Inexact Arithmetic Considerations for Direct Control and Penalty Methods: American Options under Jump Diffusion *

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Abstract

Solutions of Hamilton Jacobi Bellman (HJB) Partial Integro Differential Equations (PIDEs) arising in financial option problems are not necessarily unique. In order to ensure convergence of a numerical scheme to the viscosity solution, it is common to use a positive coefficient discretization for such PIDEs. However in finite precision arithmetic one often encounters difficulties in solving the discretized nonlinear algebraic equations. In this paper we focus on a specific HJB PIDE, arising from pricing American options under jump diffusion. We use two formulations of this problem, the first a penalty method and the second a direct control formulation. In each case we use a positive coefficient discretization which implies that a fixed point policy iteration will converge when used to solve the nonlinear discretized algebraic equations, under very mild restrictions on parameters. However, when using finite precision arithmetic, we observe that convergence may not occur for either formulation, even if the theoretical conditions are satisfied. We estimate bounds for the penalty parameter (penalty method) and the scaling parameter (direct control formulation) so that convergence of the fixed point policy iteration in inexact arithmetic can be expected. Numerical tests verify that these bounds are conservative. The lower bound is of more practical importance, and conveniently this has a very simple form. We remark that similar issues also arise in more complicated HJB PIDES in finance, for example when pricing American options under regime switching or guaranteed minimum withdrawal benefits (GMWB) under jump diffusion.

Keywords: American options, jump diffusion, inexact arithmetic

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1 Introduction

Penalty methods have been suggested for American option pricing problems in [15, 21, 35, 11, 26]. These techniques have also been applied to singular [9, 17] and impulse [6] control problems,

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transaction cost problems [10], and other Hamilton Jacobi Bellman (HJB) PDEs in finance [32]. Such methods are simple to implement, and make no assumptions about the connectedness of the controlled/uncontrolled regions. It is also straightforward to apply penalty methods to multi-dimensional problems [36, 21], jump diffusions [12] and regime switching [23, 18]. However, with penalty methods there is always the question of the selection of the dimensionless penalty parameter.

An alternative approach is based on a direct control formulation [4, 19, 33]. Superficially, a direct control method does not appear to require a scaling parameter as is required for the penalty method. However, since any iterative method for solution of the discretized equations requires comparing two (or more) terms and finding the maximum, there is an implicit scaling parameter [19] which affects convergence. This is particularly obvious if the terms being maximized have different units, which is often the case.

After discretizing the original HJB equation in the time and the space-like directions, a nonlinear set of algebraic equations must be solved at each timestep. In order to ensure convergence to the viscosity solution of the HJB equation, a positive coefficient discretization is used [14]. It is then straightforward to prove that policy iteration for the solution of the algebraic equations will converge [4, 19]. However, experimental computations show that in inexact arithmetic, the policy iteration may not converge for some choices of the penalty parameter or the direct control scaling parameter.

In the case of an American option with jump diffusion, a full policy iteration is not feasible, since the discretization of the jump term results in a dense matrix. The fixed point policy iteration algorithm [12] requires only a sparse matrix solve and a dense matrix-vector multiplication at each iteration. The matrix-vector multiply can be efficiently carried out using an FFT [13]. Provided a positive coefficient discretization is used, then convergence of the fixed point policy iteration can be guaranteed [12, 19], under very mild conditions on the scaling parameter, in exact arithmetic.

The objective of this article is to examine the effect of inexact arithmetic on the convergence of a fixed point policy iteration scheme for American options under jump diffusion. We focus exclusively on methods which do not require knowledge of the structure of the exercise region. Our analysis extends some of the results in [20] for the case of a singular control problem. Our main results are:

- We derive estimates for the upper and lower bounds for the penalty parameter (penalty formulation) and the scaling parameter (scaled direct control formulation) so that convergence of the fixed point policy iteration can be expected in the presence of inexact arithmetic effects.
- The lower bound estimate is of more practical importance than the upper bound estimate, and conveniently this bound has a very simple form. Numerical tests indicate that this bound is conservative, but not too restrictive.

In addition, as secondary results, we observe that the computed solution for the scaled direct control formulation is insensitive to the choice of scaling parameter over a very wide range (fifteen orders of magnitude) while the penalty formulation solution is affected by penalization error if the parameter is too large. Also, utilizing the properties of inexact arithmetic, it is possible to convert existing software (which uses a penalty method) to use the scaled direct control formulation in just a few lines of code. Finally, our analysis can also be applied to other HJB equations, such as singular control problems [20].

2 American Options under Jump Diffusion

Let the price of the underlying risky asset be S, which follows the risk neutral process

$$dS = (r - \lambda\kappa)Sdt + \sigma S dZ + (\xi - 1)S dq \ , \tag{2.1}$$

where dZ is the increment of a Weiner process, r is the risk free rate, and σ is the volatility. Here λ is the jump intensity representing the mean arrival rate of the Poisson process:

$$dq = \begin{cases} 0 & \text{with probability } 1 - \lambda dt \\ 1 & \text{with probability } \lambda dt \end{cases} \ , \tag{2.2}$$

with ξ a random variable representing the jump size of S. When a jump occurs, $S \rightarrow \xi S$. We assume that ξ follows a log-normal distribution $p(\xi)$ given by

$$p(\xi) = \frac{1}{\sqrt{2\pi}\zeta\xi} \exp\Big(-\frac{(\log(\xi) - \nu)^2}{2\zeta^2}\Big) \ , \tag{2.3}$$

with parameters ζ and ν, $\kappa = E[\xi - 1]$, where $E[\cdot]$ is the expectation, and $E[\xi] = \exp(\nu + \zeta^2/2)$ given the distribution function $p(\xi)$ in (2.3).

Define $\tau = T - t$ where t is the forward time, and T is the expiry time of the contract and set $V = V(W, A, \tau)$ to be the no-arbitrage value of the contingent claim. The no-arbitrage price of the claim is then given by

$$\min\Bigg[V_\tau - \mathcal{L}V - \lambda\mathcal{J}V, V - V^* \Bigg] = 0 \ , \tag{2.4}$$

where $V^*(S)$ is the payoff. Here the operators $\mathcal{L}, \mathcal{J}$ are defined as

$$\begin{aligned} \mathcal{L}V &= \frac{\sigma^2}{2} S^2 V_{SS} + (r - \lambda\kappa) S V_S - (r + \lambda) V \\ &= \frac{\sigma^2}{2} S^2 D_{SS} V + (r - \lambda\kappa) S D_S V - (r + \lambda) V \\ \mathcal{J}V &= \int_0^\infty V(\xi S, \tau) p(\xi) \ d\xi \ . \end{aligned} \tag{2.5}$$

For computational purposes we localize the problem to the domain $(S, \tau) \in [0, S_{\max}] \times [0, T]$. The boundary conditions for equation (2.4) are

$$\begin{aligned} V(S, 0) = V^*(S) \quad &; \quad \tau = 0 \\ \min\Bigg[V_\tau - rV, V - V^* \Bigg] \quad &; \quad S = 0 \\ V(S_{\max}, \tau) = V^*(S_{\max}) \quad &; \quad S = S_{\max} \\ V_{SS} \rightarrow 0 \quad &; \quad S \rightarrow S_{\max} \ . \end{aligned} \tag{2.6}$$

2.1 Direct Control Formulation

We introduce a scaling parameter Ω into equation (2.4) and rewrite (2.4) in control form [4, 19]

$$\max_{\varphi \in \{0,1\}} \Bigg[\Omega \ \varphi(V^* - V) - (1 - \varphi)(V_\tau - \mathcal{L}V - \lambda\mathcal{J}V) \Bigg] = 0 \ . \tag{2.7}$$

Although the scaling parameter has no effect on the exact solution of (2.7), it does affect convergence of the iterative method used to solve the discretized equations in finite precision arithmetic.

2.2 Penalty Formulation

Penalty methods were first suggested for solution of equation (2.4) in [12] with the idea now applied to various other problems in finance [9, 10, 21, 7, 26, 32]. The penalty approach rewrites equation (2.4) in the form

$$\lim_{\varepsilon\to 0}\left[V_\tau - \mathcal{L}V - \lambda\mathcal{J}V - \max_{\varphi\in\{0,1\}}\varphi\left(\frac{V^*-V}{\varepsilon}\right)\right] = 0 \ . \tag{2.8}$$

It is straightforward to show that equation (2.8) is consistent [22], in the viscosity sense, with equation (2.4). For suppose $\psi(S,\tau)$ is a smooth test function, with bounded derivatives of all orders. Then replacing V in equation (2.8) by ψ, and removing the control φ gives

$$\lim_{\varepsilon\to 0}\left[\psi_\tau - \mathcal{L}\psi - \lambda\mathcal{J}\psi - \max\left(\frac{V^*-\psi}{\varepsilon},0\right)\right] = 0 \ . \tag{2.9}$$

Rearranging equation (2.9), noting that $\varepsilon > 0$, then gives

$$\lim_{\varepsilon\to 0}\min\left[\psi_\tau - \mathcal{L}\psi - \lambda\mathcal{J}\psi, \psi - V^* + \varepsilon(\psi_\tau - \mathcal{L}\psi - \lambda\mathcal{J}\psi)\right] = 0 \ . \tag{2.10}$$

Taking the limit as $\varepsilon \to 0$ gives an equation consistent with equation (2.4). A more precise argument for consistency in the viscosity sense is given in [3] for a more general case of an impulse control problem.

3 Discretization

Define a set of nodes $S_1,\ldots,S_{i_{\max}}$, and discrete times $\tau^n = n\Delta\tau$. Let V_i^n be the approximate solution of equation (2.4) and set $V^n = [V_1,\ldots,V_{i_{\max}}]'$.

Let $\mathcal{L}^h, \mathcal{J}^h, D^h_{SS}, D^h_S$ be the discrete forms of the operators $\mathcal{L}, \mathcal{J}, D_{SS}, D_S$ and define

$$\mathcal{L}_i^h V_i^n \;=\; \begin{cases} -rV_i^n & i=1 \\ \frac{\sigma^2}{2}S_i^2 D^h_{SS}V_i^n + (r-\lambda\kappa)S_i D^h_S V_i^n - (r+\lambda)V_i^n & 2\le i\le \hat{i} \\ \frac{\sigma^2}{2}S_i^2 D^h_{SS}V_i^n + rS_i D^h_S V_i^n - rV_i^n & \hat{i} < i < i_{\max} \\ 0 & i = i_{\max} \end{cases} . \tag{3.1}$$

We use standard three point central, forward and backward differencing so that the positive coefficient condition is satisfied [30, 14, 17] with central differencing used as much as possible [30]. Linear behaviour of the solution is assumed for $i > \hat{i}$ [13, 29]. The integral term $\mathcal{J}V$ is discretized via transformation into a correlation integral combined with a use of the midpoint rule as described in detail in [13, 29]. For notational convenience, we define $\mathcal{J}_i^h$ as

$$\mathcal{J}_i^h V_i^n \;=\; \begin{cases} [\mathcal{J}^h V^n]_i & 2\le i\le\hat{i} \\ 0 & \text{otherwise} \end{cases} . \tag{3.2}$$

Let $(\Delta S)_{\max} = \max_i(S_{i+1}-S_i)$ and $(\Delta\tau)_{\max} = \max(\tau^{n+1}-\tau^n)$. We suppose that the grid and timesteps are selected so that

$$(\Delta S)_{\max} = \hat{C}h \ \ ; \ \ (\Delta\tau)_{\max} = \widetilde{C}h \ , \tag{3.3}$$

with $\hat{C}, \widetilde{C}$ being positive constants.

Observe that the discretization method is at least first order correct. Hence, taking into account the definitions (2.5) and (3.1), and noting that $\mathcal{J}_i^h$ represents a discrete probability density (on a truncated domain) [13, 29], we obtain the following results. If e is the $i_{\max}$ length vector $[1, 1, \ldots, 1]'$, then since $D_S^h\ e_i = 0$ and $D_{SS}^h\ e_i = 0$ we have

$$
\begin{aligned}
\mathcal{L}_i^h e_i \quad &= \quad \begin{cases} -r & i = 1 \text{ or } \hat{i} < i < i_{\max} \\ -(r+\lambda) & 2 \le i \le \hat{i} \\ 0 & i = i_{\max} \end{cases} \\
\mathcal{J}_i^h e_i \quad &\le \quad \begin{cases} 1 & 2 \le i \le \hat{i} \\ 0 & \text{otherwise} \end{cases} \ .
\end{aligned} \tag{3.4}
$$

3.1 Discretization: Direct Control Formulation

We use fully implicit ($\theta = 1$) or Crank Nicolson ($\theta = 1/2$) to discretize equation (2.7), using the discrete forms of the operators as discussed in Section 3,

$$
\begin{aligned}
(1-\varphi_i^{n+1})\left(\frac{V_i^{n+1}}{\Delta\tau} - \theta\mathcal{L}_i^h V_i^{n+1}\right) \quad &+ \quad \Omega\ \varphi_i^{n+1} V_i^{n+1} \\
&= \quad (1-\varphi_i^{n+1})\frac{V_i^n}{\Delta\tau} + \Omega\ \varphi_i^{n+1} V_i^* + (1-\varphi_i^{n+1})\lambda\theta\mathcal{J}_i^h V_i^{n+1} \\
&\qquad + (1-\varphi_i^{n+1})(1-\theta)\big[\mathcal{L}_i^h V_i^n + \lambda\mathcal{J}_i^h V_i^n\big] \ ; \qquad i < i_{\max} \\
\frac{V_i^{n+1}}{\Delta\tau} \quad &= \quad \frac{V_i^*}{\Delta\tau} \ ; \qquad\qquad i = i_{\max} \ ,
\end{aligned} \tag{3.5}
$$

where

$$
\begin{aligned}
\{\varphi_i^{n+1}\} \quad \in \quad \underset{\varphi\in\{0,1\}}{\arg\max} \bigg\{ &\Omega\ \varphi(V_i^* - V_i^{n+1}) - (1-\varphi)\bigg(\frac{V_i^{n+1} - V_i^n}{\Delta\tau} \\
&- \theta\big(\mathcal{L}_i^h V_i^{n+1} + \lambda\mathcal{J}_i^h V_i^{n+1}\big) - (1-\theta)\big(\mathcal{L}_i^h V_i^n + \lambda\mathcal{J}_i^h V_i^n\big)\bigg)\bigg\} \ .
\end{aligned} \tag{3.6}
$$

3.2 Discretization: Penalty Method

If $\varepsilon = C\Delta\tau$, where $C > 0$ is a constant, then the following is a consistent discretization of (2.8),

$$
\begin{aligned}
\frac{V_i^{n+1}}{\Delta\tau} - \theta\mathcal{L}_i^h V_i^{n+1} + \frac{\varphi_i^{n+1}}{\varepsilon} V_i^{n+1} \quad &= \quad \frac{V_i^n}{\Delta\tau} + \frac{\varphi_i^{n+1}}{\varepsilon} V_i^* + \lambda\theta\mathcal{J}_i^h V_i^{n+1} \\
&\qquad + (1-\theta)\big[\mathcal{L}_i^h V_i^n + \lambda\mathcal{J}_i^h V_i^n\big] \ ; \qquad i < i_{\max} \\
\frac{V_i^{n+1}}{\Delta\tau} \quad &= \quad \frac{V_i^*}{\Delta\tau} \ ; \qquad i = i_{\max},
\end{aligned} \tag{3.7}
$$

where

$$
\varphi_i^{n+1} \in \underset{\varphi\in\{0,1\}}{\arg\max} \left\{ \frac{\varphi}{\varepsilon}(V_i^* - V_i^{n+1}) \right\} \ . \tag{3.8}
$$

4 Solving the Discretized Equations

At each timestep we must solve the nonlinear equations (3.5) or (3.7). We can write both sets of equations in terms of nonlinear matrix operators. Let $\mathcal{A}$, $\mathcal{B}$ be $i_{\max} \times i_{\max}$ matrices, and $\mathcal{C}$ be an $i_{\max}$ length vector, which are defined for both the scaled direct control and penalty formulations in Appendix A. For each timestep let U denote the vector of the unknown solution V^{n+1} and $Q = [\varphi_1, \ldots, \varphi_{i_{\max}}]$ be an indexed set of controls with each $\varphi_j \in \{0, 1\}$. Then, the discretized equations (3.5) and (3.7) can be written as

$$\begin{aligned} (\mathcal{A}(Q) - \mathcal{B}(Q))\, U &= \mathcal{C}(Q) \\ \text{with each} \quad Q_i &\in \operatorname*{arg\,max}_{Q \in Z} \Big[-(\mathcal{A}(Q) - \mathcal{B}(Q))U + \mathcal{C}(Q) \Big]_i \end{aligned} \tag{4.1}$$

where Z is the set of admissible controls. Observe that $\mathcal{A}$ is sparse, but $\mathcal{B}$ is dense, since it represents the discretization of the jump term $\mathcal{J}$.

Remark 4.1. *Note that $[\mathcal{A}(Q)]_{i,j}, [\mathcal{B}(Q)]_{i,j}, [\mathcal{C}(Q)]_i$ depend only on Q_i.*

It is useful to note the following properties of $\mathcal{A}, \mathcal{B}$.

Proposition 4.1. *Suppose a positive coefficient discretization [14] is used. Then*

(a) $\mathcal{B}(Q) \geq 0$.

(b) The i^{th} row sums for $\mathcal{A}(Q^k)$ and $\mathcal{B}(Q^k)$ are

Direct Control:

$$\begin{aligned} Row\_Sum\ (\ \mathcal{A}(Q^k)\ )_i &= \begin{cases} (1-\varphi_i^k)\left(\frac{1}{\Delta\tau} + \theta r\right) + \varphi_i^k \Omega & i = 1 \quad or \\ & i = \hat{i}+1, \ldots, i_{\max}-1 \\ (1-\varphi_i^k)\left(\frac{1}{\Delta\tau} + \theta(r+\lambda)\right) + \varphi_i^k \Omega & 2 \leq i \leq \hat{i} \\ 1/(\Delta\tau) & i = i_{\max} \end{cases} \\ Row\_Sum\ (\mathcal{B}(Q^k)\ )_i &\leq \begin{cases} (1-\varphi_i^k)\theta\lambda & 2 \leq i \leq \hat{i} \\ 0 & otherwise \end{cases}, \end{aligned} \tag{4.2}$$

Penalty Method:

$$\begin{aligned} Row\_Sum\ (\mathcal{A}(Q^k)\ )_i &= \begin{cases} \frac{1}{\Delta\tau} + \theta r + \frac{\varphi_i^k}{\varepsilon} & i = 1 \quad or \quad i = \hat{i}+1, \ldots, i_{\max}-1 \\ \frac{1}{\Delta\tau} + \theta(r+\lambda) + \frac{\varphi_i^k}{\varepsilon} & 2 \leq i \leq \hat{i} \\ 1/(\Delta\tau) & i = i_{\max} \end{cases} \\ Row\_Sum\ (\mathcal{B}(Q^k)\ )_i &\leq \begin{cases} \theta\lambda & 2 \leq i \leq \hat{i} \\ 0 & otherwise \end{cases}. \end{aligned} \tag{4.3}$$

(c) The matrices $\mathcal{A}(Q) - \mathcal{B}(Q)$ and $\mathcal{A}(Q)$ in equation (4.1) are strictly diagonally dominant M matrices [28].

Proof. Part (a) follows from the discretization method for $\mathcal{J}$ [13, 29], and the definition of $\mathcal{B}(Q)$ in Appendix A. Part(b) follows from properties (3.4), equations (3.5), (3.7) and Appendix A. Since a positive coefficient discretization is used [14], part (c) follows from (b) and [28]. □

4.1 Fixed Point Policy Iteration

Since $\mathcal{B}$ is dense, direct application of policy iteration to solve equation (4.1) is not feasible. Various methods have been suggested for solution of equations of this type [12, 2, 27, 19]. For the purposes of investigating floating point errors, we will focus on the fixed point policy iteration discussed in [12, 19]. Fixed point policy iteration was also used for American options under regime switching in [18]. The regime switching case has some similarities with the jump diffusion case, since full policy iteration is not efficient for either problem. The fixed point-policy iteration is given in Algorithm 4.1.

Algorithm 4.1 Fixed Point-Policy Iteration

$U^0 =$ Initial solution vector of size N

for $k = 0, 1, 2, \ldots$ until converge **do**

$$Q_\ell^k \in \arg\max_{Q_\ell \in Z} \left\{ -\left[\mathcal{A}(Q) - \mathcal{B}(Q)\right] U^k + \mathcal{C}(Q) \right\}_\ell$$

Solve $\mathcal{A}(Q^k) U^{k+1} = \mathcal{B}(Q^k) U^k + \mathcal{C}(Q^k)$

if $k > 0$ and $\max_\ell \dfrac{|U_\ell^{k+1} - U_\ell^k|}{\max\left[scale, |U_\ell^{k+1}|\right]} < tolerance$ **then**

break from the iteration

end if

end for

The term *scale* in Algorithm 4.1 is used to ensure that unrealistic levels of accuracy are not enforced. As an example, if options are priced in dollars, then a typical value of $scale = 1.0$. Each iteration of Algorithm 4.1 requires a sparse matrix solve (in this case a tridiagonal system) and a dense matrix-vector multiply $\mathcal{B}(Q^k)U^k$. This dense matrix-vector multiply can be carried out efficiently using an FFT as described in [13].

Theorem 4.1 (Convergence of Fixed Point-Policy Iteration)**.** *Suppose:*

(a) The matrix $\mathcal{A}(Q)$ is an M matrix [28].

(b) The matrices $\mathcal{A}(Q)$, $[\mathcal{A}(Q)]^{-1}$ and the vector $\mathcal{C}(Q)$ are bounded independent of Q.

(c) There is a constant $C_1 < 1$ such that

$$\|\mathcal{A}(Q^k)^{-1}\mathcal{B}(Q^{k-1})\|_\infty \leq C_1 \quad \text{and} \quad \|\mathcal{A}(Q^k)^{-1}\mathcal{B}(Q^k)\|_\infty \leq C_1 \ . \tag{4.4}$$

Then the fixed point-policy iteration in Algorithm 4.1 converges.

Proof. See [19]. □

Corollary 4.1. *The fixed point-policy iteration converges unconditionally for the penalty discretization (3.7) and converges for the scaled direct control discretization (3.5) if*

$$\Omega \ > \ \theta\lambda \ . \tag{4.5}$$

Proof. This follows from the definitions of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ in Appendix A, Proposition 4.1, and Theorem 4.1, following the same steps as used in [19] for a regime switching problem. □

5 Floating Point Considerations: Example

To motivate our discussion of the floating point issues surrounding the iterative solution of discretized HJB equations, we first consider the simple case of an American option with no jumps. Formally, we set $\lambda = 0$ in equations (2.4-2.5) with the resulting discretized equations then of the form (4.1) with $\mathcal{B} = 0$.

In this case, it is trivial to verify that Algorithm 4.1 converges, since for $\mathcal{B} = 0$, this reduces to pure policy iteration. To be precise, policy iteration applied to equation (4.1) with $\mathcal{B} = 0$ is given in Algorithm 5.1.

Algorithm 5.1 Policy Iteration

$U^0 =$ Initial solution vector of size N
for $k = 0, 1, 2, \ldots$ until converge **do**
  $Q_\ell^k \in \arg\max_{Q_\ell \in Z} \left\{ -\mathcal{A}(Q)U^k + \mathcal{C}(Q) \right\}_\ell$
  Solve $\mathcal{A}(Q^k)U^{k+1} = \mathcal{C}(Q^k)$
  if $k > 0$ and $\max_\ell \dfrac{|U_\ell^{k+1} - U_\ell^k|}{\max\left[scale, |U_\ell^{k+1}|\right]} < tolerance$ **then**
    break from the iteration
  end if
end for

For policy iteration applied to an American option problem with no jumps, we can obtain the following result [15, 4, 32].

Theorem 5.1. *If Algorithm 5.1 is applied to the discretized form of equation (2.4-2.5), with $\lambda = 0$, using either a penalty or a scaled direct control formulation, and*

(a) $\mathcal{A}(Q)$ is an M matrix.

(b) $\mathcal{A}(Q)$, $\mathcal{A}(Q)^{-1}$ and $\mathcal{C}(Q)$ are bounded independent of Q,

then the policy iteration Algorithm 5.1 converges in a finite number of steps. Furthermore, convergence is monotone, non-decreasing (after the first iteration), that is,

$$U^{k+1} \geq U^k \; ; \; k > 0 \; . \tag{5.1}$$

As an example, consider the problem given in Table 5.1. This problem was solved on a sequence of grids, as given in Table 5.2. Crank-Nicolson timestepping was used with the Rannacher modification [25] and with variable timestepping [15].

For the penalty formulation (3.7), we use a penalty factor of the form $\varepsilon = C\Delta\tau$, where C is dimensionless. In the case of the direct control method (3.5), the scaling factor Ω should have the units of inverse time, so that quantities with the same units are being compared in the $\max(\cdot)$ expression in equation (2.7). It is convenient to choose $\Omega = 1/(C\Delta\tau)$ where C is dimensionless, so that $\Omega = 1/\varepsilon$.

Since the penalty method is consistent for any C, such that $\varepsilon = C\Delta\tau$, then any $C > 0$ will result in convergence to the solution as $\Delta\tau \to 0$. Table 5.3 verifies this for three different choices

Expiry Time	.25
Exercise	American
Strike (Put) K	100
Risk free rate r	.02
Volatility σ	.20

TABLE 5.1: *Data for the an American put, no jumps ($\lambda = 0$).*

Refine	S Nodes	Timesteps
0	129	39
1	257	71
2	513	140
3	1025	276
4	2049	546
5	5097	1087
6	10193	2167

TABLE 5.2: *Grid/timestep data for convergence study, American put, no jumps ($\lambda = 0$). Other data in Table 5.1. On each grid refinement, new fine grids are inserted between each two coarse grid nodes, and the timestep control parameter is halved.*

of C. Table 5.4 compares the performance of the penalty method and the scaled direct control formulation, as a function of the scaling parameter Ω or the penalty parameter ε, for a fixed grid size. From Table 5.4, we can see that the scaled direct control method (when the iteration converges) is unaffected by the size of Ω over eight orders of magnitude. On the other hand, the penalty solution is affected when ε is large, at a finite grid size. This is, of course, due to the error induced by the term $\varepsilon(\psi_\tau - \mathcal{L}\psi - \lambda \mathcal{J}\psi)$ in equation (2.10), which will be present at any finite grid size.

Observe that for sufficiently small ε or $1/\Omega$, the policy iteration for both penalty and scaled direct control methods does not converge. From Theorem 5.1 we learn that policy iteration must converge for this problem in exact arithmetic and so the lack of convergence in Table 5.4 is a result of using floating point arithmetic. In particular, analysis of the cases where policy iteration did not converge revealed that the iterates oscillated, at levels above the convergence tolerance, and so the exact arithmetic convergence property (5.1) was violated.

We can rewrite Algorithm 5.1 in the form

$$\mathcal{A}(Q^k)(U^{k+1} - U^k) \quad = \quad \max_{Q \in Z} \left\{ -\mathcal{A}(Q)U^k + \mathcal{C}(Q) \right\} . \tag{5.2}$$

The analysis in [14] shows that the right hand side of equation (5.2) is always non-negative for $k > 0$. However, in inexact arithmetic, we have verified that this is not always true, which results in the oscillatory iterates and nonconvergence of the iteration. Consequently, the main source of finite precision arithmetic error appears to be due to the computation of the right hand side of equation (5.2).

It is now desirable to carry out some analysis to explain the observations in Table 5.4. This

Refine	$\varepsilon = 10^{-6}\Delta\tau$		$\varepsilon = 10^{-2}\Delta\tau$		$\varepsilon = \Delta\tau$	
	Itns/Step	Value	Itns/Step	Value	Itns/Step	Value
0	2.54	3.765795756	2.56	3.765735290	2.58	3.760452288
1	2.72	3.767678056	2.72	3.767643630	2.65	3.764771892
2	2.70	3.768152726	2.68	3.768134992	2.64	3.766668367
3	2.68	3.768272342	2.70	3.768263624	2.55	3.767533565
4	2.57	3.768302463	2.51	3.768298209	2.20	3.767938281
5	2.20	3.768310012	2.13	3.768307954	2.04	3.768130740
6	2.03	3.768311910	2.03	3.768310918	2.05	3.768223491

TABLE 5.3: *Convergence study, American put, no jumps ($\lambda = 0$). Other data in Table 5.1. Penalty method (3.7). Value at $S = 100$. tolerance $= 10^{-6}$.*

ε or	*tolerance* $= 10^{-6}$		*tolerance* $= 10^{-8}$	
$1/\Omega$	Direct Control	Penalty	Direct Control	Penalty
$10^{-2}\Delta\tau$	3.768310012	3.768307954	3.768310012	3.768307954
$10^{-3}\Delta\tau$	3.768310012	3.768309783	3.768310012	3.768309783
$10^{-4}\Delta\tau$	3.768310012	3.768309989	3.768310012	3.768309989
$10^{-5}\Delta\tau$	3.768310012	3.768310010	3.768310012	3.768310010
$10^{-6}\Delta\tau$	3.768310012	3.768310012	3.768310012	3.768310012
$10^{-7}\Delta\tau$	3.768310012	3.768310012	3.768310012	3.768310012
$10^{-8}\Delta\tau$	3.768310012	3.768310012	3.768310012	****
$10^{-9}\Delta\tau$	3.768310012	3.768310012	****	****
$10^{-10}\Delta\tau$	****	****	****	****

TABLE 5.4: *Option value at $S = 100$, refinement level 5. Comparison of penalty parameter ε and direct control scaling parameter Ω for the penalty discretization (3.7) and the direct control discretization (3.5), no jumps ($\lambda = 0$). Other data in Table 5.1. $****$ indicates failure to converge after 100 iterations in any timestep.*

analysis should give us

- a conservative, order of magnitude estimate of the largest value of Ω (smallest value of ε) which can be safely used in either the penalty or scaled direct control formulation,
- an estimate which depends on the convergence tolerance, consistent with results in Table 5.4.

We remind the reader that the consistency analysis in equation (2.10) indicates that a small value of ε is advantageous for the penalty method, but there is not any particular advantage (in terms of solution accuracy at a fixed grid size) in selecting Ω large for the scaled direct control formulation.

5.1 Floating Point Considerations: Analysis

We return now to the case of American options with jump diffusion, where we use the fixed point policy iteration in Algorithm 4.1. We rewrite Algorithm 4.1 in the form

$$\mathcal{A}(Q^k)(U^{k+1} - U^k) \quad = \quad \max_{Q \in Z} \Big\{ -\big[\mathcal{A}(Q) - \mathcal{B}(Q)\big] U^k + \mathcal{C}(Q) \Big\} \ . \tag{5.3}$$

The analysis in [19] shows that in exact arithmetic, this iteration always converges, hence the right hand side of equation (5.3) should converge to zero. However, we have observed in our numerical experiments that in any case where the Algorithm 4.1 failed to converge, the computed value of the right hand side of equation (5.3) oscillated in sign, with a non-decreasing magnitude. Define the residual of the solution of the linear system at iteration $k+1$ as

$$r^{k+1} = \mathcal{B}(Q^k) U^k + \mathcal{C}(Q^k) - \mathcal{A}(Q^k) U^{k+1} \ . \tag{5.4}$$

Numerical examination of $\mathcal{A}(Q^k)^{-1} r^{k+1}$ indicated that this was small compared to the convergence tolerance. This suggests that the main source of finite precision error is the numerically computed right hand side of equation (5.3).

Let $fl(x)$ denote the floating point representation of a real number x, that is,

$$fl(x) \quad = \quad x(1 + \delta_x) \quad \text{with} \quad |\delta_x| \leq \delta \ , \tag{5.5}$$

where δ is the machine precision. Define the floating point error vector Δe_δ^k as

$$\Delta e_\delta^k = \max_{Q \in Z} \bigg\{ fl^{cum} \Big(-\mathcal{A}(Q) U^k + \mathcal{B}(Q) U^k + \mathcal{C}(Q) \Big) \bigg\} - \max_{Q \in Z} \bigg\{ \Big(-\mathcal{A}(Q) U^k + \mathcal{B}(Q) U^k + \mathcal{C}(Q) \Big) \bigg\} \ , \tag{5.6}$$

where $fl^{cum}(\cdot)$ denotes the accumulated effect of floating point errors from all arithmetic operations in $(\cdot)$. Suppose that in exact arithmetic, Algorithm 4.1 would terminate at step $k+1$. Let U^k be the iterates computed in exact arithmetic, and let ΔU_δ^k denote the floating point error in U^{k+1} generated by Δe_δ^k. Then, from equations (5.3) and (5.6), we have

$$\begin{aligned} \mathcal{A}(Q^k) \left[U^{k+1} - U^k + \Delta U_\delta^k \right] \quad &= \quad \max_{Q \in Z} \Big\{ -\mathcal{A}(Q) U^k + \mathcal{B}(Q) U^k + \mathcal{C}(Q) \Big\} + \Delta e_\delta^k \\ \Delta U_\delta^k \quad &= \quad \mathcal{A}(Q^k)^{-1} \Delta e_\delta^k \ . \end{aligned} \tag{5.7}$$

Now, if

$$\max_i \left[\frac{\mid [\Delta U_\delta^k]_i \mid}{\max(|U_i^{k+1}|, scale)} \right] \quad > \quad tolerance \quad , \tag{5.8}$$

then Algorithm 4.1 may not converge. Consequently, we should choose parameters such that

$$\max_i \left[\frac{\mid [\Delta U_\delta^k]_i \mid}{\max(|U_i^{k+1}|, scale)} \right] \quad = \quad \max_i \left[\frac{\mid [\mathcal{A}(Q^k)^{-1} \Delta e_\delta^k]_i \mid}{\max(|U_i^{k+1}|, scale)} \right] < \ tolerance \ . \tag{5.9}$$

5.2 Approximation of Equation (5.9)

If we attempt to provide a rigorous bound for equation (5.9), then the result will be far too pessimistic to be useful. Instead we proceed in a somewhat more heuristic manner in order to obtain a more practically useful bound.

We restrict attention to the typical situation where the grid spacing and timestep are reduced proportionally to a discretization parameter h, as in equation (3.3), and consider the limit $h \to 0$. Recalling that $\mathcal{A}^k$ is a strictly diagonally dominant M matrix, we can write $\mathcal{A}^k$ as

$$\mathcal{A}^k \;=\; D + P \quad \text{where} \quad [D]_{ii} = \; Row\_Sum\;(\mathcal{A}^k\;)_i\;. \tag{5.10}$$

Thus D is diagonal, with entry $D_i > 0$ on the i^{th} row, and if $e = [1, \ldots, 1]'$, then $Pe = 0$. Consequently, $[I + D^{-1}P]e = e$ which gives

$$e \;=\; [I + D^{-1}P]^{-1} e\;, \tag{5.11}$$

implying that $Row\_Sum\;([I + D^{-1}P]^{-1}\;)_i = 1$ for each i. Now, from equation (5.7), we have that

$$\mathcal{A}^k \Delta U_\delta^k = \Delta e_\delta^k \tag{5.12}$$

so that

$$D^{-1}\mathcal{A}^k \Delta U_\delta^k \;=\; D^{-1}\Delta e_\delta^k\;, \tag{5.13}$$

which gives

$$\Delta U_\delta^k \;=\; [I + D^{-1}P]^{-1}\big(D^{-1}\Delta e_\delta^k\big)\;. \tag{5.14}$$

Let

$$g_{i,j} \;=\; \Big[[I + D^{-1}P]^{-1}\Big]_{i,j}\;, \tag{5.15}$$

so that, since $[I + D^{-1}P]$ is an M matrix, and noting equation (5.11), then equation (5.14) becomes

$$\begin{aligned}\Big[\Delta U_\delta^k\Big]_i \;&=\; \sum_j g_{i,j} D_j^{-1}\Big[\Delta e_\delta^k\Big]_j \\ g_{i,j} &\geq 0\;\;;\;\;\sum_j g_{i,j} = 1\;.\end{aligned} \tag{5.16}$$

Let $\|D^{-1}\|_\infty = \max_i |D_i^{-1}|$, and let $[\Delta \hat{e}_\delta^k]_j$ be an upper bound estimate for $|\;[\Delta e_\delta^k]_j\;|$. Then, equation (5.16) becomes

$$\left|\;\Big[\Delta U_\delta^k\Big]_i\;\right| \;\leq\; \|D^{-1}\|_\infty \sum_j g_{i,j}[\Delta \hat{e}_\delta^k]_j\;. \tag{5.17}$$

Define the computational domain $\Phi = [0, S_{\max}]$. At any iteration k, the domain can be considered to be the union of disjoint sets

- Φ_a : the set of points where the American constraint is active. In this case if $S_i \in \Phi_a$ then $\varphi_i^k = 1$ for both penalty and direct control methods (see Section 3).
- $\Phi - \Phi_a$: the set of points where the American constraint is not active, that is, if $S_i \in \Phi - \Phi_a$, then $\varphi_i^k = 0$.

If $S_i \in (\Phi - \Phi_a)$, then we denote the set to which S_i belongs by $(\Phi - \Phi_a)_{S_i}$, and the boundaries of this region by $\partial(\Phi - \Phi_a)_{S_i}$. Note that the set $\Phi - \Phi_a$ will consist (in general) of the union of disjoint sets.

For $S_i \in \Phi_a$, then for the direct control method we have that $g_{i,j} = \delta_{i,j}$, and so equation (5.17) becomes

$$\left| \left[\Delta U_\delta^k\right]_i \right| \quad \leq \quad \|D^{-1}\|_\infty [\Delta \hat{e}_\delta^k]_i \quad ; \quad S_i \in \Phi_a \ . \tag{5.18}$$

In the case of the penalty method things are somewhat more involved. In this case, from properties (5.16), we have that

$$\left\| \sum_j g_{i,j} \left[\Delta \hat{e}_\delta^k\right]_j \right\|_\infty \leq \|\Delta \hat{e}_\delta^k\|_\infty \ . \tag{5.19}$$

As well note that ($(\Delta S)_{\min} = \min_i (S_{i+1} - S_i)$)

$$| \ [D^{-1}P]_{i,j} \ | \quad = \quad O\left(\frac{\varepsilon \Delta \tau}{(\Delta S)^2_{\min}} \right) = O(C) \quad ; \quad S_i \in \Phi_a \ , \tag{5.20}$$

assuming that the penalty parameter is $\varepsilon = C\Delta\tau$, C is small, and that the grid is refined as in equation (3.3). Straightforward computation then shows that

$$\left[[I + D^{-1}P]^{-1} \Delta \hat{e}_\delta^k \right]_i \quad \leq \left[\Delta \hat{e}_\delta^k\right]_i + O(C)\|\Delta \hat{e}_\delta^k\|_\infty \quad ; \quad S_i \in \Phi_a \ , \tag{5.21}$$

and hence equation (5.17) becomes

$$\left| \left[\Delta U_\delta^k\right]_i \right| \quad \leq \quad \|D^{-1}\|_\infty \left([\Delta \hat{e}_\delta^k]_i + O(C)\|\Delta \hat{e}_\delta^k\|_\infty \right) \quad ; \quad S_i \in \Phi_a \ . \tag{5.22}$$

In the following, we drop the $O(C)$ term in equation (5.22) and assume equation (5.18) holds for both direct control and penalty methods. Consider the equation

$$[I + D^{-1}P]W = f \tag{5.23}$$

where f is an arbitrary vector. Examination of the discretized equations shows that, as $h \to 0$ (see equation (3.3)), for $S_i \in \Phi - \Phi_a$, we can consider the discrete equations $[I + D^{-1}P]_{i,j} = g_{i,j}$ to be a discrete approximation to the Green's function solution of the equation

$$W_\tau - \theta \left(\frac{\sigma^2 S^2}{2} W_{SS} + (r - \lambda\kappa) W_S \right) = 0 \quad ; \quad W(S, 0) = f(S) \ . \tag{5.24}$$

In equation (5.24) there is no term $(r+\lambda)W$ since the scaling by D in equation (5.15) has effectively removed this term. For $S \in \Phi_a$, we can consider equation (5.23) as specifying that

$$W = f \quad ; \quad S \in \Phi_a \ . \tag{5.25}$$

The Green's function $G(S, \tau, S', \tau')$ of equation (5.24) [16] is the formal solution to

$$\begin{aligned} G_\tau - \theta\left(\frac{\sigma^2 S^2}{2} G_{SS} + (r - \lambda\kappa) G_S\right) &= \delta(S - S')\delta(\tau - \tau') \quad ; \quad S \in (\Phi - \Phi_a)_S \\ \lim_{(\tau - \tau') \to 0} G(S, \tau, S', \tau') &= \delta(S - S') \quad ; \quad S \in (\Phi - \Phi_a)_S \\ G(S^*, \tau, S', \tau') &= 0 \quad ; \quad S^* \in \partial(\Phi - \Phi_a)_S, \end{aligned} \tag{5.26}$$

where $\delta(\cdot)$ denotes a Dirac function. The solution of the equation

$$\begin{aligned} W_\tau - \theta\left(\frac{\sigma^2 S^2}{2} W_{SS} + (r - \lambda\kappa) W_S\right) &= 0 \quad ; \quad S \in (\Phi - \Phi_a) \\ W(S^*, \tau) &= q(S) \quad ; \quad S^* \in \partial(\Phi - \Phi_a)_S \\ W(S, 0) &= f(S) \end{aligned} \tag{5.27}$$

for arbitrary $q(S)$ is then given by

$$W(S, \tau) = \int_{(\Phi - \Phi_a)_S} G(S, \tau, S', 0) f(S') \ dS' + \int_0^\tau \int_{\partial(\Phi - \Phi_a)_S} P(S, \tau, S', \tau') q(S') \ d\tau' \ dS' \ , \tag{5.28}$$

where $P(S, \tau, S', \tau')$ is the Poisson function [16]. The Poisson function allows us to handle non-zero Dirichlet boundary conditions. We remind the reader that in this context, the Poisson function has nothing to do with a Poisson process. The term *Poisson function* is an unfortunate but standard terminology.

Note that

$$\begin{aligned} & G(S, \tau, S', \tau') \geq 0 \quad ; \quad P(S, \tau, S', \tau') \geq 0 \\ & \int_{(\Phi - \Phi_a)_S} G(S, \tau, S', 0) \ dS' + \int_0^\tau \int_{\partial(\Phi - \Phi_a)_S} P(S, \tau, S', \tau') \ d\tau' \ dS' = 1 \ . \end{aligned} \tag{5.29}$$

which is the continuous analogue of properties (5.16).

Now, as $h \to 0$, we expect that (setting $q(S) = f(S)$ from equation (5.25))

$$\sum_j g_{i,j} f(S_j) \ \to \ \int_{(\Phi - \Phi_a)_{S_i}} G(S_i, \Delta\tau, S', 0) f(S') \ dS' + \int_0^{\Delta\tau} \int_{\partial(\Phi - \Phi_a)_{S_i}} P(S_i, \Delta\tau, S', \tau') f(S') \ d\tau' \ dS'. \tag{5.30}$$

From the property of the continuous Green's function [16] ($S \in (\Phi - \Phi_a)$)

$$\lim_{\Delta\tau \to 0} \left[\int_{(\Phi - \Phi_a)_S} G(S, \Delta\tau, S', 0) f(S') \ dS' + \int_0^{\Delta\tau} \int_{\partial(\Phi - \Phi_a)_S} P(S, \Delta\tau, S', \tau') f(S') \ d\tau' \ dS' \right] = f(S) \ , \tag{5.31}$$

we can conclude that, as $h \to 0$ (from equations (5.30) and (5.31)),

$$\lim_{h \to 0} \sum_j g_{i,j} f(S_j) = f(S_i) \quad ; \quad S_i \in \Phi - \Phi_a \ . \tag{5.32}$$

More precise estimates of equation (5.32) are given in Appendix B.

Set $f(S_i) = [\Delta \hat{e}_\delta^k]_i$ in equation (5.32). We then obtain (using equations (5.17) and (5.32)) that

$$\max_i \left[\frac{|[\Delta U_\delta^k]_i|}{\max(|U_i^{k+1}|, scale)} \right] \leq \max_i \left[\frac{\|D^{-1}\|_\infty [\Delta \hat{e}_\delta^k]_i}{\max(|U_i^{k+1}|, scale)} \right] + \text{ small terms} \ . \tag{5.33}$$

Assuming we are close to convergence, so that $U_i^{k+1} \simeq U_i^k$, then we obtain the final estimate for bound (5.9)

$$\max_i \left[\frac{\|D^{-1}\|_\infty [\Delta \hat{e}_\delta^k]_i}{\max(|U_i^k|, scale)} \right] < \ tolerance \ . \tag{5.34}$$

5.3 Bounds on Floating Point Errors

From equation (5.6), we have that

$$\begin{aligned} |\Delta e_\delta^k| &\leq \max_{Q \in Z} \left\{ \left| fl^{cum}\Big(-\mathcal{A}(Q)U^k + \mathcal{B}(Q)U^k + \mathcal{C}(Q) \Big) - \Big(-\mathcal{A}(Q)U^k + \mathcal{B}(Q)U^k + \mathcal{C}(Q) \Big) \right| \right\} \\ &= \Delta \hat{e}_\delta^k \ . \end{aligned} \tag{5.35}$$

For the scaled direct control formulation, as $\Omega \to \infty$, the floating point error bound $\Delta \hat{e}_\delta^k$ (5.35) will be dominated by

$$\Omega(U_i^k - V_i^*) \ . \tag{5.36}$$

This is because near the exercise region, $U_i^k \simeq V_i^*$. In this case, we are subtracting two almost equal floating point numbers (which is highly prone to round off error amplification) and then multiplying by a large number.

Now, examine the error induced by computing term (5.36) in finite precision arithmetic

$$\begin{aligned} [\Delta \hat{e}_\delta^k]_i &\simeq \left| fl \Big[fl(\Omega) fl(fl(U_i^k) - fl(V_i^*)) \Big] - \Omega(U_i^k - V_i^*) \right| \\ &\leq \Omega\delta(|U_i^k| + |V_i^*|) + 3\delta\Omega|U_i^k - V_i^*| + O(\delta^2) \ , \end{aligned} \tag{5.37}$$

where δ is the unit roundoff. Ignoring the second order terms, and assuming that $V_i^* = U_i^k(1 + a_i)$, then equation (5.37) becomes

$$\begin{aligned} [\Delta \hat{e}_\delta^k]_i &\simeq \Omega|U_i^k|(2 + |a_i|)\delta + 3|a_i|\Omega|U_i^k| \\ &= \Omega|U_i^k|(2 + 4|a_i|)\delta \ . \end{aligned} \tag{5.38}$$

Assume that $|a_i| \ll 1$ (which will be true near the exercise region) so that equation (5.38) becomes

$$[\Delta \hat{e}_\delta^k]_i \simeq 2\Omega|U_i^k|\delta \ . \tag{5.39}$$

For the penalty formulation, we obtain (5.39) but with $\Omega = 1/\varepsilon$.

For the case where $\Omega \to 0$, the floating point error in $\Delta \hat{e}^k_\delta$ (5.6) will be dominated by the term

$$\theta \frac{\sigma^2 S_i^2}{2} D^h_{SS} U_i^k \ , \tag{5.40}$$

since computing the numerical second derivative will produce the largest errors. Following a similar argument as used in the derivation of equation (5.39), we obtain

$$[\Delta \hat{e}^k_\delta]_i \ \simeq \ 2\theta\delta \frac{\sigma^2 S_i^2}{(\Delta S)_i^2} |U_i^k| \ , \tag{5.41}$$

where $(\Delta S)_i = \min(S_{i+1} - S_i, S_i - S_{i-1})$. For details of the derivation of equation (5.41) see [20].

5.4 Ω Large, ε small

From equation (5.39), if $\Omega \to \infty$ ($\varepsilon \to 0$), then

$$\begin{aligned} [\Delta \hat{e}^k_\delta]_i \ &\simeq \ 2\Omega\delta |U_i^k| \\ &\leq \ 2\Omega\delta \max(|U_i^k|, scale) \ . \end{aligned} \tag{5.42}$$

From Proposition 4.1, assuming that $\varepsilon = 1/\Omega = C\Delta\tau$, $C \ll 1$, we can see that the worst case for equation (5.34) will occur when $\varphi_i^k = 0$, in which case, for both penalty and direct control formulations

$$\max_i |D_i^{-1}| \leq \Delta\tau \ . \tag{5.43}$$

Substituting equations (5.42-5.43) into equation (5.34), and assuming that

$$\Omega = \frac{1}{\varepsilon} = \frac{1}{C\Delta\tau} \ , \tag{5.44}$$

we obtain

$$C \ > \ \frac{2\delta}{tolerance} \ . \tag{5.45}$$

Assuming that

$$\delta \simeq \ 10^{-16} \ \ (\text{ double precision }) \ , \tag{5.46}$$

then we obtain from equation (5.45)

$$C \ > \ \begin{cases} 2 \times 10^{-8} & tolerance = 10^{-8} \\ 2 \times 10^{-10} & tolerance = 10^{-6} \end{cases} \ . \tag{5.47}$$

This estimate is consistent with the results in Table 5.4.

5.5 Ω Small

From equation (5.41) we have that, for the scaled direct control formulation with $\Omega \to 0$,

$$
\begin{aligned}
[\Delta \hat{e}^k_\delta]_i &\simeq 2\theta\delta \frac{\sigma^2 S_i^2}{(\Delta S)_i^2} |U_i^k| \\
&\leq 2\theta\delta \frac{\sigma^2 S_i^2}{(\Delta S)_i^2} \max(|U_i^k|, scale) \ .
\end{aligned} \tag{5.48}
$$

From Proposition 4.1, for the case $\Omega \to 0$, the worst case will occur when $\varphi_i^k = 1$, so that we have

$$
\max_i |D_i^{-1}| \leq \frac{1}{\Omega} \ . \tag{5.49}
$$

Substituting equations (5.48) (5.49) into equation (5.34), and assuming that

$$
\Omega = \frac{1}{C\Delta\tau} \ , \tag{5.50}
$$

gives

$$
C < \frac{1}{\Delta\tau} \left(\frac{tolerance}{\delta} \right) \min_i \left(\frac{(\Delta S)_i^2}{2\theta S_i^2 \sigma^2} \right) \ . \tag{5.51}
$$

In addition, from equation (4.5), assuming equation (5.50) holds, then

$$
C < \frac{1}{\theta\lambda\Delta\tau} \ . \tag{5.52}
$$

Combining equations (5.51, 5.52) gives

$$
C < \min \left[\frac{1}{\theta\lambda\Delta\tau}, \frac{1}{\Delta\tau} \left(\frac{tolerance}{\delta} \right) \min_i \left(\frac{(\Delta S)_i^2}{2\theta S_i^2 \sigma^2} \right) \right] \ . \tag{5.53}
$$

5.6 A Note on Implementation

We remark here that given a penalty method implementation, it is trivial to generate a scaled direct control implementation. In this case, we can use the properties of inexact arithmetic to our advantage. The discretized equations (3.7) are used in both cases, but for the scaled direct control formulation, φ is determined from equation (3.6) (instead of equation (3.8)). In equation (3.6), we define

$$
\Omega = \frac{1}{C\Delta\tau} \tag{5.54}
$$

with C satisfying conditions (5.45) and (5.53). In equations (3.7), we set

$$
\varepsilon = C_2 \Delta\tau \ , \tag{5.55}
$$

with $C_2 = \sqrt{\epsilon_{small}}$, where ϵ_{small} is the smallest positive double precision number, e.g. $\simeq 10^{-308}$. We take the square root here to avoid any possible overflow problems. Effectively, when $\varphi = 0$, we solve the unconstrained PDE. When $\varphi = 1$, the very small ε eliminates the other terms in equation (3.7) (in finite precision arithmetic), so that this equation becomes $V_i^{n+1} = V_i^*$.

Expiry Time	.25
Exercise	American
Payoff	Butterfly
K_1, K_2	90, 110
Risk free rate r	.05
Volatility σ	.15
Jump Intensity λ	.1
Log jump mean ν	-.90
Log jump stnrd dev ζ	.45

TABLE 6.1: *Data for the an American butterfly*

.

5.7 Generality of Condition (5.34)

The condition (5.34) is general and can be applied to other HJB equations. We need the following properties to hold

- The discretized equations are of the form (4.1).
- The matrix $\mathcal{A}(Q)$ can be split as in equation (5.10).
- The result (5.32) holds.

For each specific PDE, it is only necessary to estimate $\|D^{-1}\|_\infty$ and $\Delta \hat{e}_\delta$. For an example application of condition (5.34) to a singular control problem, see [20].

6 Numerical Results: Jump Diffusion

We consider the case of an American option with jump diffusion, with the data in Table 6.1. We take the payoff to be a butterfly

$$V^* = \max(S - K_1, 0) - 2\max(S - (K_1 + K_2)/2, 0) + \max(S - K_2, 0) \ , \qquad (6.1)$$

and assume the existence of an American contract with payoff (6.1), which can only be early exercised as a unit. This contract has been used as severe test case by several authors [1, 26, 24]. In the no-jump case, the exercise region is not simply connected to the boundary, hence the direct method in [5] cannot be used (at least in straightforward fashion) and an iterative method is required. A classical iterative method is described in [8].

The variable timestep selector described in [12] is used combined with Crank Nicolson timestepping and the Rannacher modification suggested in [25]. This problem is solved on a sequence of (unequally spaced) grids. At each grid refinement, a new fine grid node is inserted between each two coarse grid nodes, and the timestep control parameter is halved. Table 6.2 shows the number of nodes and timesteps for various levels of refinement. Table 6.3 shows a convergence study for the American butterfly case, which demonstrates approximately second order convergence. The value at $t = 0$ is shown in Figure 6.1.

Refine	S Nodes	Timesteps
0	129	35
1	257	70
2	513	137
3	1025	271
4	2049	537
5	5097	1068
6	10193	2130

TABLE 6.2: *Grid/timestep data for convergence study, American butterfly. Data in Table 6.1. On each grid refinement, new fine grids are inserted between each two coarse grid nodes, and the timestep control parameter is halved.*

Refine	Itns/step	Value	Ratio
0	3.2	5.249893574	N/A
1	3.0	5.251270846	N/A
2	2.98	5.251520409	5.5
3	2.98	5.251585969	3.8
4	2.91	5.251601866	4.1
5	2.65	5.251605835	4.0
6	2.43	5.251606872	3.8

TABLE 6.3: *Convergence study, American butterfly, data in Table 6.1. Penalty formulation (3.7) used. Value at $S = 105$. Penalty parameter $\varepsilon = 10^{-6}\Delta\tau$. Crank Nicolson timestepping with the Rannacher modification used. Ratio is the ratio of successive changes in the solution.*

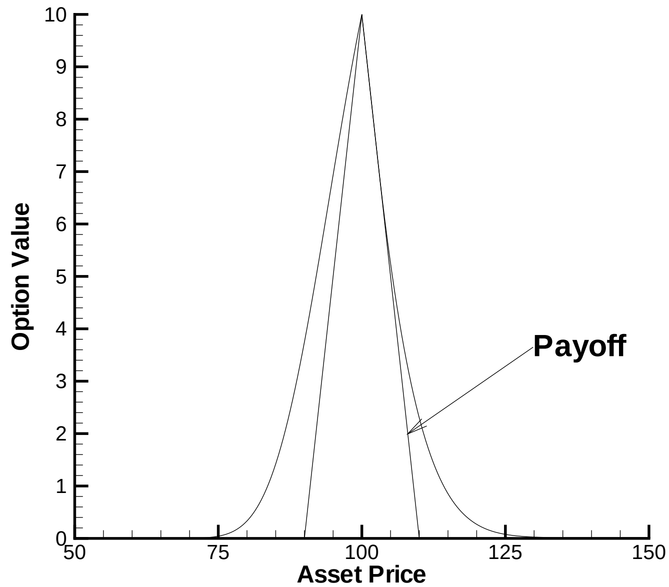

FIGURE 6.1: *American butterfly, jump diffusion. Data in Table 6.1.*

6.1 Bounds on C

The lower bounds for C for both penalty and direct control methods are given from equation (5.47).

For a level 5 discretization, we will estimate the upper bounds from the following data

$$\begin{aligned} \theta &= .5 \\ (\Delta\tau)_{\max} &= 3 \times 10^{-3} \\ \lambda &= .1 \\ \sigma &= .2 \\ \left(\frac{(\Delta S)_i}{S_i}\right)_{\min} &= 1.5 \times 10^{-4} \\ \delta &= 10^{-16}\ . \end{aligned} \tag{6.2}$$

Bound (5.52) is then

$$C \quad < \quad \frac{2}{3} \times 10^4 \simeq 10^4 \tag{6.3}$$

while bound (5.51) gives

$$C \quad < \quad \begin{cases} \frac{75}{4} \times 10^3 \simeq 10^4 & tolerance = 10^{-8} \\ \frac{75}{4} \times 10^5 \simeq 10^6 & tolerance = 10^{-6} \end{cases} . \tag{6.4}$$

Table 6.4 shows that the lower bounds for $C = \varepsilon/\Delta\tau$, from equation (5.47), $tolerance = 10^{-8}$ are fairly sharp for the penalty method, but conservative for the scaled direct control formulation. Table

6.4 also shows that the upper bound (6.4) is quite conservative for the direct control formulation. Note that for the penalty method, the consistency error becomes quite large for $C > 10^{-1}$ (see equation (2.10)), hence results for $C > 1$ are not shown. However, observe that the number of iterations per step for the scaled direct control method increases sharply for $C > 1$. Recall that condition (5.52) is sufficient but not necessary for convergence. It would seem that as we near this upper bound, the rate of convergence degrades. Nevertheless, Table 6.4 shows the remarkable fact that the computed solution for the scaled direct control formulation is unchanged (to 10 digits) for C varying by fifteen orders of magnitude.

From a practical perspective, it would seem that the lower bound for C is of primary interest. To verify that the lower bound (5.47) has the correct behaviour as a function of the convergence tolerance, Table 6.5 shows the results for $tolerance = 10^{-6}$. The observed lower bound for C does decrease, relative to the values in Table 6.4, as expected.

The largest value of C where convergence occurs in Tables 6.4 and 6.5 should be about 10^4 from equation (6.3) in both cases if the non-convergence was due to the violation of equation (5.52). However, the maximum value of C where convergence occurs increases in Table 6.5 compared to Table 6.4, which is what we expect if the non-convergence is due to floating point errors (i.e. violation of condition (5.51)). However, in both cases, non-convergence occurs at significantly larger values of C then predicted from equation (5.51).

On the basis of numerous tests, a useful rule of thumb is to select the lower bound for C to be two orders of magnitude larger than the bound (5.45). In several years of experiments, we have never seen this fail. In the case of the penalty method, this size for C produces a consistency error which is usually much smaller than the discretization error for typical grid sizes. In the case of the Direct Control formulation, this choice of C seems to minimize the number of nonlinear iterations.

Table 6.6 shows the results for the American butterfly, with tolerance 10^{-6}, and a coarser grid (refinement 3) compared to Table 6.5. In this case, we expect that the lower bound for C should be the same for both Tables, based on the bound (5.47). If the upper bound for C is determined by equation (5.51), then it is expected that the upper bound will increase for the coarser grid. We can observe this trend (approximately) in Table 6.5 and Table 6.6. However, for both Table 6.5 and Table 6.6, we remind the reader that the upper bound is not sharp. Note as well that the average number of nonlinear iterations per step decreases as the grid and timestep are refined. This trend is consistent across many tests.

7 Conclusions

Discretization of the HJB PIDE for American options under jump diffusion gives rise to a system of nonlinear algebraic equations at each timestep. If a positive coefficient discretization is used, then in exact arithmetic a fixed point policy iteration method is unconditionally convergent for a penalty formulation and conditionally convergent for a scaled direct control formulation. However, in inexact arithmetic, the fixed point policy iteration may not converge, even though the theoretical conditions are satisfied.

We have determined upper and lower bound estimates for the penalty parameter (penalty formulation) and the scaling parameter (direct control formulation) so that convergence can be expected in the presence of floating point errors. Numerical experiments show that these estimates are the correct order of magnitude. In practice, the lower bound is more important, and the expression for the lower bound estimate has a very simple form.

ε or $1/\Omega$	Direct Control		Penalty	
	Itns/step	Value	Itns/step	Value
$10^7\Delta\tau$	****	****		
$10^6\Delta\tau$	8.95	5.251605841		
$10^5\Delta\tau$	9.30	5.251605841		
$10^4\Delta\tau$	9.90	5.251605841		
$10^3\Delta\tau$	10.1	5.251605841		
$10^2\Delta\tau$	9.91	5.251605841		
$10^1\Delta\tau$	8.56	5.251605841		
$\Delta\tau$	4.65	5.251605841	2.63	5.247591885
$10^{-1}\Delta\tau$	2.75	5.251605841	2.65	5.251199230
$10^{-2}\Delta\tau$	2.46	5.251605841	2.65	5.251562864
$10^{-3}\Delta\tau$	2.46	5.251605841	2.65	5.251600928
$10^{-4}\Delta\tau$	2.46	5.251605841	2.65	5.251605297
$10^{-5}\Delta\tau$	2.46	5.251605841	2.65	5.251605786
$10^{-6}\Delta\tau$	2.46	5.251605841	2.65	5.251605835
$10^{-7}\Delta\tau$	2.46	5.251605841	2.65	5.251605841
$10^{-8}\Delta\tau$	2.46	5.251605841	****	****
$10^{-9}\Delta\tau$	2.46	5.251605841	****	****
$10^{-10}\Delta\tau$	****	****	****	****

TABLE 6.4: *Option value at $S = 105$, refinement level 5, American butterfly, data in Table 6.1. $****$ indicates failure to converge after 100 iterations in any timestep. tolerance $= 10^{-8}$.*

ε or $1/\Omega$	Direct Control		Penalty	
	Itns/step	Value	Itns/step	Value
$10^{+8}\Delta\tau$	****	****		
$10^{+7}\Delta\tau$	8.33	5.251605841		
$10^{+6}\Delta\tau$	8.83	5.251605841		
$10^{+5}\Delta\tau$	9.18	5.251605841		
$10^{+4}\Delta\tau$	9.76	5.251605841		
$10^{+3}\Delta\tau$	9.95	5.251605841		
$10^{+2}\Delta\tau$	9.81	5.251605841		
$10^{+1}\Delta\tau$	8.42	5.251605841		
$\Delta\tau$	4.48	5.251605841		
$10^{-1}\Delta\tau$	2.49	5.251605841	2.19	5.251199230
$10^{-2}\Delta\tau$	2.12	5.251605841	2.30	5.251562864
$10^{-3}\Delta\tau$	2.12	5.251605841	2.31	5.251600928
$10^{-4}\Delta\tau$	2.12	5.251605841	2.33	5.251605297
$10^{-5}\Delta\tau$	2.12	5.251605841	2.33	5.251605786
$10^{-6}\Delta\tau$	2.12	5.251605841	2.33	5.251605835
$10^{-7}\Delta\tau$	2.12	5.251605841	2.33	5.251605840
$10^{-8}\Delta\tau$	2.12	5.251605841	2.33	5.251605841
$10^{-9}\Delta\tau$	2.12	5.251605841	2.33	5.251605841
$10^{-10}\Delta\tau$	2.12	5.251605841	****	****
$10^{-11}\Delta\tau$	****	****	****	****

TABLE 6.5: *Option value at $S = 105$, refinement level 5, American butterfly, data in Table 6.1. $****$ indicates failure to converge after 100 iterations in any timestep. tolerance $= 10^{-6}$.*

ε or $1/\Omega$	Direct Control		Penalty	
	Itns/step	Value	Itns/step	Value
$10^{+11}\Delta\tau$	****	****		
$10^{+10}\Delta\tau$	3.62	5.251585989		
$10^{+9}\Delta\tau$	3.62	5.251585989		
$10^{+8}\Delta\tau$	3.62	5.251585989		
$10^{+7}\Delta\tau$	3.66	5.251585989		
$10^{+6}\Delta\tau$	3.94	5.251585989		
$10^{+5}\Delta\tau$	4.11	5.251585989		
$10^{+4}\Delta\tau$	4.48	5.251585989		
$10^{+3}\Delta\tau$	4.78	5.251585989		
$10^{+2}\Delta\tau$	4.71	5.251585989		
$10^{+1}\Delta\tau$	4.71	5.251585989		
$\Delta\tau$	2.82	5.251585989		
$10^{-1}\Delta\tau$	2.27	5.251585989	2.46	5.249944409
$10^{-2}\Delta\tau$	2.25	5.251585989	2.43	5.251409455
$10^{-3}\Delta\tau$	2.25	5.251585989	2.43	5.251565568
$10^{-4}\Delta\tau$	2.25	5.251585989	2.43	5.251584012
$10^{-5}\Delta\tau$	2.25	5.251585989	2.43	5.251585791
$10^{-6}\Delta\tau$	2.25	5.251585989	2.43	5.251585969
$10^{-7}\Delta\tau$	2.25	5.251585989	2.43	5.251585987
$10^{-8}\Delta\tau$	2.25	5.251585989	2.43	5.251585989
$10^{-9}\Delta\tau$	2.25	5.251585989	2.43	5.251585989
$10^{-10}\Delta\tau$	2.25	5.251585989	2.43	5.251585989
$10^{-11}\Delta\tau$	****	****	****	****

TABLE 6.6: *Option value at* $S = 105$, *refinement level* 3, *American butterfly, data in Table 6.1.* $****$ *indicates failure to converge after* 100 *iterations in any timestep. tolerance* $= 10^{-6}$.

The direct control solution is very insensitive to the choice of scaling parameter, compared to the penalty formulation. However, the number of iterations per timestep required for the scaled direct control formulation does depend on the scaling parameter. As long as the direct control scaling parameter is selected within fairly large bounds, the effect on the computed solution and the number of iterations per timestep is fairly small.

The number of iterations required for solution of the nonlinear iterations for the penalty method is insensitive to the choice of the penalty parameter (see also [34]). Nevertheless, a poor choice for the penalty parameter will result in poor convergence as the grid and timestep are refined. However, with our recommended choice for the penalty parameter (two orders of magnitude larger than the lower bound estimate), the consistency error due to the finite penalty parameter is small compared to the discretization error at practical grid sizes and timesteps.

Appendix

A Matrix Form of the Discretized Equations

The discretized nonlinear equations (3.5) and (3.7) can be represented as nonlinear matrix equations. Let $\mathcal{A}, \mathcal{B}$ be $i_{max} \times i_{\max}$ matrices, and $\mathcal{C}$ be an $i_{\max}$ length vector.

A.1 Matrix Form: Direct Control

Equation (3.5) can be written in terms of matrices $\mathcal{A}, \mathcal{B}$ and vector $\mathcal{C}$ defined as operating on the $i_{\max}$ length vector U ($i < i_{\max}$)

$$\begin{aligned} [\mathcal{A}(\varphi_i^k)U]_i = [\mathcal{A}^k U]_i &= (1-\varphi_i^k)\left(\frac{U_i}{\Delta\tau} - \theta\mathcal{L}_i^h U_i\right) + \varphi_i^k \Omega\ U_\ell \\ \Big[\mathcal{B}(\varphi_i^k)U\Big]_i = [\mathcal{B}^k U]_i &= (1-\varphi_i^k)\lambda\theta\mathcal{J}_i^h U_i^{n+1} \\ \mathcal{C}(\varphi_i^k) = \quad \mathcal{C}_i^k &= (1-\varphi_i^k)\frac{V_i^n}{\Delta\tau} + \varphi_i^k \Omega\ V_i^* \\ &\quad +(1-\varphi_i^k)(1-\theta)\big[\mathcal{L}_i^h V_i^n + \lambda\mathcal{J}_i^h V_i^n\big]\ . \end{aligned} \tag{A.1}$$

A.2 Matrix Form: Penalty Method

Equation (3.7) can also be written in terms of $\mathcal{A}, \mathcal{B}$ and vector $\mathcal{C}$ defined as ($i < i_{\max}$)

$$\begin{aligned} [\mathcal{A}(\varphi_i^k)U]_i = [\mathcal{A}^k U]_i &= \frac{U_i}{\Delta\tau} - \theta\mathcal{L}_i^h U_i + \frac{\varphi_i^k}{\varepsilon} U_i \\ \Big[\mathcal{B}(\varphi_i^k)U\Big]_i = [\mathcal{B}^k U]_i &= \lambda\theta\mathcal{J}_i^h U_i^{n+1} \\ \mathcal{C}(\varphi_i^k) = \qquad \mathcal{C}_i^k &= \frac{V_i^n}{\Delta\tau} + \frac{\varphi_i^k}{\varepsilon} V_i^* + (1-\theta)\big[\mathcal{L}_i^h V_i^n + \lambda\mathcal{J}_i^h V_i^n\big]\ . \end{aligned} \tag{A.2}$$

A.3 Dirichlet Condition

At $i = i_{\max}$, we define (for both discretizations)

$$[\mathcal{A}^k U]_i = \frac{U_{i_{\max}}}{\Delta\tau}\ ; \qquad [\mathcal{B}^k U]_i = 0\ ; \qquad \mathcal{C}_i^k = \frac{V_{i_{\max}}^*}{\Delta\tau}\ . \tag{A.3}$$

B Approximation (5.32)

In this Appendix, we give a heuristic argument to show that approximation (5.32) is reasonable.

To make equation (5.32) more precise, we need the following results

(i) the rate of convergence as $h \to 0$ in equation (5.30) is required; and

(ii) the precise form of the Green's function for equation (5.26) must be known.

We will use equation (5.32) to bound the effect of floating point errors. At this point, we will now proceed in a very informal manner, to provide a non-rigorous justification of equation (5.32). Assuming that a consistent finite difference method is used in equation (3.1), then we expect that

$$\begin{aligned}\sum_j g_{i,j} f(S_j) &= \int_{(\Phi-\Phi_a)_S} G(S,\Delta\tau,S',0) f(S')\, dS' \\ &\quad + \int_0^{\Delta\tau} \int_{\partial(\Phi-\Phi_a)_S} P(S,\Delta\tau,S',\tau') f(S')\, d\tau'\, dS' + O(h)\ .\end{aligned} \tag{B.1}$$

Equation (B.1) simply states that our finite difference approximation converges at least at a first order rate to the exact solution. The Green's function for equation (5.26) for the domain $\Phi = [0,\infty)$, $\Phi_a = \emptyset$ is well known, and is given by [31]

$$G_\infty(S,\Delta\tau,S',0) = \frac{1}{\sigma S' \sqrt{2\pi(\Delta\tau)}} \exp\left(- \frac{(\log(S/S') + (r - \lambda\kappa - \sigma^2/2)\Delta\tau)^2}{2\sigma^2 \Delta\tau} \right)\ . \tag{B.2}$$

Now, the actual Green's function $G(S,\Delta\tau,S',0)$ for $S \in (\Phi - \Phi_a)$ can be written as

$$\begin{aligned} G(S,\Delta\tau,S',0) &= G_\infty(S,\Delta\tau,S',0) + (\text{ terms required for boundary conditions }) \\ &= G_\infty(S,\Delta\tau,S',0) - \int_0^{\Delta\tau} \int_{\partial(\Phi-\Phi_a)_S} P(S,\Delta\tau,S'',\tau'') G_\infty(S'',\tau'',S',0)\, d\tau''\, dS''\ .\end{aligned} \tag{B.3}$$

For any point $S \in (\Phi - \Phi_a)$, $G_\infty(S,\Delta\tau,S',0) \to \delta(S-S')$, $\Delta\tau \to 0$, hence from equation (5.31) the integral term involving the Poisson function in equation (B.3) tends to zero as $\Delta\tau \to 0$. As a result, there exists $\gamma > 0$ such that

$$\begin{aligned}&\int_{(\Phi-\Phi_a)_S} G(S,\Delta\tau,S',0) f(S')\, dS' + \int_0^{\Delta\tau} \int_{\partial(\Phi-\Phi_a)_S} P(S,\Delta\tau,S',\tau') q(S')\, d\tau'\, dS' \\ &= \int_{(\Phi-\Phi_a)_S} G_\infty(S,\Delta\tau,S',0) f(S')\, dS' + O((\Delta\tau)^\gamma)\ .\end{aligned} \tag{B.4}$$

From equation (B.2), we have that G_∞ can be made arbitrarily small by choosing a C_3 sufficiently large so that

$$|\log(S'/S)| \geq C_3 \sqrt{\Delta\tau}\ , \tag{B.5}$$

which implies that G_∞ is non-negligible only if

$$|S' - S| \quad \leq \quad C_3 S\sqrt{\Delta\tau} + O(\Delta\tau) \ . \tag{B.6}$$

Assume that $f(S) > 0$ and $f(S), f'(S)$ are bounded. Then we have that, $\forall \eta > 0$, and for $\Delta\tau$ sufficiently small, there exists $C_4(\eta)$ such that

$$\int_{(\Phi-\Phi_a)_S} G_\infty(S, \Delta\tau, S', 0) f(S')\ dS' \leq f(S) + O(C_4(\eta) S\sqrt{\Delta\tau}) + \eta \ . \tag{B.7}$$

Combining equation (B.1), equation (B.4), and (B.7) gives (for arbitrary small η)

$$\sum_j g_{i,j} f(S_j) \leq f(S_i) + O(h) + O((\Delta\tau)^\gamma) + O(C_4(\eta) S_i\sqrt{\Delta\tau}) + \eta \ . \tag{B.8}$$

Letting $f(S_i) = [\Delta\hat{e}^k_\delta]_i$, then equation (B.8) gives us

$$\sum_j g_{i,j}[\Delta\hat{e}^k_\delta]_j \quad \leq \quad [\Delta\hat{e}^k_\delta]_i + O(h) + O((\Delta\tau)^\gamma) + O(C_4(\eta) S_i\sqrt{\Delta\tau}) + \eta \ . \tag{B.9}$$

Substituting equation (B.9) into equation (5.17) gives

$$\left|\left[\Delta U^k_\delta\right]_i\right| \quad \leq \quad \|D^{-1}\|_\infty\left([\Delta\hat{e}^k_\delta]_i + O(h) + O((\Delta\tau)^\gamma) + O(C_4(\eta) S_i\sqrt{\Delta\tau}) + \eta \ ,\right) \quad ; \quad S_i \in (\Phi - \Phi_a) \tag{B.10}$$

From equation (5.18), the same result holds for $S_i \in \Phi_a$. As a result, for all $S_i \in \Phi$, (B.10) gives us

$$\begin{aligned} \max_i\left[\frac{|\ [\Delta U^k_\delta]_i\ |}{\max(|U^{k+1}_i|, scale)}\right] \quad &\leq \quad \max_i\left[\frac{\|D^{-1}\|_\infty[\Delta\hat{e}^k_\delta]_i}{\max(|U^{k+1}_i|, scale)}\right] \\ &\quad + \max_i\left[\frac{\|D^{-1}\|_\infty\big(O(h) + O((\Delta\tau)^\gamma) + O(C_4(\eta) S_i\sqrt{\Delta\tau}) + \eta\big)}{\max(|U^{k+1}_i|, scale)}\right] \\ &\simeq \quad \max_i\left[\frac{\|D^{-1}\|_\infty[\Delta\hat{e}^k_\delta]_i}{\max(|U^{k+1}_i|, scale)}\right] \ , \end{aligned} \tag{B.11}$$

which is equation (5.33).

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