Computation of Numerical Padé-Hermite and Simultaneous Padé Systems I: Near Inversion of Generalized Sylvester Matrices

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Abstract. We present new formulae for the "near" inverses of striped Sylvester and mosaic Sylvester matrices. The formulae assume computation over floating-point rather than exact arithmetic domains. The near inverses are expressed in terms of numerical Padé-Hermite systems and simultaneous Padé systems. These systems are approximants for the power series determined from the coefficients of the Sylvester matrices. The inverse formulae provide good estimates for the condition numbers of these matrices, and serve as primary tools in a companion paper for the development of a fast, weakly stable algorithm for the computation of Padé-Hermite and simultaneous Padé systems and, thereby, also for the numerical inversion of striped and mosaic Sylvester matrices.

Key Words. striped Sylvester inverses, mosaic Sylvester inverses, Padé-Hermite approximants, simultaneous Padé approximants, numerical stability

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1. Introduction. Let $n = [n_0, \ldots, n_k]$ be a vector of integers with $n_\beta \geq 0$, $0 \leq \beta \leq k$. A striped Sylvester matrix of order $\|n\|$ is given by

$$
(1) \qquad \mathcal{M}_n = \left[\begin{array}{ccc|c|ccc} a_0^{(0)} & & & & a_k^{(0)} & & \\ & \ddots & & & & \ddots & \\ \vdots & & a_0^{(0)} & \cdots & \vdots & & a_k^{(0)} \\ & & \vdots & & & & \vdots \\ a_0^{(\|n\|-1)} & \cdots & a_0^{(\|n\|-n_0)} & & a_k^{(\|n\|-1)} & \cdots & a_k^{(\|n\|-n_k)} \end{array} \right],
$$

where $\|n\| = n_0 + \ldots + n_k$ and where the coefficients $a_\beta^{(\ell)} \in \mathcal{R}$, the field of real numbers. Assume that $a_0^{(0)} \neq 0$. In this paper, we present a formula for the inverse of $\mathcal{M}_n$ expressed in terms of Padé-Hermite and simultaneous Padé systems.

Padé-Hermite and simultaneous Padé systems [7, 9] are approximants for the vector $A^t(z) = [a_0(z), \ldots, a_k(z)]$ of power series associated with the coefficients of $\mathcal{M}_n$, namely,

$$
a_\beta(z) = \sum_{\ell=0}^{\|n\|-1} a_\beta^{(\ell)} z^\ell, \text{ with } 0 \leq \beta \leq k.
$$

They provide necessary and sufficient conditions for $\mathcal{M}_n$ to be nonsingular, and generalize the notions of Padé-Hermite and simultaneous Padé approximants. Padé-Hermite and simultaneous Padé approximants were introduced by Hermite in the last

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century and has been widely studied by several authors (for a bibliography, see, for example [1, 2, 3, 4, 15]).

The inverse formula given here is intended to be applied in a numerical setting; it accommodates errors that may have been made in the computation of Padé-Hermite and simultaneous Padé systems. That is, the formula gives the "near" inverse for $\mathcal{M}_n$ since it expresses the inverse in terms of Padé-Hermite and simultaneous Padé systems which are computed using floating-point arithmetic. There are other closed formulae (c.f. [12, 16, 18, 19, 20]) for $\mathcal{M}_n^{-1}$. The formula given here is different in that it expresses the inverse directly in terms of numerical Padé-Hermite and simultaneous Padé systems.

The near inverse formula depends on the computation of both Padé systems. It is possible to determine a simultaneous Padé system from its "dual" Padé-Hermite system via the adjoint operation [6, 15]. In a numerical setting, however, this does not provide enough control over the resulting floating-point errors [14]. Instead, simultaneous Padé systems can be computed independently. Whereas a Padé-Hermite system can be obtained by solving a system of linear equations with $\mathcal{M}_n$ as the coefficient matrix, a simultaneous Padé system can be similarly and independently obtained with a coefficient matrix that is a specialized mosaic Sylvester matrix. This specialized mosaic Sylvester matrix of order $k\|n\|$ is defined to be

$$\mathcal{M}_n^* = \begin{bmatrix} \mathcal{S}_{0,1}^* & \cdots & \mathcal{S}_{0,k}^* \\ \vdots & & \vdots \\ \mathcal{S}_{k,1}^* & \cdots & \mathcal{S}_{k,k}^* \end{bmatrix}, \tag{2}$$

where $\mathcal{S}_{\alpha,\beta}^*$ are matrices of size $(\|n\| - n_\alpha) \times \|n\|$, with

$$\mathcal{S}_{0,\beta}^* = -\begin{bmatrix} a_\beta^{(0)} & & \cdots & & a_\beta^{(\|n\|-1)} \\ & \ddots & & & \vdots \\ & & a_\beta^{(0)} & \cdots & a_\beta^{(n_0)} \end{bmatrix},$$

$$\mathcal{S}_{\beta,\beta}^* = \begin{bmatrix} a_0^{(0)} & & \cdots & & a_0^{(\|n\|-1)} \\ & \ddots & & & \vdots \\ & & a_0^{(0)} & \cdots & a_0^{(n_\beta)} \end{bmatrix},$$

for $1 \le \beta \le k$, and with the remaining $\mathcal{S}_{\alpha,\beta}^* = 0$. The matrix $\mathcal{M}_n^*$ is closely related to $\mathcal{M}_n$. Indeed, in the special case when $k = 1$ the matrix $\mathcal{M}_n$ and the transpose of $\mathcal{M}_n^*$ coincide, up to a sign and a permutation of the rows and columns. Similarly, when $k \ge 1$ and $a_0(z) = 1$, the matrix $\mathcal{M}_n$ and the transpose of $\mathcal{M}_n^*$ are both obtained by a suitable block extension of the same matrix. In this paper, we also provide a "near" inverse formula for the matrix $\mathcal{M}_n^*$, again in terms of numerical Padé-Hermite and simultaneous Padé systems.

The inverse formulae provide "good" estimates for the condition numbers of $\mathcal{M}_n$ and $\mathcal{M}_n^*$ and these enable the stable numerical computation of Padé-Hermite and simultaneous Padé systems described in the companion paper [6]. In return, the accurate computation of Padé-Hermite and simultaneous Padé systems by the algorithm in [6] enables the effective inversion of generalized Sylvester matrices by the formulae given in this paper, as well as the solution of linear systems of equations with these as the coefficient matrices.

This paper is organized as follows. Preliminary definitions and basic facts about Padé-Hermite and simultaneous Padé systems are given in the next two sections. §3 gives a near commutativity relationship between these two systems in floating-point arithmetic, while §4 and §5 give the approximate inversion formulae for striped and mosaic Sylvester matrices. The final section gives a summary and some conclusions.

We conclude this section by defining some norms which are used in §4 and §5. Let

$$a(z) = \sum_{\ell=0}^{\infty} a^{(\ell)} z^{\ell} \in \mathcal{R}\,[[z]]\,,$$

where $\mathcal{R}[[z]]$ is the domain of power series with coefficients from $\mathcal{R}$, and define the bounded power series

$$\mathcal{R}^B[[z]] = \left\{ a(z) \;\middle|\; a(z) \in \mathcal{R}\,[[z]]\,,\ \sum_{\ell=0}^{\infty} |a^{(\ell)}| < \infty \right\}.$$

A norm for $a(z) \in \mathcal{R}^B\,[[z]]$ is

$$\|a(z)\| = \sum_{\ell=0}^{\infty} |a^{(\ell)}|.$$

$\mathcal{R}^B[[z]]$ includes the domain of polynomials $\mathcal{R}[z]$. So, for

$$s(z) = \sum_{\ell=0}^{\partial} s^{(\ell)} z^{\ell} \ \in \mathcal{R}\,[z]\,,$$

we use the norm

$$\|s(z)\| = \sum_{\ell=0}^{\partial} |\ s^{(\ell)}\ |\ .$$

For vectors and matrices over $\mathcal{R}^B[[z]]$, we use the 1-norm unless otherwise specified. So, for example, the norm for $A^t(z)$ is

$$\|A^t(z)\| = \max_{0 \le \beta \le k} \{\|a_\beta(z)\|\}$$

and the norm for $S(z) \in \mathcal{R}_{(k+1)\times(k+1)}[z]$ is

$$\|S(z)\| = \max_{0 \le \beta \le k} \left\{ \sum_{\alpha=0}^{k} \|S_{\alpha,\beta}(z)\| \right\}.$$

It is easy to verify that various compatibility conditions are satisfied. For example,

$$\|A^t(z) \cdot S(z)\| \le \|A^t(z)\| \cdot \|S(z)\|$$

and

$$\|a(z) \cdot b(z)\| \ \le\ \|a(z)\| \cdot \|b(z)\|,$$

where $b(z)$ is also a bounded power series. In addition, for $S^*(z) \in \mathcal{R}_{(k+1)\times(k+1)}[z]$ and $A^*(z) \in \mathcal{R}^B_{(k+1)\times k}[[z]]$,

$$\|S^*(z) \cdot A^*(z)\| \leq \|S^*(z)\| \cdot \|A^*(z)\|,$$

$$\|S(z) \cdot S^*(z)\| \ \leq \ \|S(z)\| \cdot \|S^*(z)\|.$$

In the subsequent development, we also make use of the inequality

$$\|a(z) \ (mod \ z^{\|n\|+1})\| \ \leq \ \|a(z)\|,$$

where

$$a(z) \ (mod \ z^{\|n\|+1}) = \sum_{\ell=0}^{\|n\|} a^{(\ell)} \, z^{\ell} + \sum_{\ell=\|n\|+1}^{\infty} 0 \cdot z^{\ell} \in \mathcal{R}^B[[z]]$$

2. Multi-dimensional Padé Systems. In this section, we introduce the notion of Padé-Hermite and simultaneous Padé systems. Let $n = [n_0, \ldots, n_k]$ and define $\|n\| = n_0 + \cdots + n_k$. Also let

$$A^t(z) = [a_0(z), \ldots, a_k(z)] \, ,$$

where

$$a_\beta(z) = \sum_{\ell=0}^{\infty} a_\beta^{(\ell)} \, z^{\ell}, \qquad \beta = 0, \ldots, k,$$

with $a_\beta^{(\ell)} \in \mathcal{R}$, the field of real numbers. Assume that $a_0^{(0)} \neq 0$, which means that $a_0^{-1}(z)$ exists. Assume also that $A^t(z)$ is scaled so that $\|a_\beta(z) \, (mod \ z^{\|n\|+1})\| = 1$, $0 \leq \beta \leq k$.

The $(k+1) \times (k+1)$ matrix of polynomials

$$S(z) = \left[\begin{array}{c|c} zp(z) & U^t(z) \\ \hline zQ(z) & V(z) \end{array} \right] = \left[\begin{array}{c|ccc} zp(z) & u_1(z) & \cdots & u_k(z) \\ \hline zq_1(z) & v_{1,1}(z) & \cdots & v_{1,k}(z) \\ \vdots & \vdots & & \vdots \\ zq_k(z) & v_{k,1}(z) & \cdots & v_{k,k}(z) \end{array} \right] \tag{3}$$

is a numerical Padé-Hermite system (NPHS) [9] of type n for $A(z)$ if the following conditions are satisfied.

I. **(Degree conditions)**: For $1 \leq \alpha, \beta \leq k$,

$$p(z) = \sum_{\ell=0}^{n_0-1} p^{(\ell)} z^{\ell}, \qquad u_\beta(z) = \sum_{\ell=0}^{n_0} u_\beta^{(\ell)} z^{\ell}, \tag{4}$$

$$q_\alpha(z) = \sum_{\ell=0}^{n_\alpha-1} q_\alpha^{(\ell)} z^{\ell}, \qquad v_{\alpha,\beta}(z) = \sum_{\ell=0}^{n_\alpha} v_{\alpha,\beta}^{(\ell)} z^{\ell}.$$

II. **(Order condition)**:

$$A^t(z) S(z) = z^{\|n\|} T^t(z) + \delta T^t(z), \tag{5}$$

where $T^t(z) = [r(z), zW^t(z)]$ with $W^t(z) = [w_1(z), \ldots, w_k(z)]$ is the residual, and where $\delta T^t(z) = [z\ \delta r(z), \delta W^t(z)]$ is the residual error, with $\delta W^t(z) = [\delta w_1(z), \ldots, \delta w_k(z)]$ and with

$$\delta r(z) = \sum_{\ell=0}^{\|n\|-2} \delta r^{(\ell)}\ z^\ell, \quad \delta w_\beta(z) = \sum_{\ell=0}^{\|n\|} \delta w_\beta^{(\ell)}\ z^\ell.$$

If $\delta T^t(z) = 0$, then $S(z)$ is an exact, rather than a numerical, Padé-Hermite system.

III. **(Nonsingularity condition)**: The constant term of $V(z)$ is a diagonal matrix,

$$V(0) = diag\ [\gamma_1, \ldots, \gamma_k], \tag{6}$$

and

$$\gamma \equiv (a_0^{(0)})^{-1} \prod_{\alpha=0}^{k} \gamma_\alpha\ \neq 0, \tag{7}$$

where $\gamma_0 = r(0)$.

Remark 1: The nonsingularity condition III is equivalent to the condition that $r(0) \neq 0$ and that $V(0)$ be a nonsingular diagonal matrix.

Remark 2: The NPHS is said to be **normalized** [9] if the nonsingularity condition III is replaced by $r(0) = 1$ and $V(0) = I_k$. This can be achieved by multiplying $S(z)$ on the right by Γ^{-1}, where

$$\Gamma = diag\,[\gamma_0, \ldots, \gamma_k]\,. \tag{8}$$

The NPHS is said to be **scaled** [14] if each column of $S(z)$ has norm equal to 1 and if, in addition, $\gamma_\beta > 0$, $0 \leq \beta \leq k$. Here, also, scaling a NPHS is accomplished by multiplying it on the right by an appropriate diagonal matrix.

Remark 3: The nonsingularity condition III, namely $\gamma \neq 0$, refers to the nonsingularity of the associated striped Sylvester matrix $\mathcal{M}_n$ defined in (1); in [9] it is shown that an exact NPHS exists iff $\mathcal{M}_n$ is nonsingular.

Remark 4: In [5, 6, 9], rather than $S(z)$, the Padé-Hermite system is defined to be $S(z) \cdot diag[z, 1, \ldots, 1]$. The notation used here is consistent with that of [16], and simplifies the presentation of some of the results.

A Padé-Hermite system gives an approximation to a vector of formal power series using matrix multiplication on the right. A simultaneous Padé system is a similar approximation using matrix multiplication on the left and with degree constraints that can be thought of as being "dual" to the degree constraints of a Padé-Hermite system.

Let[1]

$$A^*(z) = \left[\begin{array}{ccc} -a_1(z) & \cdots & -a_k(z) \\ \hline a_0(z) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & a_0(z) \end{array}\right] \tag{9}$$

be a $(k+1) \times k$ matrix of power series. The $(k+1) \times (k+1)$ matrix of polynomials

$$S^*(z) = \left[\begin{array}{c|c} v^*(z) & U^{*t}(z) \\ \hline zQ^*(z) & zP^*(z) \end{array}\right] = \left[\begin{array}{c|ccc} v^*(z) & u_1^*(z) & \cdots & u_k^*(z) \\ \hline zq_1^*(z) & zp_{1,1}^*(z) & \cdots & zp_{1,k}^*(z) \\ \vdots & \vdots & & \vdots \\ zq_k^*(z) & zp_{k,1}^*(z) & \cdots & zp_{k,k}^*(z) \end{array}\right] \tag{10}$$

is a numerical simultaneous Padé system (NSPS) [7, 9] of type n for $A^*(z)$ if the following conditions are satisfied.

I. **(Degree conditions)**: For $1 \le \alpha, \beta \le k$,

$$v^*(z) = \sum_{\ell=0}^{\|n\|-n_0} v^{*(\ell)} z^\ell, \qquad u_\beta^*(z) = \sum_{\ell=0}^{\|n\|-n_\beta} u_\beta^{*(\ell)} z^\ell, \tag{11}$$

$$q_\alpha^*(z) = \sum_{\ell=0}^{\|n\|-n_0-1} q_\alpha^{*(\ell)} z^\ell, \qquad p_{\alpha,\beta}^*(z) = \sum_{\ell=0}^{\|n\|-n_\beta-1} p_{\alpha,\beta}^{*(\ell)} z^\ell.$$

II. **(Order condition)**:

$$S^*(z)A^*(z) = z^{\|n\|} T^*(z) + \delta T^*(z), \tag{12}$$

where $T^{*t}(z) = [z\,W^*(z) | R^{*t}(z)]$ with $R^*(z)$ a $k \times k$ is the residual, and where $\delta T^{*t}(z) = [\delta W^*(z) | z\,\delta R^{*t}(z)]$ is the residual error, with $\delta R^*(z)$ a $k \times k$ matrix and

$$\delta w_\beta^*(z) = \sum_{\ell=0}^{\|n\|} \delta w_\beta^{*(\ell)}\, z^\ell, \quad \delta r_{\alpha,\beta}^*(z) = \sum_{\ell=0}^{\|n\|-2} \delta r_{\alpha,\beta}^{*(\ell)}\, z^\ell.$$

If $\delta T^*(z) = 0$, then $S^*(z)$ is an exact NSPS.

III. **(Nonsingularity condition)**: The constant term of $R^*(z)$ is a diagonal matrix

$$R^*(0) = diag\ [\gamma_1^*, \ldots, \gamma_k^*], \tag{13}$$

and

$$\gamma^* \equiv (a_0^{(0)})^{-1} \prod_{\alpha=0}^{k} \gamma_\alpha^* \ \ne 0, \tag{14}$$

[1] More generally,

$$A^*(z) = \left[\begin{array}{ccc} a_{0,1}^*(z) & \cdots & a_{0,k}^*(z) \\ \hline a_{1,1}^*(z) & \cdots & a_{1,k}^*(z) \\ \vdots & & \vdots \\ a_{k,1}^*(z) & \cdots & a_{k,k}^*(z) \end{array}\right] = \left[\begin{array}{c} B^{*t}(z) \\ \hline C^*(z) \end{array}\right]$$

with $C^*(0)$ nonsingular, but the restriction to (9), which algebraically is made without loss of generality, permits us to establish in §3 a duality relationship between Padé-Hermite systems and simultaneous Padé systems.

where $\gamma_0^* = v^*(0)$.

Remark 5: The NSPS is said to be **normalized** [7] if the nonsingularity condition III is replaced by $v^*(0) = 1$ and $R^*(0) = I_k$. This can be achieved by multiplying $S^*(z)$ on the left by Γ^{*-1}, where

$$\Gamma^* = diag\,[\gamma_0^*, \ldots, \gamma_k^*]\,. \tag{15}$$

The NSPS is said to be **scaled** when each row of $S^*(z)$ has norm equal to 1 and if, in addition, $\gamma_\alpha^* > 0$, $0 \le \alpha \le k$. Here, also, scaling a NSPS is accomplished by multiplying it on the left by an appropriate diagonal matrix.

Remark 6: The nonsingularity condition III, namely $\gamma^* \neq 0$, refers to the nonsingularity of the associated mosaic Sylvester matrix $\mathcal{M}_n^*$ defined in (2); in [7] it is shown that an exact NSPS exists iff $\mathcal{M}_n^*$ is nonsingular.

Remark 7: In [5, 6, 9], rather than $S^*(z)$, the simultaneous Padé system is defined to be $diag[1, z, \ldots, z] \cdot S^*(z)$. This is for notational convenience only.

3. Duality. Theorem 1 below gives a relationship between Padé-Hermite and simultaneous Padé systems which is crucial to the results of the subsequent sections. It generalizes earlier results of Mahler and their extensions to block matrices [10, 15, 17, 21].

Theorem 1. *If $S(z)$ is a NPHS of type n for $A(z)$ and $S^*(z)$ is a NSPS of type n for $A^*(z)$, then*

$$S^*(z) \cdot S(z) = z^{\|n\|}(a_0^{(0)})^{-1}\Gamma^*\Gamma + \theta_I(z), \tag{16}$$

where

$$\theta_I(z) \;=\; a_0^{-1}(z)\left\{\begin{bmatrix} v^*(z) \\ zQ^*(z) \end{bmatrix}\delta T^t(z) + \delta T^*(z)\left[\begin{array}{c|c} zQ(z) & V(z) \end{array}\right]\right\} \;\; (mod\; z^{\|n\|+1}).$$

Proof. The theorem (in the case that $\delta T(z) = 0$ and $\delta T^*(z) = 0$) follows from [15]. The arguments used in the following proof, however, are considerably simpler. Let

$$B^t(z) = [a_1(z), \cdots, a_k(z)].$$

Then, using (5) and (12),

$$\begin{aligned} (17)\quad & a_0(z)\, S^*(z) \cdot S(z) \\ &= a_0(z)\left\{\begin{bmatrix} v^*(z) \\ zQ^*(z) \end{bmatrix}\left[\begin{array}{c|c} zp(z) & U^t(z) \end{array}\right] + \begin{bmatrix} U^{*t}(z) \\ zP^*(z) \end{bmatrix}\left[\begin{array}{c|c} zQ(z) & V(z) \end{array}\right]\right\} \\ &= \begin{bmatrix} v^*(z) \\ zQ^*(z) \end{bmatrix}\left\{a_0(z)\left[\begin{array}{c|c} zp(z) & U^t(z) \end{array}\right] + B^t(z)\left[\begin{array}{c|c} zQ(z) & V(z) \end{array}\right]\right\} \\ &\quad + \left\{a_0(z)\begin{bmatrix} U^{*t}(z) \\ zP^*(z) \end{bmatrix} - \begin{bmatrix} v^*(z) \\ zQ^*(z) \end{bmatrix}B^t(z)\right\}\left[\begin{array}{c|c} zQ(z) & V(z) \end{array}\right] \end{aligned}$$

$$
\begin{aligned}
&= \begin{bmatrix} v^*(z) \\ zQ^*(z) \end{bmatrix} A^t(z)S(z) + S^*(z)A^*(z) \left[\ zQ(z) \mid V(z)\ \right] \\
&= z^{\|n\|} \left\{ \begin{bmatrix} v^*(z) \\ zQ^*(z) \end{bmatrix} \left[\ r(z) \mid W^t(z)\ \right] + \begin{bmatrix} zW^{*t}(z) \\ R^*(z) \end{bmatrix} \left[\ zQ(z) \mid V(z)\ \right] \right\} \\
&\quad + \begin{bmatrix} v^*(z) \\ zQ^*(z) \end{bmatrix} \delta T^t(z) + \delta T^*(z) \left[\ zQ(z) \mid V(z)\ \right].
\end{aligned}
$$

But, from (4) and (11), the degrees of $S^*(z)S(z)$ are bounded component-wise by $\|n\|$. It then follows from (17) that

$$
\begin{aligned}
S^*(z)S(z) &= z^{\|n\|}(a_0^{(0)})^{-1} \left[\begin{array}{c|c} v^*(0)r(0) & 0 \\ \hline 0 & R^*(0)V(0) \end{array} \right] + \theta_I(z) \\
&= z^{\|n\|}(a_0^{(0)})^{-1}\Gamma^*\Gamma + \theta_I(z),
\end{aligned}
$$

which is (16). ∎

Corollary 2. *If $S(z)$ is a* **normalized** *NPHS of type n for $A(z)$ and $S^*(z)$ is a* **normalized** *NSPS of type n for $A^*(z)$, then for sufficiently small*[2] *$\delta T(z)$ and $\delta T^*(z)$*

$$
S(z) \cdot S^*(z) = z^{\|n\|}(a_0^{(0)})^{-1} I_{k+1} + \theta_{II}(z), \tag{18}
$$

where

$$
\theta_{II}(z) = S(z)\ \theta_I(z)\ [z^{\|n\|}(a_0^{(0)})^{-1} I_{k+1} + \theta_I(z)]^{-1}\ S^*(z).
$$

Proof. If $\theta_I(z)$ is so small, that is, if $\delta T(z)$ and $\delta T^*(z)$ are so small, that $z^{\|n\|}(a_0^{(0)})^{-1} I_{k+1} + \theta_I(z)$ is nonsingular, then from (16)

$$
S^{*-1}(z) = S(z) \cdot [z^{\|n\|}(a_0^{(0)})^{-1} I_{k+1} + \theta_I(z)]^{-1}.
$$

So,

$$
\begin{aligned}
&S(z)S^*(z) - z^{\|n\|}(a_0^{(0)})^{-1} I_{k+1} \\
&\quad= \left\{ S(z) - z^{\|n\|}(a_0^{(0)})^{-1} S^{*-1}(z) \right\} S^*(z) \\
&\quad= S(z) \left\{ I_{k+1} - z^{\|n\|}(a_0^{(0)})^{-1}[z^{\|n\|}(a_0^{(0)})^{-1} I_{k+1} + \theta_I(z)]^{-1} \right\} S^*(z) \\
&\quad= S(z)\ \theta_I(z)\ [z^{\|n\|}(a_0^{(0)})^{-1} I_{k+1} + \theta_I(z)]^{-1}\ S^*(z).
\end{aligned}
$$

∎

Corollary 3. *The residuals $T(z)$ for a* **normalized** *NPHS of type n for $A(z)$ and $T^*(z)$ for a* **normalized** *NSPS of type n for $A^*(z)$ satisfy*

$$
T^t(z)\ S^*(z) = (a_0^{(0)})^{-1}\ A^t(z) + \theta^t_{III}(z), \tag{19}
$$

where

$$
\theta^t_{III}(z) = \left\{ A^t(z)\theta_{II}(z) - \delta T^t(z)S^*(z) \right\} / z^{\|n\|}.
$$

[2] It is adequate, for example, that condition (34) of Corollary 6 be satisfied.

Proof. From (5) and (18), it follows that

$$\begin{aligned}\left\{z^{\|n\|}\, T^t(z) + \delta T^t(z)\right\} S^*(z) &= A^t(z)\, S(z)\, S^*(z) \\ &= A^t(z) \left\{z^{\|n\|}\, (a_0^{(0)})^{-1} I_{k+1} \;+ \theta_{II}(z)\right\}\end{aligned}$$

and so (19) is true. ∎

4. The Inverse of a Striped Sylvester Matrix. In this section, a formula is given for the inverse of $\mathcal{M}_n$ expressed in terms of both $S(z)$ and $S^*(z)$. This enables estimating the condition number of $\mathcal{M}_n$ without explicitly computing $\mathcal{M}_n^{-1}$.

Associated with the NPHS $S(z)$, define the order $\|n\|$ matrices

$$\mathcal{P} = \left[\begin{array}{ccc|ccc|c|ccc} p^{(0)} & \cdots & p^{(n_0-1)} & q_1^{(0)} & \cdots & q_1^{(n_1-1)} & & q_k^{(0)} & \cdots & q_k^{(n_k-1)} \\ \vdots & \iddots & 0 & \vdots & \iddots & 0 & & \vdots & \iddots & 0 \\ p^{(n_0-1)} & \iddots & & q_1^{(n_1-1)} & \iddots & & \cdots & q_k^{(n_k-1)} & \iddots & \\ 0 & & \vdots & 0 & & \vdots & & 0 & & \vdots \\ \vdots & & & \vdots & & & & \vdots & & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & & 0 & \cdots & 0 \end{array}\right]$$

(20)

and, for $\beta = 1, 2, \ldots, k$,

$$(21)\quad \mathcal{U}_\beta = \left[\begin{array}{ccc|ccc|c|ccc} u_\beta^{(1)} & \cdots & u_\beta^{(n_0)} & v_{1,\beta}^{(1)} & \cdots & v_{1,\beta}^{(n_1)} & & v_{k,\beta}^{(1)} & \cdots & v_{k,\beta}^{(n_k)} \\ \vdots & \iddots & 0 & \vdots & \iddots & 0 & & \vdots & \iddots & 0 \\ u_\beta^{(n_0)} & \iddots & & v_{1,\beta}^{(n_1)} & \iddots & & \cdots & v_{k,\beta}^{(n_k)} & \iddots & \\ 0 & & \vdots & 0 & & \vdots & & 0 & & \vdots \\ \vdots & & & \vdots & & & & \vdots & & \\ 0 & \cdots & 0 & 0 & \cdots & 0 & & 0 & \cdots & 0 \end{array}\right].$$

Also, for any power series $a(z) = \sum_{\ell=0}^{\infty} a^{(\ell)} z^\ell$, and any integer function $f(i,j)$, $i, j = 1, 2, \ldots,$ let $\left[a^{(f(i,j))}\right]$ denote a matrix of order $\|n\|$ whose element in position (i,j) is $a^{(f(i,j))}$.

The main result of this section is Theorem 4 below which gives the inverse of $\mathcal{M}_n$ in terms of the NPHS $S(z)$ and the NSPS $S^*(x)$ of types n for $A(z)$.

THEOREM 4. *In terms of the* **normalized** *NPHS $S(z)$ and the* **normalized** *NSPS $S^*(x)$ of types n for $A(z)$, the inverse of $\mathcal{M}_n$ satisfies*

$$(22)\quad \mathcal{M}_n^{-1}\left\{\left[a_0^{(i-j)}\right] + \theta_{IV}\right\} = a_0^{(0)}\left\{\mathcal{P}^t\left[v^{*(\|n\|-i-j+1)}\right] + \sum_{\beta=1}^{k} \mathcal{U}_\beta^t \left[q_\beta^{*(\|n\|-i-j)}\right]\right\},$$

where

$$\theta_{IV} \quad = \quad a_0^{(0)}\left\{\left[(\theta_{III})_0^{(i-j)}\right] - \sum_{\alpha=0}^{k}\left[a_\alpha^{(\|n\|+i-j)}\right]\left[(\theta_{II})_{\alpha,0}^{(i-j)}\right]\right.$$

$$+ \left[\delta r^{(i+j-2)}\right]\left[v^{*(\|n\|-i-j+1)}\right] + \sum_{\beta=1}^{k}\left[\delta w_\beta^{(i+j-1)}\right]\left[q_\beta^{*(\|n\|-i-j)}\right]\Bigg\}.$$

Proof. The coefficient of z^{i+j-2}, for $i,j = 1,2,\ldots,\|n\|$, in the first component of (5), namely,

$$a_0(z)\ p(z) + \sum_{\alpha=1}^{k} a_\alpha(z)\ q_\alpha(z) = z^{\|n\|-1} r(z) + \delta r(z),$$

is

$$\sum_{\ell=0}^{n_0} a_0^{(i+j-\ell-2)} p^{(\ell)} + \sum_{\alpha=1}^{k}\sum_{\ell=0}^{n_\alpha-1} a_\alpha^{(i+j-\ell-2)} q_\alpha^{(\ell)} = r^{(-\|n\|+i+j-1)} + \delta r^{(i+j-2)}.$$

This is the $(i,j)th$ component of

$$\begin{aligned}(23)\quad \left[r^{(-\|n\|+i+j-1)}\right] + \left[\delta r^{(i+j-2)}\right] &= \left[a_0^{(\|n\|+i-j)}\right]\left[p^{(-\|n\|+i+j-2)}\right] \\ &+ \sum_{\alpha=1}^{k}\left[a_\alpha^{(\|n\|+i-j)}\right]\left[q_\alpha^{(-\|n\|+i+j-2)}\right] + \mathcal{M}_n\ \mathcal{P}^t.\end{aligned}$$

Similarly, the coefficient of z^{i+j-1}, for $i,j = 1,2,\ldots,\|n\|$, in the $(\beta+1)st$ component, $\beta = 1,\ldots,k$, of (5), namely,

$$a_0(z)\ u_\beta(z) + \sum_{\alpha=1}^{k} a_\alpha(z)\ v_{\alpha,\beta}(z) = z^{\|n\|+1} w_\beta(z) + \delta w_\beta(z),$$

is

$$\sum_{\ell=0}^{n_0} a_0^{(i+j-\ell-1)} u_\beta^{(\ell)} + \sum_{\alpha=1}^{k}\sum_{\ell=0}^{n_\alpha} a_\alpha^{(i+j-\ell-1)} v_{\alpha,\beta}^{(\ell)} = w_\beta^{(-\|n\|+i+j-2)} + \delta w_\beta^{(i+j-1)}.$$

This is the $(i,j)th$ component of

$$\begin{aligned}(24)\quad \left[w_\beta^{(-\|n\|+i+j-2)}\right] + \left[\delta w_\beta^{(i+j-1)}\right] &= \left[a_0^{(\|n\|+i-j)}\right]\left[u_\beta^{(-\|n\|+i+j-1)}\right] \\ &+ \sum_{\alpha=1}^{k}\left[a_\alpha^{(\|n\|+i-j)}\right]\left[v_{\alpha,\beta}^{(-\|n\|+i+j-1)}\right] + \mathcal{M}_n\ \mathcal{U}_\beta^t.\end{aligned}$$

Next, the coefficient of z^{i-j-1} for $i,j = 1,\ldots,\|n\|$, in the first row and first column of (18) for a normalized NPHS and a normalized NSPS, namely,

$$p(z)\ v^*(z) + \sum_{\beta=1}^{k} u_\beta(z) q_\beta^*(z) = z^{\|n\|-1}(a_0^{(0)})^{-1} + z^{-1}(\theta_{II})_{0,0}(z),$$

is

$$\sum_{\ell=0}^{n_0-1} v^{*(i-j-\ell-1)}\ p^{(l)} + \sum_{\beta=1}^{k}\sum_{\ell=0}^{n_0} q_\beta^{*(i-j-\ell-1)} u_\beta^{(\ell)} = (\theta_{II})_{0,0}^{(i-j)}.$$

This is the $(i,j)th$ component of

$$\text{(25)}\quad \left[p^{(-\|n\|+i+j-2)}\right]\left[v^{*(\|n\|-i-j+1)}\right] + \sum_{\beta=1}^{k}\left[u_\beta^{(-\|n\|+i+j-1)}\right]\left[q_\beta^{*(\|n\|-i-j)}\right] = \left[(\theta_{II})_{0,0}^{(i-j)}\right].$$

The coefficient of z^{i-j-1} in the first column and the $(\alpha+1)st$ row, $\alpha = 1,\dots,k$, of (18), namely,

$$q_\alpha(z)\ v^*(z) + \sum_{\beta=1}^{k} v_{\alpha,\beta}(z) q_\beta^*(z) = z^{-1}(\theta_{II})_{\alpha,0}(z)$$

is

$$\sum_{\ell=0}^{n_\alpha} v^{*(i-j-\ell-1)}\ q_\alpha^{(l)} + \sum_{\beta=1}^{k}\sum_{\ell=0}^{n_\alpha} q_\beta^{*(i-j-\ell-1)} v_{\alpha,\beta}^{(\ell)} = (\theta_{II})_{\alpha,0}^{(i-j)}.$$

This is the $(i,j)th$ component of

$$\text{(26)}\quad \left[q_\alpha^{(-\|n\|+i+j-2)}\right]\left[v^{*(\|n\|-i-j+1)}\right] + \sum_{\beta=1}^{k}\left[v_{\alpha,\beta}^{(-\|n\|+i+j-1)}\right]\left[q_\beta^{*(\|n\|-i-j)}\right] = \left[(\theta_{II})_{\alpha,0}^{(i-j)}\right].$$

Also, the coefficient of z^{i-j}, for $i,j = 1,\dots,\|n\|$ in the first component of (19) for a normalized NPHS and NSPS, namely,

$$r(z)v^*(z)\ +\ z^2\sum_{\beta=1}^{k} w_\beta(z) q_\beta^*(z) = (a_0^{(0)})^{-1}\ a_0(z) + (\theta_{III})_0(z).$$

is the $(i,j)th$ component of

$$\text{(27)}\quad (a_0^{(0)})^{-1}\ \left[a_0^{(i-j)}\right] + \left[(\theta_{III})_0^{(i-j)}\right] = \left[r^{(-\|n\|+i+j-1)}\right]\left[v^{*(\|n\|-i-j+1)}\right] + \sum_{\beta=1}^{k}\left[w_\beta^{(-\|n\|+i+j-2)}\right]\left[q_\beta^{*(\|n\|-i-j)}\right].$$

We are finally ready to prove the theorem. ¿From (23), (24), (25), (26) and (27),

$$\begin{aligned}
&\mathcal{M}_n\left\{\mathcal{P}^t\left[v^{*(\|n\|-i-j+1)}\right] + \sum_{\beta=1}^{k}\mathcal{U}_\beta^t\left[q_\beta^{*(\|n\|-i-j)}\right]\right\}\\
&= \left\{\left[r^{(-\|n\|+i+j-1)}\right] + \left[\delta r^{(i+j-2)}\right] - \left[a_0^{(\|n\|+i-j)}\right]\left[p^{(-\|n\|+i+j-2)}\right]\right.\\
&\quad \left. - \sum_{\alpha=1}^{k}\left[a_\alpha^{(\|n\|+i-j)}\right]\left[q_\alpha^{(-\|n\|+i+j-2)}\right]\right\}\left[v^{*(\|n\|-i-j+1)}\right]
\end{aligned}$$

$$
\begin{aligned}
& + \sum_{\beta=1}^{k} \Big\{ \Big[w_\beta^{(-\|n\|+i+j-2)} \Big] + \Big[\delta w_\beta^{(i+j-1)} \Big] - \Big[a_0^{(\|n\|+i-j)} \Big] \Big[u_\beta^{(-\|n\|+i+j-1)} \Big] \\
& - \sum_{\alpha=1}^{k} \Big[a_\alpha^{(\|n\|+i-j)} \Big] \Big[v_{\alpha,\beta}^{(-\|n\|+i+j-1)} \Big] \Big\} \Big[q_\beta^{*(\|n\|-i-j)} \Big] \\
= \; & \Big[r^{(-\|n\|+i+j-1)} \Big] \Big[v^{*(\|n\|-i-j+1)} \Big] + \sum_{\beta=1}^{k} \Big[w_\beta^{(-\|n\|+i+j-2)} \Big] \Big[q_\beta^{*(\|n\|-i-j)} \Big] \\
+ \; & \Big[\delta r^{(i+j-2)} \Big] \Big[v^{*(\|n\|-i-j+1)} \Big] + \sum_{\beta=1}^{k} \Big[\delta w_\beta^{(i+j-1)} \Big] \Big[q_\beta^{*(\|n\|-i-j)} \Big] \\
- \; & \sum_{\alpha=0}^{k} \Big[a_\alpha^{(\|n\|+i-j)} \Big] \Big[(\theta_{II})_{\alpha,0}^{(i-j)} \Big] \\
= \; & (a_0^{(0)})^{-1} \Big[a_0^{(i-j)} \Big] + \theta_{IV}.
\end{aligned}
$$

The result (22) now follows.

∎

Corollary 5 below drops the requirement in Theorem 4 that $S(z)$ and $S^*(z)$ be normalized. In particular, the corollary is valid in the case that $S(z)$ and $S^*(z)$ are scaled.

COROLLARY 5. *In terms of the (unnormalized) NPHS $S(z)$ of type n for $A(z)$ and the (unnormalized) NSPS $S^*(z)$ of type n for $A^*(z)$, the inverse of $\mathcal{M}_n$ is given by*

$$
\begin{aligned}
(28) \qquad & \mathcal{M}_n^{-1} \Big\{ \Big[a_0^{(i-j)} \Big] + \ddot{\theta}_{IV} \Big\} \\
& = a_0^{(0)} \left\{ (\gamma_0 \gamma_0^*)^{-1} \mathcal{P}^t \Big[v^{*(\|n\|-i-j+1)} \Big] + \sum_{\beta=1}^{k} (\gamma_\beta \gamma_\beta^*)^{-1} \mathcal{U}_\beta^t \Big[q_\beta^{*(\|n\|-i-j)} \Big] \right\},
\end{aligned}
$$

where

$$
\begin{aligned}
(29) \qquad \ddot{\theta}_{IV} &= a_0^{(0)} \Bigg\{ \Big[(\ddot{\theta}_{III})_0^{(i-j)} \Big] - \sum_{\alpha=0}^{k} \Big[a_\alpha^{(\|n\|+i-j)} \Big] \Big[(\ddot{\theta}_{II})_{\alpha,0}^{(i-j+1)} \Big] \\
& \quad + (\gamma_0 \gamma_0^*)^{-1} \Big[\delta r^{(i+j-2)} \Big] \Big[v^{*(\|n\|-i-j+1)} \Big] \\
& \quad + \sum_{\beta=1}^{k} (\gamma_\beta \gamma_\beta^*)^{-1} \Big[\delta w_\beta^{(i+j-1)} \Big] \Big[q_\beta^{*(\|n\|-i-j)} \Big] \Bigg\}, \\
(30) \qquad \ddot{\theta}_{III}^t(z) &= \Big\{ A^t(z) \ddot{\theta}_{II}(z) - \delta T^t(z) (\Gamma^* \Gamma)^{-1} S^*(z)) \Big\} / z^{\|n\|}, \\
(31) \qquad \ddot{\theta}_{II}(z) &= S(z)\, (\Gamma^* \Gamma)^{-1} \ddot{\theta}_I(z)\, (\Gamma^* \Gamma)^{-1} \\
& \quad \cdot [z^{\|n\|} (a_0^{(0)})^{-1} I_{k+1} + \ddot{\theta}_I(z) (\Gamma^* \Gamma)^{-1}]^{-1} S^*(z), \\
(32) \qquad \ddot{\theta}_I(z) &= a_0^{-1}(z) \Bigg\{ \begin{bmatrix} v^*(z) \\ z Q^*(z) \end{bmatrix} \delta T^t(z) \\
& \quad + \delta T^*(z) \Big[\; zQ(z) \; \Big| \; V(z) \; \Big] \Bigg\} \quad (mod\ z^{\|n\|+1})
\end{aligned}
$$

Proof. The normalized NPHS is obtained from an unnormalized one by multiplying it on the right by the diagonal matrix $diag[\gamma_0^{-1}, \ldots, \gamma_k^{-1}]$. Similarly, the normalized NSPS is obtained from an unnormalized one by multiplying it on the left by the diagonal matrix $diag[\gamma_0^{*-1}, \ldots, \gamma_k^{*-1}]$. The result now follows directly from (22).

∎

Let

$$\kappa = \sum_{\beta=0}^{k} (\gamma_\beta \gamma_\beta^*)^{-1}. \tag{33}$$

In the corollary below, we give a bound for $\mathcal{M}_n^{-1}$ in terms of κ.

COROLLARY 6. *If the residual errors $\delta T^t(z)$ and $\delta T^*(z)$ associated with the scaled $S(z)$ and the scaled $S^*(z)$ are not too large, so that*

$$\begin{aligned}\Big[(\kappa+1)(k+2)|a_0^{(0)}|(\|a_0^{-1}(z)\ (mod\ z^{\|n\|+1})\| + 1)\Big]^2 \\ \cdot \big[(k+2)\|\delta T^t(z)\| + \|\delta T^*(z)\|\big] \leq 1/8,\end{aligned} \tag{34}$$

then

$$\|\mathcal{M}_n^{-1}\|_1 \quad \leq \quad 2\kappa \cdot |a_0^{(0)}| \cdot \|a_0^{-1}(z)\ (mod\ z^{\|n\|+1})\|. \tag{35}$$

Proof. We use Corollary 5 with $S(z)$ and $S^*(z)$ scaled. We begin by finding a bound for $\ddot{\theta}_{IV}$ appearing in the inverse formula (28) for $\mathcal{M}_n$. A bound for $\ddot{\theta}_{IV}$ depends on bounds for $\ddot{\theta}_I(z)$, $\ddot{\theta}_{II}(z)$ and $\ddot{\theta}_{III}(z)$. ¿From (16),

$$\|\ddot{\theta}_I(z)\| \leq \|a_0^{-1}(z)\ (mod\ z^{\|n\|+1})\| \cdot \big\{(k+1)\|\delta T^t(z)\| + \|\delta T^*(z)\|\big\},$$

since $\|S(z)\| = 1$ and $\|S^*(z)\| \leq k+1$. ¿From (16) and (32), note that $\ddot{\theta}_I(z)$ is a matrix polynomial of at most degree $\|n\|$ and so, using (34),

$$\|a_0^{(0)} z^{\|n\|} \ddot{\theta}_I(z^{-1})(\Gamma^*\Gamma)^{-1}\| = \|a_0^{(0)} \ddot{\theta}_I(z)(\Gamma^*\Gamma)^{-1}\| \leq \kappa \cdot |a_0^{(0)}| \cdot \|\ddot{\theta}_I(z)\| \leq 1/2,$$

since $\|(\Gamma^*\Gamma)^{-1}\| \leq \kappa$. So as in Stewart [22, page 187], the inverse of $(a_0^{(0)})^{-1} I_{k+1} + z^{\|n\|}\ddot{\theta}_I(z^{-1})(\Gamma^*\Gamma)^{-1}$ exists and

$$\begin{aligned}\| \Big\{(a_0^{(0)})^{-1} I_{k+1} + z^{\|n\|}\ddot{\theta}_I(z^{-1})(\Gamma^*\Gamma)^{-1}\Big\}^{-1} \| \quad &\leq \quad \frac{|a_0^{(0)}|}{1 - \|a_0^{(0)} z^{\|n\|}\ddot{\theta}_I(z^{-1})(\Gamma^*\Gamma)^{-1}\|} \\ &\leq \quad 2|a_0^{(0)}|.\end{aligned}$$

To determine a bound for $\ddot{\theta}_{II}(z)$ in (31), let $N = \max_{0\leq\beta\leq k}\{n_\beta\}$ and observe from 18 that $\ddot{\theta}_{II}(z)$ is also a matrix polynomial, now of degree at most $\|n\| + N$. Consequently,

$$\begin{aligned}\|\ddot{\theta}_{II}(z)\| \quad &= \quad \|z^{\|n\|+N}\ddot{\theta}_{II}(z^{-1})\| \\ &= \quad \| \big\{z^N S(z^{-1})\big\}\ (\Gamma^*\Gamma)^{-1} \Big\{z^{\|n\|}\ddot{\theta}_I(z^{-1})\Big\}\ (\Gamma^*\Gamma)^{-1}\end{aligned} \tag{36}$$

$$
\begin{aligned}
& \cdot \left\{ (a_0^{(0)})^{-1} I_{k+1} + z^{\|n\|} \ddot{\theta}_I(z^{-1}) (\Gamma^*\Gamma)^{-1} \right\}^{-1} [z^{\|n\|} S^*(z^{-1})] \| \\
\leq \; & \kappa^2 \|z^N S(z^{-1})\| \cdot \|z^{\|n\|} \ddot{\theta}_I(z^{-1})\| \cdot \|z^{\|n\|} S^*(z^{-1})\| \\
& \cdot \| \left\{ (a_0^{(0)})^{-1} I_{k+1} + z^{\|n\|} \ddot{\theta}_I(z^{-1}) (\Gamma^*\Gamma)^{-1} \right\}^{-1} \| \\
\leq \; & 2\kappa^2 |a_0^{(0)}| \cdot \|S(z)\| \cdot \|\ddot{\theta}_I(z)\| \cdot \|S^*(z)\| \\
\leq \; & 2\kappa^2 (k+1) \cdot |a_0^{(0)}| \cdot \|\ddot{\theta}_I(z)\|.
\end{aligned}
$$

In addition, it now follows that a bound for $\ddot{\theta}_{III}(z)$ in (30) is given by

$$
\|\ddot{\theta}^t_{III}(z)\| \leq 2\kappa^2 (k+1) \cdot |a_0^{(0)}| \cdot \|\ddot{\theta}_I(z)\| + \kappa(k+1) \cdot \|\delta T^t(z)\|.
$$

We are now ready to give a bound for $\ddot{\theta}_{IV}$ appearing in the inverse formula (28). But, first observe that

$$
\| \left[(\ddot{\theta}_{III})_0^{(i-j)} \right] \|_1 \leq \|\ddot{\theta}^t_{III}(z)\|
$$

and that

$$
\| \sum_{\alpha=0}^{k} \left[a_\alpha^{(\|n\|+i-j)} \right] \left[(\ddot{\theta}_{II})_{\alpha,0}^{(i+j)} \right] \|_1 \leq \sum_{\alpha=0}^{k} \|(\ddot{\theta}_{II})_{\alpha,0}(z)\| \leq \|(\ddot{\theta}_{II})(z)\|.
$$

Thus,

$$
\begin{aligned}
\|\ddot{\theta}_{IV}\|_1 \;=\; & \|a_0^{(0)} \left\{ \left[(\ddot{\theta}_{III})_0^{(i-j)} \right] - \sum_{\alpha=0}^{k} \left[a_\alpha^{(\|n\|+i-j)} \right] \left[(\ddot{\theta}_{II})_{\alpha,0}^{(i-j)} \right] \right. \\
& + (\gamma_0\gamma_0^*)^{-1} \left[\delta r^{(i+j-2)} \right] \left[v^{*(\|n\|-i-j+1)} \right] \\
& \left. + \sum_{\beta=1}^{k} (\gamma_\beta\gamma_\beta^*)^{-1} \left[\delta w_\beta^{(i+j-1)} \right] \left[q_\beta^{*(\|n\|-i-j)} \right] \right\} \|_1 \\
\leq \; & |a_0^{(0)}| \cdot \left\{ \|\ddot{\theta}^t_{III}(z)\| + \|\ddot{\theta}_{II}(z)\| + (\gamma_0\gamma_0^*)^{-1} \|\delta T^t(z)\| + \sum_{\beta=1}^{k} (\gamma_\beta\gamma_\beta^*)^{-1} \|\delta T^t(z)\| \right\} \\
\leq \; & |a_0^{(0)}| \left\{ \|\ddot{\theta}^t_{III}(z)\| + \|\ddot{\theta}_{II}(z)\| + \kappa \|\delta T^t(z)\| \right\} \\
\leq \; & |a_0^{(0)}| \left\{ \kappa(k+1) \|\delta T^t(z)\| + 4\kappa^2 (k+1) |a_0^{(0)}| \cdot \|\ddot{\theta}_I(z)\| + \kappa \|\delta T^t(z)\| \right\} \\
\leq \; & \kappa(k+2) \cdot |a_0^{(0)}| \left\{ \|\delta T^t(z)\| \right. \\
& \quad \left. + 4\kappa |a_0^{(0)}| \cdot \|a_0^{-1}(z) \; (mod \; z^{\|n\|+1})\| \left[(k+1) \|\delta T^t(z)\| + \|\delta T^*(z)\| \right] \right\} \\
\leq \; & 4 \|a_0^{-1}(z) \; (mod \; z^{\|n\|+1})\| \left[(\kappa+1)(k+2) |a_0^{(0)}| \right]^2 \left[(k+2) \|\delta T^t(z)\| + \|\delta T^*(z)\| \right].
\end{aligned}
$$

It then follows from (34) that

$$
\| \left[a_0^{(i-j)} \right]^{-1} \ddot{\theta}_{IV} \|_1 \leq \|a_0^{-1}(z) \; (mod \; z^{\|n\|+1})\| \cdot \|\ddot{\theta}_{IV}\|_1 \leq 1/2,
$$

and so $I_{\|n\|} + \left[a_0^{(i-j)}\right]^{-1} \ddot{\theta}_{IV}$ is invertible. In addition,

$$\begin{aligned} \| \left\{ \left[a_0^{(i-j)}\right] + \ddot{\theta}_{IV} \right\}^{-1} \|_1 &\leq \| \left\{ I_{\|n\|} + \left[a_0^{(i-j)}\right]^{-1} \ddot{\theta}_{IV} \right\}^{-1} \left[a_0^{(i-j)}\right]^{-1} \|_1 \\ &\leq \frac{\| \left[a_0^{(i-j)}\right]^{-1} \|_1}{1 - \| \left[a_0^{(i-j)}\right]^{-1} \ddot{\theta}_{IV} \|_1} \\ &\leq 2\| \left[a_0^{(i-j)}\right]^{-1} \|_1 \\ &\leq 2\|a_0^{-1}(z)\ (mod\ z^{\|n\|+1})\|. \end{aligned}$$

Therefore, a bound for $\mathcal{M}_n^{-1}$ in (28) is given

$$\begin{aligned} \|\mathcal{M}_n^{-1}\|_1 &\leq \| \left\{ \left[a_0^{(i-j)}\right] + \ddot{\theta}_{IV} \right\}^{-1} \| \cdot \|a_0^{(0)} \ \left\{ (\gamma_0\gamma_0^*)^{-1} \mathcal{P}_n^t \left[v^{*(\|n\|-i-j+1)}\right] \right. \\ &\quad \left. + \sum_{\beta=1}^{k} (\gamma_\beta\gamma_\beta^*)^{-1} \mathcal{U}_{n,\beta}^t \left[q_\beta^{*(\|n\|-i-j)}\right] \right\} \|_1 \\ &\leq 2\kappa|a_0^{(0)}| \cdot \|a_0^{-1}(z)\ (mod\ z^{\|n\|+1})\|. \end{aligned}$$

∎

¿From (35), it follows that a bound for the 1-norm condition number of $\mathcal{M}_n$ is

$$\|\mathcal{M}_n\|_1 \cdot \|\mathcal{M}_n^{-1}\|_1 \leq 2\kappa|a_0^{(0)}| \cdot \|a_0^{-1}(z)\ (mod\ z^{\|n\|+1})\|,$$

since it is assumed that each $a_\beta(z)$ is scaled.

5. The Inverse of a Mosaic Sylvester Matrix. In this section, a formula is given for the inverse of $\mathcal{M}_n^*$ expressed in terms of both $S(z)$ and $S^*(z)$. This enables estimating the condition number of $\mathcal{M}_n^*$ without explicitly computing $\mathcal{M}_n^{*-1}$.

Associated with the NPHS $S(z)$ and the NSPS $S^*(z)$, for $\beta = 1, 2, \ldots, k$, define the $\|n\| \times k\|n\|$ matrices

$$\mathcal{V}_\beta = \left[\begin{array}{ccc|c|ccc} v_{1,\beta}^{(\|n\|-1)} & \cdots & v_{1,\beta}^{(0)} & & v_{k,\beta}^{(\|n\|-1)} & \cdots & v_{k,\beta}^{(0)} \\ \vdots & \iddots & & \cdots & \vdots & \iddots & \\ v_{1,\beta}^{(0)} & & & & v_{k,\beta}^{(0)} & & \end{array} \right],$$

$$\mathcal{Q} = \left[\begin{array}{cccc|c|cccc} q_1^{(\|n\|-2)} & \cdots & q_1^{(0)} & 0 & & q_k^{(\|n\|-2)} & \cdots & q_k^{(0)} & 0 \\ \vdots & \iddots & & & \cdots & \vdots & \iddots & & \\ q_1^{(0)} & \iddots & & & & q_k^{0)} & \iddots & & \\ 0 & & & & & 0 & & & \end{array} \right],$$

$$\mathcal{V}^* = \left[\begin{array}{ccc|ccc|c|ccc}
v^{*(1)} & \cdots & v^{*(\eta_0)} & u_1^{*(1)} & \cdots & u_1^{*(\eta_1)} & & u_k^{*(1)} & \cdots & u_k^{*(\eta_k)} \\
\vdots & \iddots & 0 & \vdots & \iddots & 0 & & \vdots & \iddots & 0 \\
v^{*(\eta_0)} & \iddots & & u_1^{*(\eta_1)} & \iddots & & \cdots & u_k^{*(\eta_k)} & \iddots & \\
0 & & \vdots & 0 & & \vdots & & 0 & & \vdots \\
\vdots & & & \vdots & & & & \vdots & & \\
0 & \cdots & 0 & 0 & \cdots & 0 & & 0 & \cdots & 0
\end{array}\right]$$

and

$$\mathcal{Q}_\beta^* = \left[\begin{array}{ccc|ccc|c|ccc}
q_\beta^{*(0)} & \cdots & q_\beta^{*(\eta_0-1)} & p_{\beta,1}^{*(0)} & \cdots & p_{\beta,1}^{*(\eta_1-1)} & & p_{\beta,k}^{*(0)} & \cdots & p_{\beta,k}^{*(\eta_k-1)} \\
\vdots & \iddots & 0 & \vdots & \iddots & 0 & & \vdots & \iddots & 0 \\
q_\beta^{*(\eta_0-1)} & \iddots & & p_{\beta,1}^{*(\eta_1-1)} & \iddots & & \cdots & p_{\beta,k}^{*(\eta_k-1)} & \iddots & \\
0 & & \vdots & 0 & & \vdots & & 0 & & \vdots \\
\vdots & & & \vdots & & & & \vdots & & \\
0 & \cdots & 0 & 0 & \cdots & 0 & & 0 & \cdots & 0
\end{array}\right],$$

where $\eta_\beta = \|n\| - n_\beta$. For $\beta = 1, 2, \ldots, k$, also define the $\|n\| \times k\|n\|$ residual error matrices

$$\delta W_\beta = [\delta \bar{W}_\beta, \mathbf{0}_{n_1}, \ldots, \mathbf{0}_{n_k}]$$

and

$$\delta R = [\delta \bar{R}, \mathbf{0}_{n_1}, \ldots, \mathbf{0}_{n_k}],$$

where

$$\delta \bar{W}_\beta = \begin{bmatrix}
\delta w_\beta^{(\|n\|-1)} & \cdots & \delta w_\beta^{(n_0)} \\
 & & \vdots \\
\vdots & & \delta w_\beta^{(0)} \\
 & \iddots & \\
\delta w_\beta^{(0)} & &
\end{bmatrix}, \quad \delta \bar{R} = \begin{bmatrix}
\delta r^{(\|n\|-2)} & \cdots & \delta r^{(n_0-1)} \\
 & & \vdots \\
\vdots & & \delta r^{(0)} \\
 & \iddots & 0 \\
\delta r^{(0)} & \iddots & \vdots \\
0 & \cdots & 0
\end{bmatrix},$$

and $\mathbf{0}_{n_\beta}$ is a $\|n\| \times \|n\| - n_\beta$ matrix of zeroes. Also, let

$$\theta = \begin{bmatrix}
\theta_{0,0} & \cdots & \theta_{0,k} \\
\vdots & & \vdots \\
\theta_{k,0} & \cdots & \theta_{k,k}
\end{bmatrix},$$

where each $\theta_{\alpha,\beta}$ is an $(\|n\| - n_\alpha) \times (\|n\| - n_\beta)$ matrix given by

$$\theta_{\alpha,\beta} = \begin{bmatrix}
(\theta_{II})_{\alpha,\beta}^{(\|n\|+1)} & \cdots & (\theta_{II})_{\alpha,\beta}^{(2\|n\|-n_\beta)} \\
\vdots & & \vdots \\
(\theta_{II})_{\alpha,\beta}^{(n_\alpha+2)} & \cdots & (\theta_{II})_{\alpha,\beta}^{(\|n\|+n_\alpha-n_\beta+1)}
\end{bmatrix}$$

with $\theta_{II}(z)$ the error appearing in (18). Finally, let $\left[a_0^{(i-j)}\right]$ denote an order $\|n\|$, lower triangular, matrix as in §4.

The main result of this section is Theorem 7 below which gives the inverse of $\mathcal{M}_n^*$ in terms of the NPHS $S(z)$ and the NSPS $S^*(z)$ of types n for $A(z)$.

THEOREM 7. *In terms of the* **normalized** *NPHS $S(z)$ and the* **normalized** *NSPS $S^*(x)$ of types n for $A(z)$, the inverse of $\mathcal{M}_n^*$ satisfies*

$$\mathcal{M}_n^{*-1}\left\{(a_0^{(0)})^{-1}I_{k\|n\|}+\theta_{IV}^*\right\}=\mathcal{Q}^t\left[a_0^{(i-j)}\right]^{-1}\mathcal{V}^*+\sum_{\beta=1}^{k}\mathcal{V}_\beta^t\left[a_0^{(i-j)}\right]^{-1}\mathcal{Q}_\beta^*, \tag{37}$$

where

$$\theta_{IV}^* \quad = \quad \theta-\delta R^t\left[a_0^{(i-j)}\right]^{-1}\mathcal{V}^*-\sum_{\beta=1}^{k}\delta W_\beta^t\left[a_0^{(i-j)}\right]^{-1}\mathcal{Q}_\beta^* \tag{38}$$

Proof. Let

$$\bar{\mathcal{Q}}=\left[\begin{array}{ccc|ccc|c|ccc} p^{(\|n\|-2)} & \cdots & p^{(n_0-1)} & q_1^{(\|n\|-2)} & \cdots & q_1^{(n_1-1)} & & q_k^{(\|n\|-2)} & \cdots & q_k^{(n_k-1)} \\ & & \vdots & & & \vdots & & & & \vdots \\ \vdots & & p^{(0)} & \vdots & & q_1^{(0)} & \cdots & \vdots & & q_k^{(0)} \\ & \cdot^{\cdot^{\cdot}} & 0 & & \cdot^{\cdot^{\cdot}} & 0 & & & \cdot^{\cdot^{\cdot}} & 0 \\ p^{(0)} & \cdot^{\cdot^{\cdot}} & & q_1^{(0)} & \cdot^{\cdot^{\cdot}} & & & q_k^{(0)} & \cdot^{\cdot^{\cdot}} & \\ 0 & & & 0 & & & & 0 & & \end{array}\right].$$

Then, the order condition (5) for an NPHS implies that

$$\mathcal{M}_n^*\cdot\mathcal{Q}^t=\bar{\mathcal{Q}}^t\cdot\left[a_0^{(i-j)}\right]-\delta R^t. \tag{39}$$

To see this, note the $(i,j)th$ component, $1\le i\le\|n\|-n_0$, $1\le j\le\|n\|$, of (39) is the coefficient of $z^{\|n\|-i-j}$ in

$$a_0(z)\;p(z)+\sum_{\alpha=1}^{k}a_\alpha(z)\;q_\alpha(z)=z^{\|n\|-1}r(z)+\delta r(z).$$

The remaining components of (39) are obvious identities.

Similarly, for $1\le\beta\le k$, let

$$\bar{\mathcal{V}}_\beta=\left[\begin{array}{ccc|ccc|c|ccc} u_\beta^{(\|n\|-1)} & \cdots & u_\beta^{(n_0)} & v_{1,\beta}^{(\|n\|-1)} & \cdots & v_{1,\beta}^{(n_1)} & & v_{k,\beta}^{(\|n\|-1)} & \cdots & v_{k,\beta}^{(n_k)} \\ & & \vdots & & & \vdots & & & & \vdots \\ \vdots & & u_\beta^{(0)} & \vdots & & v_{1,\beta}^{(0)} & \cdots & \vdots & & v_{k,\beta}^{(0)} \\ & \cdot^{\cdot^{\cdot}} & & & \cdot^{\cdot^{\cdot}} & & & & \cdot^{\cdot^{\cdot}} & \\ u_\beta^{(0)} & & & v_{1,\beta}^{(0)} & & & & v_{k,\beta}^{(0)} & & \end{array}\right].$$

Then, the coefficient of $z^{\|n\|-i-j+1}$, $1 \leq i \leq \|n\| - n_0$, $1 \leq j \leq \|n\|$, in the order condition (5) for an NPHS, namely,

$$a_0(z)\ u_\beta(z) + \sum_{\alpha=1}^{k} a_\alpha(z)\ v_{\alpha,\beta}(z) = z^{\|n\|+1} w_\beta(z) + \delta w_\beta(z),$$

gives the $(i,j)th$ component of

$$\mathcal{M}_n^* \cdot \mathcal{V}_\beta^t = \bar{\mathcal{V}}_\beta^t \cdot \left[a_0^{(i-j)}\right] - \delta W^t. \tag{40}$$

The remaining components of (40) are easy to verify.

Next, observe that Theorem 1 and Corollary 2 imply that

$$\bar{\mathcal{Q}}^t \cdot \mathcal{V}^* + \sum_{\beta=1}^{k} \bar{\mathcal{V}}_\beta^t \cdot \mathcal{Q}_\beta^* = (a_0^{(0)})^{-1} I_{k\|n\|} + \theta. \tag{41}$$

Combining (39), (40) and (41), we obtain the result (37). ∎

Corollary 8 below drops the requirement in Theorem 7 that $S(z)$ and $S^*(z)$ be normalized. In particular, the results of the corollary apply when $S(z)$ and $S^*(z)$ are scaled.

Corollary 8. *In terms of the NPHS $S(z)$ (unnormalized) of type n for $A(z)$ and the NSPS $S^*(z)$ (unnormalized) of type n for $A^*(z)$, the inverse of $\mathcal{M}_n^*$ is given by*

$$\begin{aligned} \mathcal{M}_n^{*-1} &\left\{(a_0^{(0)})^{-1} I_{k\|n\|} + \ddot{\theta}_{IV}^*\right\} \\ &= (\gamma_0\gamma_0^*)^{-1} \mathcal{Q}^t \left[a_0^{(i-j)}\right]^{-1} \mathcal{V}^* + \sum_{\beta=1}^{k} (\gamma_\beta\gamma_\beta^*)^{-1} \mathcal{V}_\beta^t \left[a_0^{(i-j)}\right]^{-1} \mathcal{Q}_\beta^*, \end{aligned} \tag{42}$$

where

$$\ddot{\theta}_{IV}^* \quad = \quad \ddot{\theta} - (\gamma_0\gamma_0^*)^{-1} \delta R^t \left[a_0^{(i-j)}\right]^{-1} \mathcal{V}^* - \sum_{\beta=1}^{k} (\gamma_\beta\gamma_\beta^*)^{-1} \delta W_\beta^t \left[a_0^{(i-j)}\right]^{-1} \mathcal{Q}_\beta^*$$

and

$$\ddot{\theta} = \begin{bmatrix} \ddot{\theta}_{0,0} & \cdots & \ddot{\theta}_{0,k} \\ \vdots & & \vdots \\ \ddot{\theta}_{k,0} & \cdots & \ddot{\theta}_{k,k} \end{bmatrix}$$

with

$$\ddot{\theta}_{\alpha,\beta} = \begin{bmatrix} (\ddot{\theta}_{II})_{\alpha,\beta}^{(\|n\|+1)} & \cdots & (\ddot{\theta}_{II})_{\alpha,\beta}^{(2\|n\|-n_\beta)} \\ \vdots & & \vdots \\ (\ddot{\theta}_{II})_{\alpha,\beta}^{(n_\alpha+2)} & \cdots & (\ddot{\theta}_{II})_{\alpha,\beta}^{(\|n\|+n_\alpha-n_\beta+1)} \end{bmatrix}$$

Proof. The normalized NPHS is obtained from an unnormalized one by multiplying it on the right by the diagonal matrix $diag[\gamma_0^{-1}, \ldots, \gamma_k^{-1}]$. Similarly, the normalized NSPS is obtained from an unnormalized one by multiplying it on the left by the diagonal matrix $diag[\gamma_0^{*-1},\ldots,\gamma_k^{*-1}]$. The result now follows directly from (37). ∎

Corollary 9. *If the conditions of Corollary 6 are satisfied, then*[3]

$$\|\mathcal{M}_n^{*-1}\|_\infty \;\leq\; 2\kappa\cdot|a_0^{(0)}|\cdot\|a_0^{-1}(z)\ (mod\ z^{\|n\|+1})\|. \tag{43}$$

Proof. ¿From (36),

$$\begin{aligned}\|\ddot{\theta}\|_\infty &\leq (k+1)\|\ddot{\theta}_{II}(z)\|\\ &\leq 2\kappa^2(k+1)^2\cdot|a_0^{(0)}|\cdot\|a_0^{-1}(z)\ (mod\ z^{\|n\|+1})\|\cdot\left\{(k+1)\|\delta T^t(z)\|+\|\delta T^*(z)\|\right\}.\end{aligned}$$

Thus,

$$\begin{aligned}\|\ddot{\theta}^*_{IV}\|_\infty &= \|\ddot{\theta}-(\gamma_0\gamma_0^*)^{-1}\delta R^t\left[a_0^{(i-j)}\right]^{-1}\mathcal{V}^*-\sum_{\beta=1}^{k}(\gamma_\beta\gamma_\beta^*)^{-1}\delta W_\beta^t\left[a_0^{(i-j)}\right]^{-1}\mathcal{Q}_\beta^*\|_\infty\\ &\leq \|\ddot{\theta}\|_\infty+\ (\gamma_0\gamma_0^*)^{-1}(k+1)\cdot\|\delta T^t(z)\|\cdot\|a_0^{-1}(z)\ (mod\ z^{\|n\|+1})\|\cdot\|S^*(z)\|\\ &\quad+\ \sum_{\beta=1}^{k}(\gamma_\beta\gamma_\beta^*)^{-1}(k+1)\cdot\|\delta T^t(z)\|\cdot\|a_0^{-1}(z)\ (mod\ z^{\|n\|+1})\|\cdot\|S^*(z)\|\\ &\leq \kappa(k+1)^2\cdot\|a_0^{-1}(z)\ (mod\ z^{\|n\|+1})\|\\ &\quad\cdot\left\{\|\delta T^t(z)\|+2\kappa|a_0^{(0)}|\cdot\left[(k+1)\|\delta T^t(z)\|+\|\delta T^*(z)\|\right]\right\}\\ &\leq 4|a_0^{(0)}|\cdot\left[(\kappa+1)(k+2)(\|a_0^{-1}(z)\ (mod\ z^{\|n\|+1})\|+1)\right]^2\\ &\quad\cdot\left[(k+2)\|\delta T^t(z)\|+\|\delta T^*(z)\|\right].\end{aligned}$$

Therefore, using the assumption (34),

$$\|\left\{(a_0^{(0)})^{-1}I_{k\|n\|}+\ddot{\theta}^*_{IV}\right\}^{-1}\|_\infty\leq 2|a_0^{(0)}|$$

and so

$$\begin{aligned}\|\mathcal{M}_n^{*-1}\|_\infty &\leq \|\left\{(a_0^{(0)})^{-1}I_{k\|n\|}+\ddot{\theta}^*_{IV}\right\}^{-1}\|_\infty\cdot\|(\gamma_0\gamma_0^*)^{-1}\mathcal{Q}^t\left[a_0^{(i-j)}\right]^{-1}\mathcal{V}^*\\ &\quad+\sum_{\beta=1}^{k}(\gamma_\beta\gamma_\beta^*)^{-1}\mathcal{V}_\beta^t\left[a_0^{(i-j)}\right]^{-1}\mathcal{Q}_\beta^*\|_\infty\\ &\leq 2\kappa|a_0^{(0)}|\cdot\|a_0^{-1}(z)\ (mod\ z^{\|n\|+1})\|.\end{aligned}$$

∎

6. Conclusions. In this paper we have presented new formulae for the "near" inverses of striped and mosaic Sylvester matrices. The formulae are given in terms of numerical Padé-Hermite and simultaneous Padé systems. They are important for

[3] The ∞-norm, rather than the 1-norm, is used here because it is more suitable for purposes in [6].

numerical computation since they incorporate errors caused by floating-point arithmetic. In particular, the formulae can be used to determine good estimates for the condition numbers of these matrices.

Our primary motivation for obtaining these formulae is the numerically stable computation of Padé-Hermite and simultaneous Padé approximants, the subject of the companion paper [6]. As such we have restricted our attention to a striped and a specific mosaic Sylvester matrices. We conjecture that a similar approach can also be used for determining near inverse formulae of other structured matrices, for example, of mosaic Hankel, Toeplitz or Sylvester matrices [13, 16]. Some preliminary work on this topic has already been done in [8].

Together with the results of [6], we believe that the formulae given in this paper can be used to stably invert striped and mosaic Sylvester matrices and to stably solve systems of linear equations with these as coefficient matrices. This matter requires formal verification, such as that reported in [11] for the case $k = 1$ and $a_0(z) = 1$.

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