitz case (i.e. when $k = 1$) this can happen if and only if A is lower triangular (cf. [12]). When $k > 1$, there are other cases where only k standard equations determine the inverse of A (cf., Example 3.1).

For each p, let h_p denote the index of the highest non-zero component of $X^{(M_p)}$ and suppose $h = max\{h_p\}$. For each index i let $R_i = min\{m_i - 1, m - h\}$. Then we show that we can always construct column vectors $X^{(M_i)}, \ldots, X^{(M_i+R_i)}$ satisfying

$$\begin{array}{rrcll} a) & A \cdot X^{(M_i+r)} & = & E^{(M_i+r)} & 0 \leq r \leq R_i \\ b) & Q \cdot X^{(M_i+r-1)} & = & X^{(M_i+r)} & 1 \leq r \leq R_i \end{array}$$

where $Q = S_m - \sum_{p=1}^{k} X^{(M_p)} \cdot F^{(M_p)} \cdot A \cdot S_m$. Note that, since the last $m - h$ components of each of the $X^{(M_p)}$ are zero the structure of Q then implies that

$$x_u^{(M_i+r-1)} = x_{u-1}^{(M_i+r-2)} = \cdots = x_{u-r+1}^{(M_i)} \quad \text{for } h + r \leq u \leq m.$$

Clearly the above holds for $r = 0$, hence assume that we have an r with $1 \leq r \leq R_i$ such that $X^{(M_i)}, \ldots, X^{(M_i+r-1)}$ satisfying $a)$ and $b)$ above have already been constructed. From

$$S_{(M)} \cdot A = A \cdot S_m + S_{(M)} \cdot A \cdot E^{(m)} \cdot F^{(m)} - \sum_{p=1}^{k} E^{(M_p)} \cdot F^{(M_p)} \cdot A \cdot S_m, \tag{6}$$

we have that

$$S_{(M)} \cdot A \cdot X^{(M_i+r-1)} = A \cdot S_m \cdot X^{(M_i+r-1)} + S_{(M)} \cdot A \cdot E^{(m)} x_m^{(M_i+r-1)}$$
$$- \sum_{p=1}^{k} E^{(M_p)} \cdot F^{(M_p)} \cdot A \cdot S_m \cdot X^{(M_i+r-1)}.$$

Since $m - r + 1 \geq h + 1 > h$ at follows that $x_{m-r+1}^{(M_i)} = 0$. The construction is true for all $j \leq r - 1$, hence the structure of Q implies that

$$x_m^{(M_i+r-1)} = x_{m-1}^{(M_i+r-2)} = \cdots = x_{m-r+1}^{(M_i)}$$

so $x_m^{(M_i+r-1)} = 0$. Since $r \leq R_i \leq m_i - 1$ we also have that $E^{(M_i+r)} = S_{(M)} \cdot E^{(M_i+r-1)}$. Therefore

$$\begin{aligned} E^{(M_i+r)} &= A \cdot S_m \cdot X^{(M_i+r-1)} - \sum_{p=1}^{k} E^{(M_p)} \cdot F^{(M_p)} \cdot A \cdot S_m \cdot X^{(M_i+r-1)} \\ &= A \cdot \left\{ S_m \cdot X^{(M_i+r-1)} - \sum_{p=1}^{k} X^{(M_p)} \cdot F^{(M_p)} \cdot A \cdot S_m \cdot X^{(M_i+r-1)} \right\} \\ &= A \cdot Q \cdot X^{(M_i+r-1)}. \end{aligned}$$

Clearly $X^{(M_i+r)} = Q \cdot X^{(M_i+r-1)}$ then satisfies conditions $a)$ and $b)$.

We are now in a position to prove Theorem 1.1. Let j be an index such that $h_j = h$ and assume that $m_j - 1 < m - h$. Then $S_{(M)} \cdot E^{(M_j+m_j-1)} = 0$ and

$$x_m^{(M_j+m_j-1)} = x_{m-1}^{(M_j+m_j-2)} = \cdots = x_{m-m_j+1}^{(M_j)} = 0$$

since $h < m - m_j + 1$. A similar argument to that given previously then implies that $A \cdot X = 0$ where $X = Q \cdot X^{(M_j+m_j-1)}$. Since

$$x_{h+m_j} = x_{h+m_j-1}^{(M_j+m_j-1)} = \cdots = x_h^{(M_j)} \neq 0$$

and $h + m_j < m + 1$, X is nonzero and hence A must be singular.

Now let j be an index such that $h_j = h$ with $m_j - 1 \geq m - h$. Suppose there exists a solution $X^{(M_j+m-h_j+1)}$ to the standard equation

$$A \cdot X^{(M_j+m-h_j+1)} = E^{(M_j+m-h_j+1)}$$

(Note that there may be more than one possibility for the choice of j). Then $x_{h_j}^{(M_j)} = x_m^{(M_j+m-h_j)}$ and using equation (6) we obtain

$$\begin{aligned}
S_{(M)} \cdot A \cdot X^{(M_j+m-h_j)} = \quad & A \cdot S_m \cdot X^{(M_j+m-h_j)} + S_{(M)} \cdot A \cdot E^{(m)} x_m^{(M_j+m-h_j)} \\
& - \textstyle\sum_{p=1}^{k} E^{(M_p)} \cdot F^{(M_p)} \cdot A \cdot S_m \cdot X^{(M_j+m-h_j)} \\
E^{(M_j+m-h_j+1)} = \quad & A \cdot S_m \cdot X^{(M_j+m-h_j)} + S_{(M)} \cdot A \cdot E^{(m)} x_{h_j}^{(M_j)} \\
& - \textstyle\sum_{p=1}^{k} E^{(M_p)} \cdot F^{(M_p)} \cdot A \cdot S_m \cdot X^{(M_j+m-h_j)} \\
A \cdot X^{(M_j+m-h_j+1)} = \quad & A \cdot S_m \cdot X^{(M_j+m-h_j)} + S_{(M)} \cdot A \cdot E^{(m)} x_{h_j}^{(M_j)} \\
& - A \cdot \textstyle\sum_{p=1}^{k} X^{(M_p)} \cdot F^{(M_p)} \cdot A \cdot S_m \cdot X^{(M_j+m-h_j)}.
\end{aligned}$$

Therefore $A \cdot Z = S_{(M)} \cdot A \cdot E^{(m)}$ for

$$\begin{aligned}
Z &= \frac{1}{x_{h_j}^{(M_j)}} \cdot \left\{ X^{(M_j+m-h_j+1)} - S_m \cdot X^{(M_j+m-h_j)} + \sum_{p=1}^{k} X^{(M_p)} \cdot F^{(M_p)} \cdot A \cdot S_m \cdot X^{(M_j+m-h_j)} \right\} \\
&= \frac{1}{x_{h_j}^{(M_j)}} \cdot \left\{ X^{(M_j+m-h_j+1)} - Q^{m-h_j+1} \cdot X^{(M_j)} \right\}.
\end{aligned}$$

and so by Lemma 2.3 A is invertible with inverse given by

$$A^{-1} = row[row[(Q+P)^{i-1} \cdot X^{(M_p)}]_{i=1}^{m_p}]_{p=1}^{k}$$

with $P = Z \cdot F^{(m)}$. □

Remark 1. When $k = 1$ Theorem 1.1 first appeared in [12].

Remark 2. It is natural to ask if it is possible that one can always use less than $k+1$ standard equations to determine both invertibility and inverse of a layered Toeplitz matrix. This is not the case as has been shown in [12] in the $k = 1$ case.

Remark 3. A mosaic Toeplitz matrix with $k = 1$, that is, with only one layer, is called a *striped Toeplitz matrix.* Since the transpose of a striped Toeplitz matrix is layered Toeplitz the results of this section (using row standard rather than column standard equations) are also valid for the striped Toeplitz case. Our methods, however, do not construct the inverse of a striped Toeplitz matrix in terms of column standard equations as was the case for layered Toeplitz matrices. Indeed it is an open question how such a representation can be constructed.

Example 3.1. Suppose A is the 5-by-5 layered Toeplitz matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & a \\ a & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & b & c \\ 1 & 1 & 0 & 1 & b \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

with $m_1 = 2$ and $m_2 = 3$ (and hence $M_1 = 1$ and $M_2 = 3$). There are solutions to $A \cdot X^{(1)} = E^{(1)}$ and $A \cdot X^{(3)} = E^{(3)}$ given by

$$X^{(1)} = \begin{bmatrix} b+1 \\ -ab^2 + b^2 + ac - a - b - c + 1 \\ ab^2 - b^2 - ab - ac + b + c - 1 \\ -ab - ac + a + b + c - 2 \\ ab + a - b - 1 \end{bmatrix} /d \quad \text{and} \quad X^{(3)} = \begin{bmatrix} 1 \\ -a^2 + a - 1 \\ a^2 - 2a + 1 \\ a^2 - ab - a + b \\ a - 1 \end{bmatrix} /d$$

with $d = a^2b - ab^2 + a^2 + b^2 - ab + ac - 2a - c + 2$.

If $a = b = c = 0$, then $S_{(M)} \cdot A \cdot E^{(5)} = 0$. Therefore A is invertible with inverse given by

$$\left[X^{(1)}\ ,\ Q \cdot X^{(1)}\ ,\ X^{(3)}\ ,\ Q \cdot X^{(3)}\ ,\ Q^2 \cdot X^{(3)} \right]$$

with

$$Q = S_5 - (X^{(1)} \cdot F^{(1)} + X^{(3)} \cdot F^{(3)}) \cdot A \cdot S_5 = \begin{bmatrix} -1/2 & -1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1/2 & 1/2 & 0 & 1 & 0 \end{bmatrix}.$$

If $a = 1$ and b and c are arbitrary, then $h_1 = 4$ and $h_2 = 2$. Since $m_1 - 1 \leq m - h_1$ and there is already a standard solution for column $M_1 + m - h_1 + 1 = 3$, A is invertible. In this case the construction used in the proof of Theorem 1.1 allows one to avoid the extra P matrix. The inverse in this case is given by

$$\left[X^{(1)}\ ,\ Q \cdot X^{(1)}\ ,\ X^{(3)}\ ,\ Q \cdot X^{(3)}\ ,\ Q^2 \cdot X^{(3)} \right]$$

where

$$Q = S_5 - (X^{(1)} \cdot F^{(1)} + X^{(3)} \cdot F^{(3)}) \cdot A \cdot S_5 = \begin{bmatrix} -1-b & -1 & -b & -1-b-c & 0 \\ 1+b & 1 & b & b+c & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

If $a = 2$, $b = -1$ and $c \neq 1$, then $h_1 = 4$ and $h_2 = 5$. Since $m_2 > m - h_2 + 1$, we need to solve the extra standard equation

$$A \cdot X^{(4)} = E^{(4)}$$

in order to determine invertibility. There is such a solution in this case so A is invertible with inverse given by

$$\left[X^{(1)}\ ,\ (Q+P) \cdot X^{(1)}\ ,\ X^{(3)}\ ,\ (Q+P) \cdot X^{(3)}\ ,\ (Q+P)^2 \cdot X^{(3)}\right]$$

where now $Q = S_5 - (X^{(1)} \cdot F^{(1)} + X^{(3)} \cdot F^{(3)}) \cdot A \cdot S_5$ and $P = (X^{(4)} - Q \cdot X^{(3)}) \cdot F^{(5)}/x_5^{(3)}$. This gives

$$P + Q = \begin{bmatrix} 0 & -\frac{1}{c-1} & \frac{1}{c-1} & -\frac{c}{c-1} & \frac{4\,c-5}{c-1} \\ 0 & \frac{3}{c-1} & -\frac{3}{c-1} & \frac{2+c}{c-1} & -\frac{6\,c-9}{c-1} \\ 1 & \frac{c-2}{c-1} & \frac{1}{c-1} & \frac{c-2}{c-1} & -\frac{1}{c-1} \\ 1 & -\frac{3}{c-1} & \frac{2+c}{c-1} & -\frac{2+c}{c-1} & \frac{2\,c^2+c-6}{c-1} \\ 0 & -\frac{1}{c-1} & \frac{1}{c-1} & -\frac{1}{c-1} & \frac{c-2}{c-1} \end{bmatrix}.$$

Note that $(P + Q) \cdot X^{(1)} = Q \cdot X^{(1)}$ since $x_5^{(1)} = 0$. □

The standard equations that are used in Theorem 1.1 all correspond to the first rows of each layer. It is also possible to use standard equations that instead use the last rows of each layer.

Theorem 3.1. Let A be a m-by-m layered Toeplitz matrix with layers of size $m_1, \ldots, m_k$. Suppose there are solutions $X^{(M_p)} = col(x_j^{(M_p)})_{j=1}^m$ to the k standard equations

$$A \cdot X^{(M_p)} = E^{(M_p)} \quad p = 1, \ldots, k,$$

where, for each p, $M_p = \sum_{i \leq p} m_i$ denotes the last row of the p-th layer. For each p, let t_p be the index of the first non-zero component of $X^{(M_p)}$ and let j be any index such that $t_j = min\{t_p\}_{p=1,\ldots,k}$.

a) If $m_j < t_j$ then A is not invertible.

b) If $m_j \geq t_j$ and there is a solution to

$$A \cdot X^{(M_j - t_j)} = E^{(M_j - t_j)}$$

then A is invertible with inverse

$$A^{-1} = row[row[(Q + P)^{i-1} X^{(M_p)}]_{i=m_p}^{1}]_{p=1}^{k}$$

where $Q = S_m^T - \sum_{p=1}^k X^{(M_p)} \cdot F^{(M_p)} \cdot A \cdot S_m^T$, $P = (X^{(M_j - t_j)} - Q^{t_j} \cdot X^{(M_j)}) \cdot F^{(1)}/x_{t_j}^{(M_j)}$ and S_m^T is the m-by-m upper shift matrix.

Proof. Let $A = [A^{(i)}]_{i=1}^{k}$ and

$$J_{(M)} = \begin{bmatrix} J_{m_1} & & & \\ & J_{m_2} & & \\ & & \ddots & \\ & & & J_{m_k} \end{bmatrix}$$

where J_i is the i-by-i matrix $J_i = [\delta_{p,q-i+1}]_{p,q=1}^{i}$ having ones along the anti-diagonal and zeros elsewhere. Then

$$J_{(M)} \cdot A \cdot J_m = \bar{A} = [\bar{A}^{(i)}]_{i=1}^{k}$$

is a layered Toeplitz matrix having components

$$\bar{a}_j^{(i)} = a_{m_i-m-j}^{(i)}.$$

The matrix $\bar{A}$ also has layers of size $m_1, \ldots, m_k$, hence the first row of the i-th layer is given by

$$\bar{M}_i = m_1 + \cdots + m_{i-1} + 1 = M_i - m_i + 1.$$

We will show that the conditions on A in Theorem 3.1 are equivalent to the conditions on $\bar{A}$ in Theorem 1.1.

For each i the standard equation

$$A \cdot X^{(M_i)} = E^{(M_i)}$$

is equivalent to

$$\bar{A} \cdot \bar{X}^{(\bar{M}_i)} = E^{(\bar{M}_i)},$$

where $\bar{X}^{(\bar{M}_i)} = J_m \cdot X^{(M_i)}$. Note that if $\bar{h}_i$ is the last component of $\bar{X}^{(\bar{M}_i)}$ then

$$\bar{h}_i = m - t_i + 1.$$

Clearly an index j such that $t_j = min\{t_i\}$ corresponds to an index j such that $\bar{h}_j = max\{\bar{h}_i\}$. If $m_j < t_j$ then $m_j < m - \bar{h}_j + 1$ so $\bar{A}$ (and hence A) is not invertible.

Similarly, from part b) the condition $m_j \geq t_j$ corresponds to $m_j \geq m - \bar{h}_j + 1$ for such a j while an equation of the form

$$A \cdot X^{(M_j-t_j)} = E^{(M_j-t_j)}$$

can be transformed into a solution of

$$\bar{A} \cdot \bar{X}^{(\bar{M}_j+m-\bar{h}_j+1)} = E^{(\bar{M}_j+m-\bar{h}_j+1)}$$

where $\bar{X}^{(\bar{M}_j+m-\bar{h}_j+1)} = J_m \cdot X^{(M_j-t_j)}$. Therefore $\bar{A}$ (and hence A) is invertible by Theorem 1.1. In this case the inverse of $\bar{A}$ is given by

$$\bar{A}^{-1} = row[row[(\bar{Q} + \bar{P})^{i-1} \cdot \bar{X}^{\bar{M}_p)}]_{i=1}^{m_p}]_{p=1}^{k}$$

with $\bar{Q} = S_m - \sum_{p=1}^{k} \cdot \bar{X}^{(\bar{M}_p)} \cdot F^{(\bar{M}_p)} \cdot \bar{A} \cdot S_m$ and $\bar{P} = (\bar{X}^{(M_j+m-h_j+1)} - \bar{Q}^{m-h_j+1} \cdot \bar{X}^{(M_j)}) \cdot F^{(m)}/\bar{x}_{h_j}^{(M_j)}$. Therefore

$$\begin{aligned} A^{-1} = J_m \cdot \bar{A}^{-1} \cdot J_{(M)} &= J_m \cdot row[row[(\bar{Q}+\bar{P})^{i-1} \cdot \bar{X}^{(\bar{M}_p)}]_{i=1}^{m_p}]_{p=1}^{k} \cdot J_{(M)} \\ &= row[row[J_m \cdot (\bar{Q}+\bar{P})^{i-1} \cdot J_m \cdot X^{(M_p)}]_{i=1}^{m_p}]_{p=1}^{k} \cdot J_{(M)} \\ &= row[row[(J_m \cdot (\bar{Q}+\bar{P}) \cdot J_m)^{i-1} X^{(M_p)}]_{i=1}^{m_p}]_{p=1}^{k} \cdot J_{(M)} \\ &= row[row[(Q+P)^{i-1} X^{(M_p)}]_{i=m_p}^{1}]_{p=1}^{k} \end{aligned}$$

which gives part b) since for example (setting $J = J_m$)

$$\begin{aligned} J \cdot S_m \cdot J - \sum_{p=1}^{k} J_m \cdot \bar{X}^{(\bar{M}_p)} \cdot F^{(\bar{M}_p)} \cdot \bar{A} \cdot S_m \cdot J &= S_m^T - \sum_{p=1}^{k} X^{(M_p)} \cdot F^{(M_p)} \cdot J \cdot \bar{A} \cdot J \cdot S_m^T \\ &= S_m^T - \sum_{p=1}^{k} X^{(M_p)} \cdot F^{(M_p)} \cdot A \cdot S_m^T \end{aligned}$$

and a similar transformation shows that $J \cdot \bar{P} \cdot J = P$. □

Remark 4. Similar results hold for layered Hankel matrices (and also striped Hankel matrices). Indeed if A is a layered Hankel matrix, then $J_{(M)} \cdot A$ is a layered Toeplitz matrix. Hence for such matrices the argument presented in Theorem 3.1 is the main technique required in translating the results in this paper to the layered and striped Hankel cases.

4 Generic Mosaic Toeplitz Matrices

Let $A = [A_{p,q}]_{p=1,q=1}^{k \ \ \ell}$, with $A_{p,q} = [a_{i-j}^{(p,q)}]_{i=1,j=1}^{m_p \ \ n_q}$ and $\sum_{p=1}^{k} m_p = \sum_{q=1}^{\ell} n_q$ be a mosaic Toeplitz matrix. In this section we consider the problem of inverting a mosaic Toeplitz matrix using only standard equations. Without loss of generality, we may assume that $k \leq \ell$; otherwise, we would consider A^T instead of A. As in previous sections, M_p will denote the first row of the p-th layer and N_q will denote the last column of the q-th stripe.

Theorem 4.1. Let A be a mosaic Toeplitz matrix and suppose there are solutions $X^{(M_p)} = col(x_j^{(M_p)})_{j=1,\cdots,m}$ to the k standard equations

$$A \cdot X^{(M_p)} = E^{(M_p)} \quad p = 1, \ldots, k, \tag{7}$$

Assume that the ℓ-by-k matrix

$$X = \begin{bmatrix} x_{N_1}^{(M_1)} & \ldots & x_{N_1}^{(M_k)} \\ \vdots & & \vdots \\ x_{N_\ell}^{(M_1)} & \ldots & x_{N_\ell}^{(M_k)} \end{bmatrix},$$

has full-rank ℓ. If, for each $m_p > 1$, there are solutions

$$A \cdot X^{(M_p+1)} = E^{(M_p+1)} \qquad p = 1, \ldots, k,$$

then A is invertible with inverse given by

$$A^{-1} = row[row[(Q+P)^{i-1}X^{(M_p)}]_{i=1}^{m_p}]_{p=1}^{k} \tag{8}$$

where

$$Q = S_{(N)} - \sum_{p=1}^{k} X^{(M_p)} \cdot F^{(M_p)} \cdot A \cdot S_{(N)},$$

and (setting $V^{(i)} = X^{(M_i+1)}$ if $m_i > 1$ and 0 otherwise)

$$P = ([V^{(1)}, \cdots, V^{(k)}] - Q \cdot [X^{(M_1)}, \cdots, X^{(M_k)}]) \cdot Y \cdot \begin{bmatrix} F^{(N_1)} \\ \vdots \\ F^{(N_\ell)} \end{bmatrix}.$$

Here , Y is any right inverse of the ℓ ranked ℓ-by-k matrix X, i.e. $X \cdot Y = I$.

Proof: By Lemma 2.3 the invertibility of A is equivalent to the existence of solutions to equations

$$A \cdot X^{(M_p)} = E^{(M_p)} \qquad p = 1, \ldots, k, \tag{9}$$

$$A \cdot Z^{(N_q)} = S_{(M)} \cdot A \cdot E^{(N_q)} \qquad q = 1, \ldots, \ell \tag{10}$$

hence we need to determine the $Z^{(N_i)}, i = 1, \cdots, q$.

For any solutions of (7) equation (1) implies that

$$S_{(M)} \cdot A \cdot X^{(M_i)} = A \cdot S_{(N)} \cdot X^{(M_i)} + \sum_{q=1}^{\ell} S_{(M)} \cdot A \cdot E^{(N_q)} \cdot x_{N_q}^{(M_i)} - \sum_{p=1}^{k} E^{(M_p)} \cdot F^{(M_p)} \cdot A \cdot S_{(N)} \cdot X^{(M_i)}.$$

Since

$$S_{(M)} \cdot A \cdot X^{(M_i)} = S_{(M)} \cdot E^{(M_i)} = \begin{cases} E^{(M_i+1)} & \text{if } m_i > 1 \\ 0 & \text{if } m_i = 1 \end{cases}$$

we see that

$$\left[S_{(M)} \cdot A \cdot E^{(N_1)}, \cdots, S_{(M)} \cdot A \cdot E^{(N_q)}\right] \cdot X = A \cdot ([V^{(1)}, \cdots, V^{(k)}] - [S_{(N)}X^{(M_1)}, \cdots, S_{(N)}X^{(M_k)}]$$

$$+ [X^{(M_1)} \cdot F^{(M_1)} \cdot A \cdot S_{(N)}, \cdots, X^{(M_k)} \cdot F^{(M_k)} \cdot A \cdot S_{(N)}] \cdot [X^{(M_1)}, \cdots, X^{(M_k)}])$$

in which $V^{(i)} = X^{(M_i+1)}$ if $m_i > 1$ and 0 otherwise.

Therefore, since X has full rank there are solutions to (10) given by

$$[Z^{(1)}, \cdots, Z^{(k)}] = ([V^{(1)}, \cdots, V^{(k)}] -$$

$$(S_{(N)} - [X^{(M_1)} \cdot F^{(M_1)} \cdot A \cdot S_{(N)}, \cdots, X^{(M_k)} \cdot F^{(M_k)} \cdot A \cdot S_{(N)}]) \cdot [X^{(M_1)}, \cdots, X^{(M_k)}]) \cdot Y$$

hence A is invertible with inverse given by (8). □

Remark 1. When $k = \ell = 1$, Theorem 4.1 is the same as the Gohberg-Krupnik formula.

Theorem 4.2. Let A be a mosaic Toeplitz matrix, M_p the last row of the p-th layer and N_q the first column of the q-th stripe. Suppose there are solutions $X^{(M_p)} = col(x_j^{(M_p)})_{j=1}^m$ of the k standard equations

$$A \cdot X^{(M_p)} = E^{(M_p)} \qquad p = 1, \dots, k \tag{11}$$

and assume that the ℓ-by-k matrix

$$X = \begin{bmatrix} x_{N_1}^{(M_1)} & \dots & x_{N_1}^{(M_k)} \\ \vdots & & \vdots \\ x_{N_\ell}^{(M_1)} & \dots & x_{N_\ell}^{(M_k)} \end{bmatrix},$$

has full-rank ℓ with right inverse Y. If, for each $m_p > 1$, there are solutions

$$A \cdot X^{(M_p - 1)} = E^{(M_p - 1)} \qquad p = 1, \dots, k, \tag{12}$$

then A is invertible with inverse given by

$$A^{-1} = row[row[(Q + P)^{i-1} X^{(M_p)}]_{i=m_p}^1]_{p=1}^k$$

where

$$Q = S_{(N)}^T - \sum_{p=1}^k X^{(M_p)} \cdot F^{(M_p)} \cdot A \cdot S_{(N)}^T,$$

and (setting $V^{(i)} = X^{(M_i - 1)}$ if $m_i > 1$ and 0 otherwise)

$$P = ([V^{(1)}, \cdots, V^{(k)}] - Q \cdot [X^{(M_1)}, \cdots, X^{(M_k)}]) \cdot Y \cdot \begin{bmatrix} F^{(N_1)} \\ \vdots \\ F^{(N_\ell)} \end{bmatrix}.$$

Proof: Indeed, let $\bar{A} = J_{(M)} \cdot A \cdot J_{(N)}$. Then $\bar{A}$ is a mosaic Toeplitz matrix $[\bar{A}_{p,q}]_{p=1,q=1}^{k \ \ \ell}$ with the entries of $\bar{A}_{p,q}$ determined by

$$\bar{a}_i^{(p,q)} = a_{m_p - n_q - i}^{(p,q)}.$$

Note that each $\bar{A}^{(p,q)}$ also has size m_p-by-n_q.

It is a simple matter to use the argument of Theorem 3.1 to show that the conditions on A in Theorem 4.2 correspond to the conditions on $\bar{A}$ in Theorem 4.1. Indeed, equations of the form (11) are equivalent to

$$\bar{A} \cdot \bar{X}^{(\bar{M}_p)} = E^{(\bar{M}_p)} \qquad p = 1, \dots, k,$$

where for each p, $\bar{M}_p = M_p - m_p + 1$ is the first row of the p-th layer of $\bar{A}$ and $\bar{X}^{(\bar{M}_p)} = J_{(N)} \cdot X^{(\bar{M}_p)}$. Note that

$$X = \begin{bmatrix} x_{N_\ell}^{(M_k)} & \dots & x_{N_\ell}^{(M_1)} \\ \vdots & & \vdots \\ x_{N_1}^{(M_k)} & \dots & x_{N_1}^{(M_1)} \end{bmatrix} = \begin{bmatrix} \bar{x}_{\bar{N}_1}^{(\bar{M}_1)} & \dots & \bar{x}_{\bar{N}_1}^{(\bar{M}_k)} \\ \vdots & & \vdots \\ \bar{x}_{\bar{N}_\ell}^{(\bar{M}_1)} & \dots & \bar{x}_{\bar{N}_\ell}^{(\bar{M}_k)} \end{bmatrix},$$

where $\bar{N}_i$ is the last column of the i-th stripe of $\bar{A}$.

Similarly, equations of the form (12) are equivalent to

$$\bar{A} \cdot \bar{X}^{(\bar{M}_p+1)} = E^{(\bar{M}_p+1)} \qquad p = 1, \ldots, k,$$

with $\bar{X}^{(\bar{M}_p+1)} = J_{(N)} \cdot X^{(\bar{M}_p-1)}$. Therefore $\bar{A}$, and hence also A, is invertible by Theorem 4.1. The inverse formula for A follows directly (using the arguments from Theorem 3.1) from the inverse formula for $\bar{A}$. $\square$

Remark 2. Theorems 4.1 and 4.2 can also be given in the case of mosaic Hankel matrices.

Remark 3. In the case of a block Toeplitz matrix there are examples where the inverse can be given in terms of solutions of standard row and standard column equations. For example the formulae of Gohberg and Heinig [3] gives the inverse once the first and last block rows and columns of the inverse are known. Inversion formulae given in terms only of solutions to standard block row and block column equations are also given in Ben-Artzi and Shalom [2] and Lerer and Tismenetsky [14]. Using appropriate row and column permutations it is easy to see that block Toeplitz matrices are same as mosaic Toeplitz matrices having constant width stripes and constant height layers. In this context the formulae of Gohberg-Heinig describes the inverse in terms of solutions of standard column equations corresponding to the first and last columns of each stripe and standard row equations corresponding to the first and last rows of each layer. It would be of interest to generalize such formulae to inverses of more general mosaic matrices. This would also be true of corresponding mosaic forms of the formulae of Ben-Artzi and Shalom and Lerer and Tismenetsky, even in the generic case. It is an open question whether or not one can construct an inverse for a non-generic mosaic Toeplitz matrix using only standard row or standard column equations.

5 Generic Toeplitz plus Hankel Matrices

In this section we show that the techniques used previously are also applicable to matrices having the structure of a Toeplitz plus Hankel matrix. In this case we obtain results of Heinig and Rost [10]. Thus let $A = T + H$ where $T = [t_{i-j}]_{i,j=1}^{m}$ is a Toeplitz matrix and $H = [h_{i+j-1}]_{i,j=1}^{m}$ is a Hankel matrix. The matrix $U = S_m + S_m^T$ takes the role of the shift matrix of previous sections.

Lemma 5.1. $A = T + H$ is invertible if and only if there are solutions to

$$\begin{array}{ll} A \cdot X^{(1)} = E^{(1)}, & A \cdot Z^{(1)} = (S^T \cdot T + S \cdot H) \cdot E^{(1)} \\ A \cdot X^{(m)} = E^{(m)}, & A \cdot Z^{(m)} = (S^T \cdot T + S \cdot H) \cdot E^{(m)} \end{array}$$

In this case

$$A^{-1} = row \left[Q^{i-1} \cdot X^{(1)} \right]_{i=1}^{m} \cdot R^{-1}$$

where

$$Q = Q_1 \cdot S + Q_2 \cdot S^T + Z^{(1)} \cdot F^{(1)} + Z^{(m)} \cdot F^{(m)}$$

with

$$\begin{array}{l} Q_1 = (I - X^{(1)} \cdot F^{(1)} \cdot T - X^{(m)} \cdot F^{(m)} \cdot H) \\ Q_2 = (I - X^{(1)} \cdot F^{(1)} \cdot H - X^{(m)} \cdot F^{(m)} \cdot T) \end{array}$$

and

$$R = row\left[U^{i-1} \cdot E^{(1)}\right]_{i=1}^{m}$$

Proof: Set $P = U$, and note that

$$\begin{aligned} P \cdot A &= A \cdot P + (S + S^T) \cdot (T + H) - (T + H) \cdot (S + S^T) \\ &= (S \cdot T - T \cdot S) + (S^T T - T \cdot S^T) + (S \cdot H - H \cdot S^T) + (S^T \cdot H - H \cdot S) \\ &= (S^T \cdot T + S \cdot H) \cdot E^{(1)} \cdot F^{(1)} + (S \cdot T + S^T \cdot H) \cdot E^{(m)} \cdot F^{(m)} \\ &\qquad - E^{(1)} \cdot F^{(1)} \cdot (T \cdot S + H \cdot S^T) - E^{(m)} \cdot F^{(m)} \cdot (H \cdot S + T \cdot S^T) \\ &= A \cdot Q. \end{aligned}$$

Since $R = row[P^{i-1} \cdot A \cdot X^{(1)}]_{i=1}^{m}$ is invertible, the result then follows directly from Lemma 2.1. □

Theorem 5.2. Let $A = T + H$ and suppose there are solutions to the four standard equation

$$\begin{array}{llll} A \cdot X^{(1)} & = E^{(1)}, & A \cdot X^{(2)} = E^{(2)} \\ A \cdot X^{(m-1)} & = E^{(m-1)}, & A \cdot X^{(m)} = E^{(m)}. \end{array}$$

Suppose the 2-by-2 matrix

$$X = \begin{bmatrix} x_1^{(1)} & x_1^{(m)} \\ x_m^{(1)} & x_m^{(m)} \end{bmatrix}$$

is invertible. Then A is invertible with inverse given by

$$A^{-1} = row\left[(Q + P)^{i-1} \cdot X^{(1)}\right]_{i=1}^{m} \cdot R^{-1}$$

where $Q = Q_1 \cdot S + Q_2 \cdot S^T$ with

$$\begin{array}{l} Q_1 = (I - X^{(1)} \cdot F^{(1)} \cdot T - X^{(m)} \cdot F^{(m)} \cdot H) \\ Q_2 = (I - X^{(1)} \cdot F^{(1)} \cdot H - X^{(m)} \cdot F^{(m)} \cdot T), \end{array}$$

$$P = ([X^{(2)}, X^{(m-1)}] - Q \cdot [X^{(1)}, X^{(m)}]) \cdot X^{-1} \cdot \begin{bmatrix} F^{(1)} \\ F^{(m)} \end{bmatrix}$$

and

$$R = row\left[U^{i-1} \cdot E^{(1)}\right]_{i=1}^{m}.$$

Proof: For notational convenience let

$$\begin{array}{lllllll} B_1 & = & S^T \cdot T + S \cdot H, & & B_2 & = & S \cdot T + S^T \cdot H, \\ B_3 & = & T \cdot S + H \cdot S^T, & & B_4 & = & H \cdot S + T \cdot S^T. \end{array}$$

Then

$$\begin{aligned} E^{(2)} = U \cdot E^{(1)} &= A \cdot U \cdot X^{(1)} + B_1 \cdot E^{(1)} \cdot x_1^{(1)} + B_2 \cdot E^{(m)} \cdot x_m^{(1)} \\ &\quad - A \cdot X^{(1)} \cdot F^{(1)} \cdot B_3 \cdot X^{(1)} - A \cdot X^{(m)} \cdot F^{(m)} \cdot B_4 \cdot X^{(1)} \\ E^{(m-1)} = U \cdot E^{(m)} &= A \cdot U \cdot X^{(m)} + B_1 \cdot E^{(1)} \cdot x_1^{(m)} + B_2 \cdot x_m^{(m)} \\ &\quad - A \cdot X^{(m)} \cdot F^{(1)} \cdot B_3 \cdot X^{(m)} - A \cdot X^{(m)} \cdot F^{(m)} \cdot B_4 \cdot X^{(m)} \end{aligned}$$

hence

$$\left[B_1 \cdot E^{(1)} \ , \ B_2 \cdot E^{(m)}\right] \cdot X$$

$$\begin{aligned} &= \left[B_1 \cdot E^{(1)} \cdot x_0^{(1)} + B_2 - E^{(m)} \cdot x_{m-1}^{(1)} \ , \ B_1 \cdot E^{(1)} \cdot x_0^{(m)} + B_2 \cdot E^{(m)} \cdot x_{m-1}^{(m)}\right] \\ &= \left[B_1 E^{(1)} F^{(1)} X^{(1)} + B_2 E^{(m)} F^{(m)} X^{(1)} \ , \ B_1 E^{(1)} F^{(1)} X^{(m)} + B_2 E^{(m)} F^{(m)} \cdot X^{(m)}\right] \\ &= A \cdot \left[Y^{(1)} \ , \ Y^{(2)}\right] \end{aligned}$$

where

$$\begin{aligned} Y^{(1)} &= X^{(2)} + X^{(1)} \cdot F^{(1)} \cdot B_3 \cdot X^{(1)} + X^{(m)} \cdot F^{(m)} \cdot B_4 \cdot X^{(1)} - U \cdot X^{(1)} \\ Y^{(2)} &= X^{(m-1)} + X^{(1)} \cdot F^{(1)} \cdot B_3 \cdot X^{(m)} + X^{(m)} \cdot F^{(m)} \cdot B_4 \cdot X^{(m)} - U \cdot X^{(m)}. \end{aligned}$$

Therefore, when X is invertible we obtain

$$[Z^{(1)} \ , \ Z^{(m)}] = [Y^{(1)} \ , \ Y^{(m)}] \cdot X^{-1}$$

which implies that A is invertible by Lemma 5.1. The inverse formula also follows directly from the previous lemma. □

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