# Inversion of Mosaic Hankel Matrices via Matrix Polynomial Systems

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**Abstract**

Heinig and Tewodros [18] give a set of components whose existence provides a necessary and sufficient condition for a mosaic Hankel matrix to be nonsingular. When this is the case they also give a formula for the inverse in terms of these components.

By converting these components into a matrix polynomial form we show that the invertibility conditions can be described in terms of matrix rational approximants for a matrix power series determined from the entries of the mosaic matrix. In special cases these matrix rational approximations are closely related to Padé and various well-known matrix-type Padé approximants. We also show that the inversion components can be described in terms of unimodular matrix polynomials. These are shown to be closely related to the $V$ and $W$ matrices of Antoulas used in his study of recursiveness in linear systems. Finally, we present a recursion which allows for the efficient computation of the inversion components of all nonsingular "principal mosaic Hankel" submatrices (including the components for the matrix itself).

**Key words:** Hankel matrices, Mosaic Hankel matrices, matrix inversion.
**Subject Classifications:** AMS(MOS): 15A09, 15A57.

# 1 Introduction

In this paper we study matrices that can be partitioned as

$$H = \begin{bmatrix} H_{1,1} & \cdots & H_{1,\ell} \\ \vdots & & \vdots \\ H_{k,1} & \cdots & H_{k,\ell} \end{bmatrix},$$

with each $H_{\alpha,\beta} = [h_{i+j}^{(\alpha,\beta)}]_{i=1,j=1}^{m_\alpha\ n_\beta}$ an $m_\alpha \times n_\beta$ Hankel matrix. We assume that the partition sizes are such that $m = \sum_{\alpha=1}^{k} m_\alpha = \sum_{\beta=1}^{\ell} n_\beta$ so that $H$ is square of size $m \times m$. Such a matrix is called a *mosaic Hankel matrix* having $k$ layers and $\ell$ stripes. We study the inversion problem for these matrices, that is, the problem of efficiently determining when $H$ is nonsingular, and when this is the case, of constructing the inverse.

Examples of mosaic Hankel matrices appear in numerous applications in many branches of mathematics. The simplest case when $k = 1$ and $\ell = 1$ represents the classical Hankel matrix. When the $m_\alpha$ and $n_\beta$ are all equal, the matrix $H$ is a simple permutation of a block Hankel matrix [25] (see Section 2). More generally, when the $m_\alpha$ are the same and all the $n_\beta$ are the same then $H$ is a matrix that can be partitioned into non-square blocks [12], [22] having a Hankel structure. Mosaic Hankel matrices having $k = 1$ are called striped Hankel matrices [17], [21] while those having $\ell = 1$ are called layered Hankel. These appear as coefficient matrices in the linear systems defining Hermite-Padé and simultaneous Padé approximants [21]. Other examples of mosaic Hankel matrices include Sylvester matrices [11] and $p$-Hankel matrices [3].

It is easy to see that $H$ has a rank decomposition of order $k + \ell$, and hence from [25] we know that the inversion problem requires solutions to $k + \ell$ linear systems with $H$ as the coefficient matrix. There are, however, a number of possibilities for such linear systems. In our case we follow the work of Heinig and Tewodros [18] where the linear systems consist of $k$ standard equations (i.e. having columns of the identity) together with $\ell$ other equations called the "fundamental" equations. Since the transpose of $H$ is also a mosaic Hankel matrix, these inverse components exist both in column form and in row form.

The inverse components can be converted into a pair of $(k + \ell)$-square matrix polynomials. The entries of each matrix polynomial are closely related to a pair of rational approximations to a certain matrix power series determined from the entries of $H$. In special cases these rational approximations are the same as Padé or well-known matrix-type Padé approximants (such as matrix Padé, Hermite-Padé and simultaneous Padé approximants). In such cases we can use existing matrix-type Padé algorithms to obtain fast and superfast methods for computing these inverse components (cf., [5], [9]).

The matrix polynomials are also shown to be closely related to the $V$ and $W$ unimodular matrices of Antoulas [2] used for the computation of minimal realizations of a matrix sequence. In this sense our work extends the results of [19], [21] and [22]. As in the last two papers, the principal tool is a commutativity relation satisfied by the matrix polynomials. By reversing the orders of the coefficients this relation gives the main criterion of the $V$ and $W$ matrices, namely that they are unimodular.

We also present a recursion which can be used to compute any nonsingular "principal mosaic Hankel" submatrices. Indeed, the recursion can be interpreted as computing the inversion components by recursively computing the components for all principal mosaic Hankel submatrices of $H$ (including $H$ itself). In all cases our methods are reliable in exact arithmetic. By this we mean that no restrictions are needed on the nonsingularity structure of submatrices of $H$. Of course one can only take advantage of this recursion if at least one of the principal mosaic Hankel submatrices is nonsingular.

The paper is organized as follows. Section 2 gives necessary and sufficient conditions for the existence of an inverse for $H$ in terms of solutions to $k+\ell$ linear equations along with an inversion formula that computes the inverse in terms of these inverse components. Section 3 converts the inverse components into matrix polynomial form. The linear equations defining the components are shown to be equivalent to certain types of matrix Padé-like approximants of a matrix power series associated to $H$. Section 4 combines these matrix polynomials together and shows the strong relationship between the inverse components and the main tools used by Antoulas. Section 5 gives a method that recursively computes the inversion components of all nonsingular principal mosaic Hankel submatrices of $H$. The last section discusses directions for further research.

# 2 Preliminaries

In this section, we give some preliminary results necessary for the subsequent development. In particular, we give necessary and sufficient conditions for a mosaic Hankel matrix to be nonsingular and show how to compute the inverse based on these conditions. The results of this section follow directly from the work of Heinig and Tewodros [18] on the inversion of mosaic matrices.

Let $N$ be a fixed integer such that $N - m_\alpha - n_\beta \geq -1$ for all $\alpha$ and $\beta$ and define $\vec{m} = (m_1, \cdots, m_k)$, $\vec{n} = (n_1, \cdots, n_\ell)$ (for example we might choose $N = \max(\vec{m}) + \max(\vec{n}) - 1 = \max_\alpha\{m_\alpha\} + \max_\beta\{n_\beta\} - 1$). For convenience we renumber the entries in the Hankel blocks so that

$$H := H(\vec{m}, \vec{n}, N-1) = [H_{\alpha,\beta}]_{\alpha=1,\beta=1}^{k\ \ \ \ell}, \quad H_{\alpha,\beta} = [a_{i+j+t}^{(\alpha,\beta)}]_{i=1,j=1}^{m_\alpha\ n_\beta}, \tag{1}$$

where $t = N - 1 - m_\alpha - n_\beta$. This indexing is chosen in this way so that the bottom right hand corner of each block in the mosaic has index $N-1$. This indexing scheme will be useful in Section 3 where the inversion components are converted into a matrix polynomial form representing rational approximants of a matrix power series. Indeed, this scheme is common when Hankel matrices appear in applications involving Padé approximation.

Let

$$H \cdot V = -\begin{bmatrix} W_1 \\ \vdots \\ W_k \end{bmatrix} \tag{2}$$

with $V$ of size $m \times \ell$ and where each $W_\alpha$ is a matrix block of size $m_\alpha \times \ell$ given by

$$W_\alpha = \begin{bmatrix} a_{N-m_\alpha+1}^{(\alpha,1)} & \cdots & a_{N-m_\alpha+1}^{(\alpha,\ell)} \\ \vdots & & \vdots \\ a_N^{(\alpha,1)} & \cdots & a_N^{(\alpha,\ell)} \end{bmatrix}, \tag{3}$$

that is, $W := H(\vec{m}, \vec{e}, N)$ with $\vec{e} = (1, ..., 1)$. The $a_N^{(\alpha,\beta)}$ in each $W_\alpha$ are allowed to be arbitrary. Similarly, let

$$H \cdot Q = E_{(m_1,\cdots,m_k)} \tag{4}$$

where $Q$ is a $m \times k$ matrix and where the $\alpha$-th column of $E_{(m_1,\cdots,m_k)}$ is the $m_1+\cdots+m_\alpha$-th column of the $m \times m$ identity matrix.

Clearly if $H$ is nonsingular then there are solutions to equations (2) and (4). Central to our work is the fact that the converse is also true.

**Theorem 2.1. (Heinig and Tewodros [18])** *$H$ is nonsingular if and only if there are solutions to equations (2) and (4).*

Consider now

$$V^* \cdot H = -\left[W_1^*, \cdots, W_\ell^*\right] \tag{5}$$

where $V^*$ is a matrix of size $k \times m$ and each $W_\beta^*$ is a matrix block of size $k \times n_\beta$ given by

$$W_\beta^* = \begin{bmatrix} a_{N-n_\beta+1}^{(1,\beta)} & \cdots & a_N^{(1,\beta)} \\ \vdots & & \vdots \\ a_{N-n_\beta+1}^{(k,\beta)} & \cdots & a_N^{(k,\beta)} \end{bmatrix}. \tag{6}$$

Also let

$$Q^* \cdot H = F_{(n_1,\cdots,n_\ell)} \tag{7}$$

where $Q^*$ is a matrix of size $\ell \times m$ and where the $\beta$-th row of $F_{(n_1,\cdots,n_\ell)}$ is the $n_1+\cdots+n_\beta$-th row of the $m \times m$ identity.

Taking transposes and substituting $H^T$ for $H$ shows that (5) and (7) are equivalent to (2) and (4). Since $H^T$ is also a mosaic Hankel matrix we have a second nonsingularity characterization.

**Theorem 2.2.** *$H$ is nonsingular if and only if there are solutions to equations (5) and (7).*

Theorems 2.1 and 2.2 both give necessary and sufficient conditions for the nonsingularity of a mosaic Hankel matrix. In addition, Theorem 2.3 below states that the solutions so constructed can actually be used to compute the inverse when it exists.

Let $E^{(i)}$ and $F^{(i)}$ denote the $i$-th column and row of the $m \times m$ identity matrix. Set

$$X^{(\beta)} = V^{(\beta)} + E^{(n_1+\cdots+n_\beta+1)} \quad \text{for} \quad 1 \leq \beta \leq \ell - 1, \quad X^{(\ell)} = V^{(\ell)}, \tag{8}$$

and

$$X^{(\alpha)*} = V^{(\alpha)*} + F^{(m_1+\cdots+m_\alpha+1)} \quad \text{for} \quad 1 \leq \alpha \leq k - 1, \quad X^{(k)*} = V^{(k)*} \tag{9}$$

where $V^{(\beta)}$ denotes the $\beta$-th column of a solution $V$ to (2) and $V^{(\alpha)*}$ denotes the $\alpha$-th row of a solution $V^*$ to (5).

**Theorem 2.3** *Suppose there are solutions $V$, $Q$, $V^*$ and $Q^*$ to equations (2), (4), (5) and (7), respectively. Let $X$ and $X^*$ be constructed from $V$ and $V^*$ as in (8) and (9), Then $H$ is nonsingular with inverse given by*

$$H^{-1} = \sum_{\alpha=1}^{k} \begin{bmatrix} x_{m-1}^{(\alpha)} & \cdots & x_1^{(\alpha)} & \delta_{\alpha,\ell} \\ \vdots & & & \\ x_1^{(\alpha)} & & & \\ \delta_{\alpha,\ell} & & & \end{bmatrix} \begin{bmatrix} q_m^{(\alpha)*} & \cdots & \cdots & q_1^{(\alpha)*} \\ & \ddots & & \vdots \\ & & \ddots & \vdots \\ & & & q_m^{(\alpha)*} \end{bmatrix}$$

$$- \sum_{\beta=1}^{\ell} \begin{bmatrix} q_{m-1}^{(\beta)} & \cdots & q_1^{(\beta)} & 0 \\ \vdots & & & \\ q_1^{(\beta)} & & & \\ 0 & & & \end{bmatrix} \begin{bmatrix} x_m^{(\beta)*} & \cdots & \cdots & x_1^{(\beta)*} \\ & \ddots & & \vdots \\ & & \ddots & \vdots \\ & & & x_m^{(\beta)*} \end{bmatrix}. \tag{10}$$

*Here $[q_m^{(\alpha)}, \ldots, q_1^{(\alpha)}]^T$ and $[q_m^{(\beta)*}, \ldots, q_1^{(\beta)*}]$ denote the $\alpha$-th column and $\beta$-th row of $Q$ and $Q^*$, respectively, and $[x_m^{(\alpha)}, \ldots, x_1^{(\alpha)}]^T$ and $[x_m^{(\beta)*}, \ldots, x_1^{(\beta)*}]$ denote the $\alpha$-th column and $\beta$-th row of $X$ and $X^*$, respectively.*

*Proof:* Theorem 2.3 follows directly from the Bezoutian representation of the inverse of a mosaic Hankel matrix given in Theorem 2.1 of Heinig and Tewodros [18]. $\square$

**Remark 1.** In the special case of layered or striped matrices these results follow from the work of Lerer and Tismenetsky [25]. The original results in the case of scalar Hankel matrices are due to Heinig and Rost [20].

**Remark 2.** Let $R$ be an $m \times m$ rectangular-block Hankel matrix with blocks of size $r \times s$ with $m = k \cdot r = \ell \cdot s$. Then $H = P \cdot R \cdot Q$ is a mosaic Hankel matrix where $P$ and $Q$ are permutation matrices such that the $i + (j-1)s$-th row of $P$ is the $j + (i-1)r$-th row of the identity and the $j+(i-1)s$-th column of $Q$ is the $i+(j-1)r$-th column of the identity matrix. The formulation of Theorems 2.1-2.3 for these matrices first appeared in Gohberg and Shalom [12] (see also [22]).

**Remark 3.** Since the existence of solutions to both (2) and (4) implies that $H$ is nonsingular, it is clear that the solutions are unique. Similarly, the existence of solutions for both (5) and (7) implies that the solutions are unique.

**Remark 4.** When the $a_N^{(\alpha,\beta)}$ are zero, rather than arbitrary, the results of Theorem 2.3 follow directly from the rank decomposition of the matrix $H$. However, formula (10) of Theorem 2.3 holds even in the case when arbitrary choices $a_N^{(\alpha,\beta)}$ are non-zero. Additional inversion formulas in the latter case can also be found in [23] (for example, the inverse can be expressed in terms of sums of products of factor-circulants).

**Example 2.4.** Let $H$ be the $4\times 4$ mosaic Hankel matrix having 3 layers and 2 stripes given by

$$H = \left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ 2 & 3 & 0 & 1 \\ \hline -1 & -2 & 1 & 1 \\ \hline 3 & 4 & 2 & 0 \end{array}\right]$$

For ease of presentation we will assume that the entries of $H$ are from the field $Z_{19}$ of integers modulo 19. This allows us to limit the growth of the numbers appearing in our examples to at most single digits. Assuming for this example that $N=3$ and that the arbitrary constants $a_3^{(\alpha,\beta)}$ are all 0, the inversion components (on the right) are given by

$$V = \begin{bmatrix} 1 & -6 \\ -2 & -7 \\ -7 & 4 \\ 4 & -5 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} 1 & -1 & -9 \\ 9 & -9 & -5 \\ 9 & -9 & 5 \\ -9 & -9 & -5 \end{bmatrix}$$

while the inverse components (on the left) are

$$V^* = \begin{bmatrix} 5 & 7 & -7 & 3 \\ -3 & -7 & 7 & -4 \\ -5 & -4 & 4 & -2 \end{bmatrix} \quad \text{and} \quad Q^* = \begin{bmatrix} 7 & 9 & -9 & -5 \\ 5 & -9 & -9 & -5 \end{bmatrix}.$$

Formula (10) then gives the inverse of $H$ to be

$$\begin{bmatrix} -2 & -6 & 4 & 0 \\ -6 & 4 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 7 & 9 & -9 & -5 \\ 0 & 7 & 9 & -9 \\ 0 & 0 & 7 & 9 \\ 0 & 0 & 0 & 7 \end{bmatrix} + \begin{bmatrix} -7 & 4 & -5 & 1 \\ 4 & -5 & 1 & 0 \\ -5 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 5 & -9 & -9 & -5 \\ 0 & 5 & -9 & -9 \\ 0 & 0 & 5 & -9 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$- \begin{bmatrix} 9 & 9 & -9 & 0 \\ 9 & -9 & 0 & 0 \\ -9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 5 & 7 & -6 & 3 \\ 0 & 5 & 7 & -6 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} -9 & -9 & -9 & 0 \\ -9 & -9 & 0 & 0 \\ -9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -3 & -7 & 7 & -3 \\ 0 & -3 & -7 & 7 \\ 0 & 0 & -3 & -7 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$$- \begin{bmatrix} -5 & 5 & -5 & 0 \\ 5 & -5 & 0 & 0 \\ -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} -5 & -4 & 4 & -2 \\ 0 & -5 & -4 & 4 \\ 0 & 0 & -5 & -4 \\ 0 & 0 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 6 & 1 & -1 & -9 \\ 7 & 9 & -9 & -5 \\ -4 & 9 & -9 & 5 \\ 5 & -9 & -9 & -5 \end{bmatrix}.$$

□

# 3 Polynomial Matrix Forms

In the special case of a Hankel matrix ($k = \ell = 1$, $m_\alpha = n_\beta = n, N = m + n$),

$$H = \begin{bmatrix} a_{m-n+1}^{(1,1)} & \cdots & a_m^{(1,1)} \\ \vdots & & \vdots \\ a_m^{(1,1)} & \cdots & a_{m+n-1}^{(1,1)} \end{bmatrix}, \tag{11}$$

(2) is called the Yule-Walker equation. When $H$ is nonsingular, it is well known that the solution of (2) defines the denominator for the Padé fraction of type $(m, n)$ for the power

series $A(z) = \sum_{i=0}^{\infty} a_i^{(1,1)} \cdot z^i$. That is, it defines a polynomial $V(z)$ of degree at most $n$ and from this a polynomial $U(z)$ of degree at most $m$ such that

$$A(z) = \frac{U(z)}{V(z)} + O(z^{m+n+1}).$$

In the Hankel case solutions to equation (4) also define certain Padé approximants. These simple observations have been very useful in constructing efficient algorithms for Hankel matrix inversion. Indeed Padé approximation was one of the main tools used by Brent, Gustavson and Yun [6] to obtain the first superfast algorithm for computing inverses of Hankel matrices.

Let us define a formal $k \times \ell$ matrix power series $A(z)$ being associated to the Hankel mosaic matrix $H = H(\vec{m}, \vec{n}, N-1)$ by

$$A(z) = \sum_{r=0}^{\infty} A_r z^r \quad \text{with} \quad A_r = H(\vec{e}, \vec{e}, r) = \begin{bmatrix} a_r^{(1,1)} & \cdots & a_r^{(1,\ell)} \\ \vdots & & \vdots \\ a_r^{(k,1)} & \cdots & a_r^{(k,\ell)} \end{bmatrix} \quad \text{for } r \leq N-1. \tag{12}$$

We may assume that $a_r^{(\alpha,\beta)} = 0$ for $r \leq N - m_\alpha - n_\beta$. As already shown for the special case of layered, striped and block Hankel matrices [21, 22], solutions of the fundamental equations (2), (4), (5) and (7) are also closely connected to the denominators of matrix-type Padé approximants for the matrix power series $A(z)$. In order to specify the correspondence, define for a vector $\vec{n} = (n_1, .., n_\ell)$ of integers (with each $n_i \geq -1$) the matrix

$$\Pi_{\vec{n}}(z) := \left[ \begin{array}{cccc|c|cccc} z^{n_1} & \cdots & z^1 & 1 & & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & & \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots & \cdots & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & & z^{n_\ell} & \cdots & z^1 & 1 \end{array} \right]$$

Then, for any matrix $Q$ for which $H(\vec{m}, \vec{n} + \vec{e}, N) \cdot Q$ is well-defined, it is straightforward to see that $Q(z) := \Pi_{\vec{n}}(z) \cdot Q$ gives a matrix polynomial with row degree rdeg $Q(z) \leq \vec{n}$, and that the $i$-th row of the $\alpha$-th block of $H(\vec{m}, \vec{n} + \vec{e}, N) \cdot Q$, $1 \leq i \leq m_\alpha$, $1 \leq \alpha \leq k$, gives the coefficient of $z^{N - m_\alpha + i}$ in the $\alpha$-th component of the product $A(z) \cdot Q(z)$. This motivates the following approximation problem.

**Definition 3.1.** *Let $A(z)$ be a matrix polynomial satisfying equation (12) and let $\vec{m} = (m_1, .., m_k)$, $\vec{n} = (n_1, .., n_\ell)$ be vectors of integers with each $m_\alpha \geq -1$ and each*

*$n_\beta \geq -1$. Define $\delta := \delta(\vec{m}, \vec{n}) := |\vec{n}| + \ell - |\vec{m}| \geq 1$, $|\vec{m}| := m_1 + .. + m_k$, $|\vec{n}| := n_1 + .. + n_\ell$. A pair of matrix polynomials $(P(z), Q(z))$ of size $k \times \delta$ and $\ell \times \delta$, respectively, is called a right hand approximant of type $(\vec{m}, \vec{n}, N)$ with respect to $A(z)$ if*

*I) rdeg $P(z) \leq N \cdot \vec{e} - \vec{m}$, rdeg $Q(z) \leq \vec{n}$, and*

*II) $A(z) \cdot Q(z) - P(z) = z^{N+1} R(z)$ with $R(z)$ a power series (called the* residual*).*

*A pair of matrix polynomials $(P^*(z), Q^*(z))$ is called a left hand approximant of type $(\vec{m}, \vec{n}, N)$ with respect to $A(z)$ if $(P^*(z)^T, Q^*(z)^T)$ is a right hand approximant of type $(\vec{n}, \vec{m}, N)$ with respect to $A(z)^T$.* □

Of course, a left hand approximant satisfies degree constraints with respect to the degree of the columns, i.e., cdeg $P^*(z) \leq N \cdot \vec{e} - \vec{n}$, cdeg $Q^*(z) \leq \vec{m}$. Note that the existence of a nontrivial approximant follows immediately from comparing the number of coefficients and unknowns in the system $H(\vec{m}, \vec{n} + \vec{e}, N) \cdot Q = 0$ of linear homogeneous equations. Moreover, we may add the canonical condition that the columns of $Q$ have to be linearly independent.

Definition 3.1 contains several well-known approximation problems such as simultaneous Padé forms ($\ell = \delta = 1$) and Hermite-Padé approximants ($k = \delta = 1$) for a vector of power series (cf. [21]). When all the $m_\alpha$ and $n_\beta$ are equal and $\delta = \ell$, the matrix $H$ is a block Hankel matrix and we obtain right matrix Padé forms. The term form rather than fraction is used to signify that $P(z)$ and $Q(z)$ may have a common factor (or, equivalently, $Q(0)$ is singular), and hence the rational form $P(z) \cdot Q(z)^{-1}$ may not satisfy the full order condition (for the scalar case, cf. Gragg [15]). Consequently, we are especially interested in approximants satisfying further classical normalized properties where also the square Hankel mosaic matrix $H = H(\vec{m}, \vec{n}, N-1)$ is involved. Note that, for these particular vectors of nonnegative integers $\vec{m}$, $\vec{n}$, we have $\delta(\vec{m}, \vec{n}) = \ell$, $\delta(\vec{m} - \vec{e}, \vec{n} - \vec{e}) = k$, $\delta(\vec{n}, \vec{m}) = k$ and $\delta(\vec{n} - \vec{e}, \vec{m} - \vec{e}) = \ell$.

**Theorem 3.2.** *There exists a right hand approximant $(U(z), V(z))$ of type $(\vec{m}, \vec{n}, N)$ with* normalized *denominator, (i.e., $V(0) = I_\ell$, the $\ell \times \ell$ identity matrix) with residual $W(z)$ if and only if there exists a solution of (2). In addition, there exists a right hand approximant $(P(z), Q(z))$ of type $(\vec{m} - \vec{e}, \vec{n} - \vec{e}, N-2)$ with a* normalized *residual $R(z)$ (i.e.,*

$R(0) = I_k$), *if and only if there exists a solution to equation (4). Similarly, there exists a left hand approximant* $(U^*(z), V^*(z))$ *of type* $(\vec{m}, \vec{n}, N)$ *with* normalized *denominator, (i.e.,* $V^*(0) = I_k$) *with residual* $W^*(z)$ *if and only if there exists a solution to equation (5). Finally, there exists a left hand approximant* $(P^*(z), Q^*(z))$ *of type* $(\vec{m} - \vec{e}, \vec{n} - \vec{e}, N - 2)$ *with residual* $R^*(z)$ *which is* normalized, *(i.e.,* $R^*(0) = I_\ell$) *if and only if there exists a solution of (7).*

*Proof:* In order to obtain the denominator $V(z)$ of an right hand approximant $(U(z), V(z))$ of type $(\vec{m}, \vec{n}, N)$, we use the scalar matrices $V, V_0, V_1$ defined by $V(z) = \Pi_{\vec{n}}(z) \cdot V = V_0 + z \cdot \Pi_{\vec{n}-\vec{e}}(z) \cdot V_1$ and solve the system of linear equations

$$H(\vec{m}, \vec{n} + \vec{e}, N) \cdot V = H(\vec{m}, \vec{n}, N - 1) \cdot V_1 + H(\vec{m}, \vec{e}, N) \cdot V_0 = 0.$$

Consequently, an approximant with normalized denominator exists if and only if the above system with $V_0 = V(0) = I_\ell$ has a solution which is identical to (2). For the second equivalence, notice that a right hand denominator $Q(z) =: \Pi_{\vec{n}-\vec{e}}(z) \cdot Q$ of type $(\vec{m} - \vec{e}, \vec{n} - \vec{e}, N - 2)$ with corresponding residual $R$ satisfies

$$H(\vec{m} - \vec{e}, \vec{n}, N - 2) \cdot Q = 0 \quad \text{and} \quad H(\vec{m}, \vec{n}, N - 1) \cdot Q = R(0).$$

The last two equivalences follow immediately from the fact that $A(z)^T$ is a matrix power series associated with the Hankel mosaic matrix $H^T = H(\vec{m}, \vec{n}, N - 1)^T$. $\square$

**Remark.** Theorem 3.2 combined with Theorem 2.1 implies that the mosaic Hankel matrix $H = H(\vec{m}, \vec{n}, N - 1)$ is nonsingular if and only if there are matrix polynomials $U(z)$, $V(z)$, $P(z)$ and $Q(z)$ satisfying the properties of Theorem 3.2. Also, as mentioned in the previous section, the corresponding matrix polynomials are unique when they exist. Similar statements are also true for the left hand counterparts.

**Example 3.3.** Let $H$ be the matrix from Example 2.4 with $N = \max(m_\alpha) + \max(n_\beta) - 1 = 3$. Then one matrix power series $A(z)$ satisfying (12) is given by

$$A(z) = \begin{bmatrix} 1 + 2\,z + 3\,z^2 & z^2 \\ -z - 2\,z^2 & z + z^2 \\ 3\,z + 4\,z^2 & 2\,z \end{bmatrix} + O(z^4)$$

(so that the arbitrary elements $a_3^{(i,j)}$ are all 0). In this case Example 2.4 and the constructions in the proof of Theorem 3.2 gives

$$U(z) = \begin{bmatrix} 1 & -7\,z \\ -z + 4\,z^2 & z + 3\,z^2 \\ 3\,z + 6\,z^2 & 2\,z + 7\,z^2 \end{bmatrix}, \quad V(z) = \begin{bmatrix} 1 - 2\,z + z^2 & -7\,z - 6\,z^2 \\ 4\,z - 7\,z^2 & 1 - 5\,z + 4\,z^2 \end{bmatrix},$$

$$P(z) = \begin{bmatrix} 9 & -9 & -5 \\ z & 0 & 0 \\ 9\,z & -7\,z & -6\,z \end{bmatrix} \text{ and } Q(z) = \begin{bmatrix} 9 + z & -9 - z & -5 - 9\,z \\ -9 + 9\,z & -9 - 9\,z & -5 + 5\,z \end{bmatrix}.$$

The residuals are

$$W(z) = \begin{bmatrix} -4 & 5 \\ -9 & -3 \\ 4 & -5 \end{bmatrix} + O(z) \text{ and } R(z) = \begin{bmatrix} 1 - 7\,z & 7\,z & -3\,z \\ 7\,z & 1 - 7\,z & 4\,z \\ 4\,z & -4\,z & 1 + 2\,z \end{bmatrix} + O(z^2).$$

On the left side,

$$U^*(z) = \begin{bmatrix} 1 + 9\,z & 0 \\ -8\,z & z \\ -z & 2\,z \end{bmatrix} \text{ and } V^*(z) \begin{bmatrix} 1 + 7\,z + 5\,z^2 & -7\,z & 3\,z \\ -7\,z - 3\,z^2 & 1 + 7\,z & -4\,z \\ -4\,z - 5\,z^2 & 4\,z & 1 - 2\,z \end{bmatrix}.$$

Similarly,

$$P^*(z) = \begin{bmatrix} 9 & 0 \\ -9 & 0 \end{bmatrix} \text{ and } Q^*(z) = \begin{bmatrix} 9 + 7\,z & -9 & -5 \\ -9 + 5\,z & -9 & -5 \end{bmatrix}$$

with

$$W^*(z) = \begin{bmatrix} -4 & 5 \\ -9 & -3 \\ 4 & -5 \end{bmatrix} + O(z) \text{ and } R^*(z) = \begin{bmatrix} 1 + 2\,z & 7\,z \\ -4\,z & 1 + 5\,z \end{bmatrix} + O(z^2).$$

□

# 4 Matrix Polynomial Systems

In this section, the matrix polynomials from Theorem 3.2 are combined into two $(k + \ell) \times (k + \ell)$ matrix polynomials $\mathcal{V}(z)$ and $\mathcal{W}(z)$ associated to solutions of (2), (4), (5) and (7). These matrices are then shown to be closely related to the $V$ and $W$ matrices of Antoulas [2], used as the main tools in the study of recursiveness in linear system theory. For special cases such as the case of striped or layered Hankel matrices, these matrix polynomials correspond to various Padé-like systems ([8], [9], [21]). The motivation and approach used closely follows from similar results given in [21] and [22].

Set

$$\mathcal{V}(z) = \begin{bmatrix} -zP(z) & U(z) \\ -zQ(z) & V(z) \end{bmatrix} \quad \text{and} \quad \mathcal{W}(z) = \begin{bmatrix} V^*(z) & -U^*(z) \\ zQ^*(z) & -zP^*(z) \end{bmatrix}, \tag{13}$$

matrix polynomials of size $(k + \ell) \times (k + \ell)$. Note that in terms of rows

$$\text{rdeg}(\mathcal{V}(z)) \leq [N - m_1, \cdots, N - m_k, n_1, \cdots, n_\ell]^T \tag{14}$$

and in terms of columns

$$\text{cdeg}(\mathcal{W}(z)) \leq [m_1, \cdots, m_k, N - n_1, \cdots, N - n_\ell]. \tag{15}$$

In addition, these matrix polynomials satisfy the order conditions

$$[I, \ - A(z)] \cdot \mathcal{V}(z) = z^N [R(z), \ - zW(z)] \tag{16}$$

and

$$\mathcal{W}(z) \cdot \begin{bmatrix} A(z) \\ I \end{bmatrix} = z^N \begin{bmatrix} zW^*(z) \\ R^*(z) \end{bmatrix} \tag{17}$$

for any matrix power series $A(z)$ satisfying (12).

As mentioned in the previous section, it is possible to define the $\mathcal{V}$ and $\mathcal{W}$ matrices satisfying the degree constraints (14) and (15) along with the order conditions (16) and (17) for all mosaic Hankel matrices. However, if these matrix polynomials also satisfy the normalization conditions

$$V(0) = R^*(0) = I_\ell \ \text{ and } \ V^*(0) = R(0) = I_k, \tag{18}$$

then the $H$ is nonsingular and the $\mathcal{V}$ and $\mathcal{W}$ matrices are unique.

**Theorem 4.1.** *Let $\mathcal{V}(z)$ and $\mathcal{W}(z)$ be as above. Then*

$$\mathcal{W}(z) \cdot \mathcal{V}(z) = z^N \cdot I_{k+\ell}, \tag{19}$$

$$\mathcal{V}(z) \cdot \mathcal{W}(z) = z^N \cdot I_{k+\ell}. \tag{20}$$

*Furthermore, the residuals satisfy $W(z) \cdot R^*(z) = R(z) \cdot W^*(z)$.*

*Proof:* Let

$$\mathcal{A}(z) = \begin{bmatrix} I & -A(z) \\ 0 & -I \end{bmatrix}, \tag{21}$$

where $A(z)$ is a matrix power series satisfying (12). Then, $\mathcal{A}(z)$ is its own inverse, and from (16), (17)

$$\mathcal{A}(z) \cdot \mathcal{V}(z) = \begin{bmatrix} z^N R(z) & -z^{N+1} W(z) \\ zQ(z) & -V(z) \end{bmatrix}, \quad \mathcal{W}(z) \cdot \mathcal{A}(z) = \begin{bmatrix} V^*(z) & z^{N+1} W^*(z) \\ zQ^*(z) & -z^N R^*(z) \end{bmatrix}.$$

Therefore,

$$\begin{aligned} \mathcal{W}(z) \cdot \mathcal{V}(z) &= \mathcal{W}(z) \cdot \mathcal{A}(z) \cdot \mathcal{A}(z) \cdot \mathcal{V}(z) \\ &= \begin{bmatrix} V^*(z) & z^{N+1} W^*(z) \\ zQ^*(z) & -z^N R^*(z) \end{bmatrix} \cdot \begin{bmatrix} z^N R(z) & -z^{N+1} W(z) \\ zQ(z) & -V(z) \end{bmatrix} \\ &= z^N \begin{bmatrix} V^*(z) & zW^*(z) \\ zQ^*(z) & -R^*(z) \end{bmatrix} \cdot \begin{bmatrix} R(z) & -zW(z) \\ zQ(z) & -V(z) \end{bmatrix}. \end{aligned}$$

But, from (14) and (15), the component-wise bounds for the degrees of $\mathcal{W}(z) \cdot \mathcal{V}(z)$ are given by

$$\text{degree}(\mathcal{W}(z) \cdot \mathcal{V}(z)) \leq N. \tag{22}$$

It then follows that

$$\mathcal{W}(z) \cdot \mathcal{V}(z) = z^N \left[ \begin{array}{c|c} V^*(0)R(0) & 0 \\ \hline 0 & R^*(0)V(0) \end{array} \right] = z^N I_{k+\ell}, \tag{23}$$

which is (19). To obtain (20), multiply both sides of (19) on the left by $\mathcal{V}(z)$ and on the right by its inverse. Finally, the last assertion follows from the identity

$$z^N \cdot I_{k+\ell} = \mathcal{A}(z) \cdot \mathcal{V}(z) \cdot \mathcal{W}(z) \cdot \mathcal{A}(z) = \begin{bmatrix} z^N R(z) & -z^{N+1} W(z) \\ zQ(z) & -V(z) \end{bmatrix} \cdot \begin{bmatrix} V^*(z) & z^{N+1} W^*(z) \\ zQ^*(z) & -z^N R^*(z) \end{bmatrix}.$$

$\square$

Set

$$\begin{aligned} \overline{\mathcal{V}}(z) &= diag\,(z^{N-m_1}, \cdots, z^{N-m_k}, z^{n_1}, \cdots, z^{n_\ell}) \cdot \mathcal{V}(z^{-1}) \\ \overline{\mathcal{W}}(z) &= \mathcal{W}(z^{-1}) \cdot diag\,(z^{m_1}, \cdots, z^{m_k}, z^{N-n_1}, \cdots, z^{N-n_\ell}). \end{aligned} \tag{24}$$

Thus $\overline{\mathcal{V}}(z)$ reverses the order of the coefficients of the matrix polynomial $\mathcal{V}(z)$ (on a per row basis), while $\overline{\mathcal{W}}(z)$ reverses the order of the coefficients of the matrix polynomial $\mathcal{W}(z)$ (on a per column basis). Let us use the same partition of $\overline{\mathcal{V}}(z)$ and $\overline{\mathcal{W}}(z)$ as in (13), namely,

$$\overline{\mathcal{V}}(z) = \begin{bmatrix} -\overline{P}(z) & \overline{U}(z) \\ -\overline{Q}(z) & \overline{V}(z) \end{bmatrix} \text{ and } \overline{\mathcal{W}}(z) = \begin{bmatrix} \overline{V}^*(z) & -\overline{U}^*(z) \\ \overline{Q}^*(z) & -\overline{P}^*(z) \end{bmatrix}.$$

Moreover, due to the condition $a_r^{(\alpha,\beta)} = 0$ for $r \leq N - m_\alpha - n_\beta$ on the initial coefficients of our power series $A$, we may define

$$\overline{A}(z^{-1}) := z^N \cdot diag\,(z^{-m_1}, \cdots, z^{-m_k}) \cdot A(z) \cdot diag\,(z^{-n_1}, \cdots, z^{-n_\ell})$$

and notice that $\overline{A}$ is a stictly proper power series in $z^{-1}$, i.e., $\overline{A}(z) = \overline{A}_1 z^{-1} + \overline{A} z^{-2} + \ldots$. Note that the entry $\overline{a}_k^{(\alpha,\beta)}$ of $\overline{A}_k$ is obtained by indexing the mosaic Hankel matrix such that the top left hand corner of each Hankel submatrix has index 1.

Theorem 4.1 then gives:

**Corollary 4.2.** *$\overline{\mathcal{V}}(z)$ and $\overline{\mathcal{W}}(z)$ are unimodular matrix polynomials which are inverses of each other. They induce the following partial realization for the strictly proper power series $\overline{A}$*

$$\begin{aligned} &\overline{U}(z) \cdot \overline{V}(z)^{-1} = \overline{V}^*(z)^{-1} \cdot \overline{U}^*(z) \\ &= \overline{A}(z) + z^{-1} \cdot \text{diag}\,(z^{-m_1}, \cdots, z^{-m_k}) \cdot R(z^{-1}) \cdot \text{diag}\,(z^{-n_1}, \cdots, z^{-n_\ell}) \end{aligned} \tag{25}$$

*with a power series $R$. In the special case of rectangular block Hankel matrices, i.e., $m_1 = \cdots = m_k$ and $n_1 = \cdots = n_\ell$, the matrices $\overline{\mathcal{V}}$ and $\overline{\mathcal{W}}$ reduce to the $V$ and $W$ matrices of Antoulas [2].*

*Proof:* Because of the degree constraints (14) and (15), it is clear that both $\overline{\mathcal{V}}(z)$

and $\overline{\mathcal{W}}(z)$ are indeed matrix polynomials. ¿From (19) and (20) of Theorem 4.1, we have

$$\overline{\mathcal{V}}(z) \cdot \overline{\mathcal{W}}(z) = I_{k+\ell} \;\text{ and }\; \overline{\mathcal{W}}(z) \cdot \overline{\mathcal{V}}(z) = I_{k+\ell}.$$

Hence, the two matrix polynomials are both unimodular and inverses of each other. Note also that

$$\overline{U}(z) \cdot \overline{V}(z)^{-1} = \overline{V}^*(z)^{-1} \cdot \overline{U}^*(z) \quad \text{ and } \quad \overline{V}(z)^{-1} \cdot \overline{Q}(z) = \overline{Q}^*(z) \cdot \overline{V}^*(z)^{-1}.$$

Moreover, since $U(z) \cdot V(z)^{-1} = A(z) + z^{N+1} \cdot R(z)$ with a power series $R$, we have

$$\begin{aligned} \overline{U}(z) \cdot \overline{V}(z)^{-1} &= \mathit{diag}\,(z^{N-m_1}, \cdots, z^{N-m_k}) \cdot U(z^{-1}) \cdot V(z^{-1})^{-1} \cdot \mathit{diag}\,(z^{-n_1}, \cdots, z^{-n_\ell}) \\ &= \mathit{diag}\,(z^{N-m_1}, \cdots, z^{N-m_k}) \cdot A(z^{-1}) \cdot \mathit{diag}\,(z^{-n_1}, \cdots, z^{-n_\ell}) \\ &\quad + z^{-1} \cdot \mathit{diag}\,(z^{-m_1}, \cdots, z^{-m_k}) \cdot R(z^{-1}) \cdot \mathit{diag}\,(z^{-n_1}, \cdots, z^{-n_\ell}) \\ &= \overline{A}(z) + z^{-1} \cdot \mathit{diag}\,(z^{-m_1}, \cdots, z^{-m_k}) \cdot R(z^{-1}) \cdot \mathit{diag}\,(z^{-n_1}, \cdots, z^{-n_\ell}) \end{aligned}$$

giving the desired approximation properties. In order to get the equality to Antoulas' matrices [2, p.1123], it remains to show that the expression $\overline{V}(z)^{-1} \cdot \overline{Q}(z)$ is strictly proper rational. But this follows from

$$\bar{V}(z)^{-1} \cdot \bar{Q}(z) = z^{-1} \cdot V(z^{-1})^{-1} \cdot Q(z^{-1}) \quad \text{and} \quad \bar{Q}^*(z) \cdot \bar{V}^*(z)^{-1} = z^{-1} Q^*(z^{-1}) V^*(z^{-1})^{-1}.$$

□

**Remark 1.** In the case of square-block matrices, Theorem 4.1 first appeared in [24]. In that paper the relations (19) and (20) were used as commutativity relationships between certain left and right matrix Padé approximants (note that (19) in particular implies that the Padé type fractions $U(z) \cdot V(z)^{-1}$ and $V^*(z)^{-1} \cdot U^*(z)$ are equal). These relationships in turn were used to develop inversion formulas for square-block Hankel matrices. Similar identities in the case of layered or striped block Hankel matrices were shown in [21] to be a matrix generalization of fundamental duality identities of Mahler [26] between simultaneous Padé and Hermite-Padé approximants. In the present setting Theorem 4.1 generalizes the striped and layered cases found in [21] and the rectangular-block case found in [22].

**Remark 2.** In the scalar Hankel matrix case relationships (19) and (20) were used in [10] as tools in proving that the Cabay-Meleshko Padé algorithm was (weakly) stable over floating point arithmetic.

**Remark 3.** It can also be shown that the matrix $\mathcal{V}(z)$ (and similarly the matrix $\mathcal{W}(z)$ for the transposed problem) forms a particular sigma basis for a suitable power Hermite Padé approximation problem in the sense of [4, 5], but satisfying more refined degree and normalization constraints.

**Remark 4.** For solving a generalized partial realization problem as given in the assertion of Corollary 4.2, one looks for an equivalent linearized problem. In extension to the considerations given in [2, p.1124], we should suppose for the rational expression $\overline{U}(z) \cdot \overline{V}(z)^{-1}$ that $diag\,(z^{-n_1}, .., z^{-n_\ell}) \cdot \overline{V}(z)$ is column reduced with column degree $\vec{\kappa}$, since this leads to the equivalent condition

$$\overline{U}(z) = \overline{A}(z) \cdot \overline{V}(z) + z^{-1} \cdot diag\,(z^{-m_1}, \cdots, z^{-m_k}) \cdot R'(z^{-1}) \cdot diag\,(z^{\kappa_1}, \cdots, z^{\kappa_\ell})$$

with a power series $R'$, and we are dealing with minimal partial realizations if the numbers $\kappa_j$ (also called *Kronecker indices*) are as small as possible.

# 5 Recursive Computation of Mosaic Inverses

Identification of the inversion components in terms of matrix-type Padé approximants provide efficient and reliable algorithms for the components, for example using the algorithms of [5] or [9]. In this section we describe an alternate method of computing the inversion components in terms of the inversion components of submatrices of a mosaic Hankel matrix. When viewed as computing minimal partial realizations, the recursion is similar to the recursion used by Antoulas.

Algorithms for computing the inverses of structured matrices are often described in terms of computing inverses of submatrices along a given computational path. For example, in the $k = \ell = 1$ case of a Hankel matrix (11) the work of [7] combined with [24] gives an algorithm to invert $H$ by constructing the inverse components of the nonsingular principal submatrices

$$H_{(i)} = \begin{bmatrix} a^{(1,1)}_{m-n+1} & \cdots & a^{(1,1)}_{m-i} \\ \vdots & & \vdots \\ a^{(1,1)}_{m-i} & \cdots & a^{(1,1)}_{m+n-2i-1} \end{bmatrix},$$

that is, the nonsingular Hankel submatrices along the diagonal. In this section we describe a recursion in terms of nonsingular "principal mosaic Hankel" submatrices along the diagonal.

For simplicity of presentation, we use the following description of a diagonal path passing through a vector. For an arbitrary vector $\vec{v} = (v_1, \cdots, v_s)$ of integers and $t$ an integer, we define a vector $\vec{v}(t) = (v_1(t), \cdots, v_s(t))$ with $v_i(t) = \max\{0, v_i - t\}$. The vectors $\vec{v}(t)$ describe an $s$-dimensional "diagonal" line passing through $v$.

Let $H$ be a mosaic Hankel matrix determined by the integer vectors $\vec{m}$ and $\vec{n}$ and indexed as in (2) using $N = \max(\vec{m}) + \max(\vec{n}) - 1$, i.e., $H = H(\vec{m}, \vec{n}, N-1)$. Let $\sigma$ and $\tau$ be two integers such that $|\vec{m}(\sigma)| = |\vec{n}(\tau)|$ and let $H_{(\sigma,\tau)}$ be an abbreviation for the mosaic Hankel submatrix $H(\vec{m}(\sigma), \vec{n}(\tau), \hat{N})$ of $H$ with $\hat{N} := \max(\vec{m}(\sigma)) + \max(\vec{n}(\tau)) - 1 = N - \sigma - \tau$ (for example, $H = H_{(0,0)}$). The matrix $H_{(\sigma,\tau)}$ matrix is what we refer to as a "principal mosaic Hankel" submatrix of $H$. In fact, $H_{(\sigma,\tau)}$ is build up by taking principal submatrices of suitable size from each block of $H$.

Suppose now that $H_{(\sigma,\tau)}$ is nonsingular. Let $A(z)$ be a matrix power series satisfying (12) for the original matrix $H$. Then $A(z)$ also satisfies (12) for $H_{(\sigma,\tau)}$. Because $H_{(\sigma,\tau)}$ is nonsingular, there are matrix polynomials $\hat{P}(z)$, $\hat{Q}(z)$, $\hat{U}(z)$ and $\hat{V}(z)$ along with residual power series $\hat{R}(z)$ and $\hat{W}(z)$ satisfying the conditions for right hand approximants of Theorem 3.2 with $A(z)$. By Theorem 3.2, we have that $\hat{R}(0) = I_k$ and so we can form the matrix power series

$$A^{\#}(z) = z \cdot \hat{R}^{-1}(z) \cdot \hat{W}(z).$$

Let $H^{\#}$ be the mosaic Hankel matrix of type $\vec{m} - \vec{m}(\sigma)$ and $\vec{n} - \vec{n}(\tau)$ associated with the first $N^{\#}$ terms of $A^{\#}(z)$, where $N^{\#} = \sigma + \tau = \max(\vec{m} - \vec{m}(\sigma)) + \max(\vec{n} - \vec{n}(\tau))$.

The recursion in this case is then given by

**Theorem 5.1.** *Suppose the principal mosaic submatrix $H_{(\sigma,\tau)}$ of $H$ is nonsingular. Then $H$ is nonsingular if and only if $H^{\#}$ is nonsingular. In this case the corresponding inversion components satisfy*

$$\mathcal{V}(z) = \hat{\mathcal{V}}(z) \cdot \mathcal{V}^{\#}(z), \tag{26}$$

*and*

$$\mathcal{W}(z) = \mathcal{W}^{\#}(z) \cdot \hat{\mathcal{W}}(z). \tag{27}$$

*Proof:* Let $\overline{A}(z) = [I, \ -A(z)]$. Then

$$\begin{aligned}
\overline{A}(z) \cdot \hat{\mathcal{V}}(z) \cdot \mathcal{V}^{\#}(z) &= z^{\hat{N}}[\hat{R}(z), \ -z\hat{W}(z)] \cdot \mathcal{V}^{\#}(z) \\
&= z^{\hat{N}}\hat{R}(z) \cdot [I, \ -A^{\#}(z)] \cdot \mathcal{V}^{\#}(z) \\
&= z^{\hat{N}+N^{\#}}\hat{R}(z) \cdot [R^{\#}(z), \ -zW^{\#}(z)] \\
&= z^{N}[R(z), \ -zW(z)]
\end{aligned}$$

where

$$R(z) = \hat{R}(z) \cdot R^{\#}(z) \ \text{ and } \ W(z) = \hat{R}(z) \cdot W^{\#}(z). \tag{28}$$

Thus, the order condition is satisfied.

The product also satisfies the correct degree constraints. Consider the degree of row $\alpha$ where $1 \leq \alpha \leq k$. Then the degree of row $\alpha$ of the product is bounded by

$$\hat{N} - m_{\alpha}(\sigma) + \max(rdeg(\mathcal{V}^{\#}(z)))$$

which is easily seen to be at most $N - m_{\alpha}$. For row $\beta$ where $k+1 \leq \beta \leq k+\ell$, if $n_{\beta}(\tau) > 0$ then row $\beta$ of the product is bounded by

$$n_{\beta} - \tau + \max(rdeg(\mathcal{V}^{\#}(z)))$$

which again is easily shown to be at most $n_{\beta}$. On the other hand, if $n_{\beta}(\tau) = 0$ then the $\beta$-th row of $\hat{\mathcal{V}}(z)$ is 0 except for a 1 in column $\beta$. Therefore the degree of row $\beta$ of the product is bounded by the degree of row $\beta$ of $\mathcal{V}^{\#}(z)$, that is, by $n_{\beta} - n_{\beta}(\tau) = n_{\beta}$. Hence, the degree bounds hold in all cases.

¿From (28) we see that $R(0) = I_k$ if and only if $R^{\#}(0) = I_k$. In addition, let $V(z)$ be the lower right $\ell \times \ell$ submatrix of $\hat{\mathcal{V}}(z) \cdot \mathcal{V}^{\#}(z)$. From the partitioning of both $\hat{\mathcal{V}}(z)$ and $\mathcal{V}^{\#}(z)$ as in (13), it follows that

$$V(z) = \hat{V}(z)V^{\#}(z) - z\hat{Q}(z)U^{\#}(z).$$

Hence, $V(0) = I_\ell$ if and only if $V^{\#}(0) = I_\ell$. Therefore the product is normalized as in (18) if and only if $\mathcal{V}^{\#}$ is normalized as in (18). Thus, $H$ is nonsingular if and only if $H^{\#}$ is nonsingular. In addition, in this case the $\mathcal{V}$ matrices are unique; hence (26) holds. Equation (27) follows from (26) and Theorem 4.1. □

**Remark 1.** When $k = \ell = 1$ (i.e. the Hankel case) the recursion computes all inverses of nonsingular principal submatrices. When $k = \ell$ and $m_\alpha = n_\beta$ for all $\alpha, \beta$ the algorithm computes (up to permutations of the rows and columns) the inverse of a block Hankel matrix. In this case, the recursion computes the inverses of all nonsingular principal block submatrices. The complexity of an algorithm based on this recursion is $O(m^2)$ (cf., [7]).

**Remark 2.** Theorem 5.1 shows that the recursion solves the inversion problem by solving the inversion problem for two smaller mosaic matrices. One can develop an efficient algorithm based only on the recursion in Theorem 5.1 (see [21] [22] for special cases) by using Gaussian elimination to compute the inversion components of the initial nonsingular principal mosaic Hankel submatrix and then recursively proceeding using the recursion. This is particularly useful when floating-point, rather than exact arithmetic is used [10], and one is looking for well-conditioned rather than nonsingular principal submatrices. The overhead for this is the generation of the residual matrix power series to a prescribed number of terms, the computation of the inversion components in each case and finally combining the two matrix polynomials together. The efficiency of this approach depends both on the number of nonsingular principal mosaic submatrices and the mosaic structure of the original matrix itself.

**Remark 3.** The cost of computing the quotient matrix power series $A^{\#}(z)$ to $K$ terms is $k^2 \cdot \ell \cdot K^2 + O(k \cdot \ell \cdot K)$ operations. On the other hand, from Theorem 4.1 we know that

$$\hat{R}^{-1}(z) \cdot \hat{W}(z) = \hat{W}^{*}(z) \cdot \hat{R}^{*-1}(z). \tag{29}$$

Hence, if $\ell < k$ then one can use the right side of (29) for computing $A^{\#}(z)$ at a cost of $k \cdot \ell^2 \cdot K^2 + O(k \cdot \ell \cdot K)$ operations.

**Remark 4.** As was pointed out by one of the reviewers, there are alterate forms for the recursion of Theorem 5.1. Note that the inversion components for a given mosaic Hankel matrix $H = H(\vec{m}, \vec{n}, N)$ depend on a given choice for the arbitrary coefficients in $a_N$. The recursion given in Theorem 5.1 is such that for any given step this arbitrary coefficient is determined by the matrix of power series $A(z)$ for the full mosaic Hankel matrix.

It is also possible to define alternate recursions for other specifications of the arbitrary coefficient $a_{\hat{N}}$ at a given recursive step. For example, suppose $H_{(\sigma,\tau)} = H(\vec{m}(\sigma), \vec{n}(\tau), \hat{N})$ is a nonsingular principal mosaic submatrix of $H$ and that $\hat{A}(z)$ is an associated matrix power series for $H_{(\sigma,\tau)}$ satisfying $\hat{a}_{\hat{N}} = 0$. Note that $\hat{A}(z)$ is determined solely from the elements of the principal mosaic submatrix rather than the entire mosaic Hankel matrix $H$ (as was used for the recursion in Theorem 5.1). Let $\hat{Q}(z)$, $\hat{V}(z)$, $\hat{R}(z)$ and $\hat{W}(z)$ be the (unique) matrix polynomials corresponding to the inversion components of $H_{(\sigma,\tau)}$ and $\hat{A}(z)$, and set

$$A^{\#}(z) = (\hat{R}(z) + z\tilde{A}(z)\hat{Q}(z))^{-1} \cdot (z\hat{W}(z) + \tilde{A}(z)\hat{V}(z))$$

with $\tilde{A}(z) = z^{-N} \cdot (A(z) - \hat{A}(z))$. Here $A(z)$ is the matrix power series for the entire matrix $H$. Then it can be shown that the recursion given in Theorem 5.1 also holds for the corresponding $H^{\#}$.

In this case it is always true that $\hat{a}_{\hat{N}} = 0$ at every step. This form of the recursion is useful when approaching the inversion problem using the notion of a rank decomposition of a structured matrix. Our approach is more natural when using the correspondence with the inversion problem and existing algorithms for rational approximation.

**Example 5.2.** Let $H$ be the $7 \times 7$ mosaic Hankel matrix having 3 layers of size $(5,1,1)$ and 2 stripes of size $(4,3)$ given by

$$H = \left[\begin{array}{cccc|ccc} 1 & 2 & 2 & -2 & 0 & 2 & 1 \\ 2 & 2 & -2 & 3 & 2 & 1 & -9 \\ 2 & -2 & 3 & 8 & 1 & -9 & -1 \\ -2 & 3 & 8 & -2 & -9 & -1 & 0 \\ 3 & 8 & -2 & -4 & -1 & 0 & -9 \\ \hline 1 & 3 & 8 & 2 & -7 & 8 & 1 \\ \hline 0 & 5 & -9 & 9 & -8 & 4 & 1 \end{array}\right].$$

As before, we assume that arithmetic is over the field $Z_{19}$. We use

$$A(z) = \left[\begin{array}{cc} 1+2\,z+2\,z^2-2\,z^3+3\,z^4+8\,z^5-2\,z^6-4\,z^7 & 2\,z^2+z^3-9\,z^4-z^5-9\,z^7 \\ z^4+3\,z^5+8\,z^6+2\,z^7 & -7\,z^5+8\,z^6+z^7 \\ 5\,z^5-9\,z^6+9\,z^7 & -8\,z^5+4\,z^6+z^7 \end{array}\right] + O(z^9)$$

as a power series that satisfies (12) for $H$. Let $\sigma = \tau = 2$. Then the principal mosaic Hankel submatrix of type $(3,0,0)$ and $(2,1)$ is

$$H_{(2,2)} = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 2 & 2 \\ 2 & -2 & 1 \end{array}\right]$$

which is nonsingular. The inverse components of $H_{(2,2)}$ are determined to be

$$\hat{\mathcal{V}}(z) = \left[\begin{array}{ccc|cc} 4\,z & 0 & 0 & 1-9\,z & -8\,z \\ 0 & z^4 & 0 & z^4 & 0 \\ 0 & 0 & z^4 & 0 & 0 \\ \hline 4\,z-8\,z^2 & 0 & 0 & 1+8\,z+z^2 & -8\,z-5\,z^2 \\ 4\,z & 0 & 0 & -8\,z & 1+3\,z \end{array}\right]$$

and

$$\hat{\mathcal{W}}(z) = \left[\begin{array}{ccc|cc} 1-8\,z-z^2+z^3 & 0 & 0 & -1+6\,z-4\,z^2 & -2\,z^2-4\,z^3 \\ 4\,z-2\,z^2-4\,z^3 & 1 & 0 & -4\,z-6\,z^2 & -8\,z^3 \\ 0 & 0 & 1 & 0 & 0 \\ \hline -4\,z+2\,z^2+4\,z^3 & 0 & 0 & 4\,z+6\,z^2 & 8\,z^3 \\ -4\,z-7\,z^2+3\,z^3 & 0 & 0 & 4\,z-4\,z^2 & 8\,z^3 \end{array}\right].$$

Note that these are computed with the arbitrary elements $a_4^{(\alpha,\beta)}$ given by

$$A_4 = \begin{bmatrix} -2 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Computing the first few terms of the residuals $\hat{R}(z)$ and $\hat{W}(z)$ using $A(z)$ with $\hat{\mathcal{V}}(z)$ and $\hat{\mathcal{W}}(z)$ gives

$$\hat{R}(z) = \begin{bmatrix} 1+8\,z-4\,z^2-4\,z^3-2\,z^4 & 0 & 0 \\ -4\,z+5\,z^2-2\,z^3-5\,z^4 & 1 & 0 \\ -7\,z^2+3\,z^3+2\,z^4 & 0 & 1 \end{bmatrix} + O(z^5)$$

and

$$\hat{W}(z) = \begin{bmatrix} 7-3\,z+7\,z^2-4\,z^4 & -4-6\,z+5\,z^2-4\,z^3+z^4 \\ -8-6\,z+5\,z^2-3\,z^3+2\,z^4 & 4-4\,z+3\,z^2+4\,z^3+9\,z^4 \\ 5+5\,z^2-2\,z^3+9\,z^4 & -8-3\,z+3\,z^2-5\,z^3-7\,z^4 \end{bmatrix} + O(z^5)$$

so that

$$A^{\#}(z) = \begin{bmatrix} 7\,z-2\,z^2-6\,z^3-8\,z^4 & -4\,z+7\,z^2+9\,z^3-7\,z^4 \\ -8\,z+3\,z^2-3\,z^4 & 4\,z-z^2-6\,z^3-3\,z^4 \\ 5\,z-3\,z^3+z^4 & -8\,z-3\,z^2-6\,z^3-z^4 \end{bmatrix} + O(z^5).$$

Note that the corresponding mosaic Hankel matrix of type $(2,1,1)$ and $(2,2)$ indexed by

$N^{\#} = 4$ is given by

$$H^{\#} = \left[\begin{array}{rr|rr} 7 & -2 & -4 & 7 \\ -2 & -6 & 7 & 9 \\ \hline 3 & 0 & -1 & -6 \\ \hline 0 & -3 & -3 & -6 \end{array}\right].$$

$H^{\#}$ is nonsingular with inverse components on the right (in matrix polynomial form) given by

$$\mathcal{V}^{\#}(z) = \left[\begin{array}{ccc|cc} -8\,z^2 & -4\,z^2 & -3\,z^2 & 7\,z - 7\,z^2 & -4\,z + 5\,z^2 \\ -7\,z^2 + 3\,z^3 & -2\,z^2 + 5\,z^3 & -6\,z^2 - 4\,z^3 & -8\,z - 2\,z^2 + 4\,z^3 & 4\,z + 9\,z^2 - 4\,z^3 \\ z^2 + 3\,z^3 & -5\,z^2 + 6\,z^3 & 8\,z^2 - 8\,z^3 & 5\,z - 5\,z^2 + 5\,z^3 & -8\,z + 8\,z^2 + 9\,z^3 \\ \hline -4\,z + z^2 & 6\,z - 6\,z^2 & 9\,z - 2\,z^2 & 1 - 9\,z + 8\,z^2 & -8\,z - 6\,z^2 \\ -5\,z - 5\,z^2 & 2\,z + 9\,z^2 & 7\,z + 9\,z^2 & -5\,z - 6\,z^2 & 1 - 4\,z + 3\,z^2 \end{array}\right]$$

whereas the inverse components on the left are given by

$$\mathcal{W}^{\#}(z) = \left[\begin{array}{ccc|cc} 1 - 3\,z + 9\,z^2 & -2\,z & -9\,z & -7\,z - 5\,z^2 & 4\,z - 7\,z^2 \\ 9\,z^2 & 1 - 2\,z & -7\,z & 8\,z - 3\,z^2 & -4\,z - 9\,z^2 \\ 9\,z - 5\,z^2 & 6\,z & 1 - 8\,z & -5\,z + 6\,z^2 & 8\,z + 8\,z^2 \\ \hline 4\,z + z^2 & -6\,z & -9\,z & 7\,z^2 & 6\,z^2 \\ 5\,z + 5\,z^2 & -2\,z & -7\,z & 3\,z^2 & -9\,z^2 \end{array}\right].$$

Multiplying as in (26) and (27) gives the $\mathcal{V}$ and $\mathcal{W}$ matrix polynomials for $H$. From this we obtain the $V(z)$, $Q(z)$, $V^*(z)$ and $Q^*(z)$ matrix polynomials and hence the solutions to equations (2), (4), (5) and (7). These are given by

$$V = \begin{bmatrix} -1 & -4 \\ 6 & -8 \\ 5 & -2 \\ -1 & 3 \\ 4 & 1 \\ 3 & 1 \\ 6 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 5 & 0 & 4 \\ 1 & 7 & -7 \\ -9 & -7 & 5 \\ 4 & -6 & -9 \\ -2 & -2 & 7 \\ 7 & -5 & 4 \\ 5 & -2 & -7 \end{bmatrix}$$

and

$$V^* = \begin{bmatrix} 6 & 0 & -1 & -2 & 8 & -2 & -9 \\ 8 & 0 & 1 & 2 & 4 & -2 & -7 \\ 5 & 7 & -8 & -8 & 9 & 6 & -8 \end{bmatrix}, \quad Q^* = \begin{bmatrix} 9 & -1 & 5 & 2 & 4 & -6 & -9 \\ 9 & 1 & 2 & -5 & 5 & -2 & -7 \end{bmatrix}.$$

The inverse formula (10) from Theorem 2.3 then gives

$$H^{-1} = \begin{bmatrix} -7 & -5 & -3 & 5 & 5 & 0 & 4 \\ -2 & 7 & 2 & -6 & 1 & 7 & -7 \\ -8 & 0 & -5 & -7 & -9 & -7 & 5 \\ 9 & -1 & 5 & 2 & 4 & -6 & -9 \\ -5 & -2 & -2 & -1 & -2 & -2 & 7 \\ 9 & -6 & -1 & -4 & 7 & -5 & 4 \\ 9 & 1 & 2 & -5 & 5 & -2 & -7 \end{bmatrix}.$$

□

# 6 Conclusions

In their study of the inversion problem for mosaic Hankel matrices, Heinig and Tewodros [18] give a set of linear equations that both provide necessary and sufficient conditions for the existence of an inverse along with the tools required to compute the inverse when it exists. In this paper we have converted the solutions of these linear equations into a matrix polynomial form. These matrix polynomials are closely related to matrix-type Padé approximants of a related matrix power series. It is shown that they satisfy an important commutativity relationship. This commutativity relationship is then used to show that these matrix polynomials are, up to a reordering of coefficients, the same as the $V$ and $W$ matrices of Antoulas [2]. A method is also described that recursively solves the inversion problem for "principal mosaic Hankel" submatrices. All our results hold for arbitrary mosaic Hankel matrices - no other extra conditions are required.

There are still a number of open research topics in this area. Our approach leads to a computational technique that recursively computes the inverses along a type of diagonal path of mosaic Hankel submatrices. As such this can be called a mosaic Hankel solver. It is of interest to develop a mosaic Toeplitz solver that computes the inverses along a type of anti-diagonal path of mosaic submatrices. This could be possible by a generalization of the scalar Toeplitz solver of Gutknecht [16].

It would be of interest to extend the results to structured matrices. In particular, this would give efficient inversion algorithms for these matrices without any addition restrictions. In addition, it would of interest to extend our work to inversion of matrices such as generalized Loewner matrices that appear in rational interpolation problems, rather than in rational approximation problems (cf. [1]).

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