Reduction of Elliptic Integrals to Legendre Normal Form

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Abstract

It is well known that any elliptic integral can be transformed into a linear combination of elementary functions and Legendre's three Elliptic functions. Methods for transforming these integrals to the Legendre form are described in numerous papers and textbooks. However, when it comes to actually designing and implementing such a reduction algorithm the existing methods require significant modification before they can be used in practical problems. As an example, in all cases these algorithms require the need for computing the roots of polynomials of arbitrary degree. Symbolic root-solving either fails or produces expressions for the roots that are unwieldy.

In this paper we describe two methods for reducing elliptic integrals to their Legendre normal form in a computer algebra system. In both approaches presented here, the factorization of the polynomials are delayed and an exact, symbolic closed form solution is computed. These exact forms can then be used for numerical evaluation to arbitrary precision using such methods as the AGM algorithm.

Key words: Legendre Normal Form, Elliptic Integral.

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1 Introduction

Let

$$E := \int_a^b R(x,y)dx \tag{1}$$

with $R(x,y) \in K(x,y)$ a rational polynomial in x and y over a field K and where $y^2 \in K[x]$ is a polynomial in x over K. The field K is assumed to be a subfield of the real numbers. We are interested in closed form solutions for such integrals in the case when y^2 has degree 3 or 4, that is, for the case of elliptic integrals.

Liouville's Principle (see for example [16, 17]) states that if the integral of an elementary function (that is a function made up of exponentials, logarithms and algebraics) can be expressed in closed form as an elementary function, then it can be expressed in a form containing only those exponential, logarithmic and algebraic quantities found in the integrand and logarithms of those quantities. This is the case, for example, when y^2 has degree 1 or 2. Elliptic integrals, however, are non-elementary. In order to describe a closed form solution of an integral that having the form (1), we must introduce three additional non-elementary quantities. Legendre determined one possible choice for these quantities by showing that any elliptic integral could be expressed in terms of three canonical elliptic integrals. These are Legendre's normal integrals of the first, second and third kinds:

$$\begin{aligned} F(x,k) &= \int_0^x \frac{1}{\sqrt{(1-k^2t^2)(1-t^2)}}dt, \\ E(x,k) &= \int_0^x \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}dt, \\ \Pi(x,n,k) &= \int_0^x \frac{1}{1-nx^2}\frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}}dt. \end{aligned}$$

It can be shown that these integrals are not elementary [12], [17, page 35–37]. However, there are cases where some integrals that appear to have the form (1) are pseudo-elliptic and degenerate to a form that can be expressed in terms of elementary functions.

Mathematical algorithms for converting an elliptic integral to a closed form solution in terms of Legendre's elliptic integrals have been known since the last century. Our interest comes from the desire to design and implement such an algorithm in a modern computer algebra system. Such systems allow both symbolic and numeric computation. In the case of numerical computation, the systems allow computations to arbitrary precisions. In addition, these systems allow for exact numerical computation over such domains as the rational numbers along with algebraic extensions of the rationals.

Ng and Polajnar [14] studied methods based on the classical approaches of Legendre, Jacobi and Weierstrass along with a more recent approach by Carlson [8, 9] based on hypergeometric functions. Their goal was the implementation to a normal form in the system Macsyma. In our case, we have chosen to reduce our integrals to Legendre Normal form. This form always exists, is well known and is easy to use for numerical evaluation once the form has been found. Our methods can also be used to implement reductions to alternate normal forms, in particular, to the Jacobi and Weierstrass forms.

In this paper we present and compare three algorithms for computing a Legendre normal form of an elliptic integral. The first is one described in most classical texts on the subject. The second algorithm has some significant improvements and is the algorithm that we have used in versions 3 and 4 of the MAPLE computer algebra system. The final algorithm will be the one in use in versions of MAPLE after release 4.

2 Design Goals

In any approach to computation that includes both design and implementation it is useful to list the goals that should be ideally met in a practical algorithm. In our case our goals are:

1: Numerical Accuracy

We can view the reduction to Legendre normal form as the first step in a hybrid symbolic/numeric algorithm for numerical approximation of elliptic integrals. Indeed, one of the main uses of a Legendre normal form is the ability to get accurate numerical approximations of such integrals. Direct methods for numerical approximation of elliptic integrals using traditional methods such as quadrature are limited because of the potential poles of the integrand. On the other hand, the Legendre elliptic integrals of the first, second and third kinds allow for efficient numerical evaluation over the complex plane. These methods, some based on the Arithmetic Geometric Mean (AGM) algorithm, are discussed in the articles by Bulrisch [3, 4, 6, 5].

2: Efficient Conversion to Normal Form

While this goal may be considered to be obvious for most readers, the methods for achieving this are somewhat unique in the symbolic/numeric environments found in computer algebra systems. In this environment efficiency is rarely measured by the number of numerical operations. In a computer algebra system the cost of each numerical operation is not constant, as is the case in environments supporting only floating point operations, since individual numbers can vary greatly in size. In this case, one wishes to reduce the size of coefficients appearing in intermediate computations. In addition, if the need for algebraic numbers are required, then the algebraic extensions should be as simple as possible. For example, arithmetic operations in the domain $Q(\sqrt{2})$ are considerably simpler than in the domain $Q(\sqrt{2+\sqrt{3+\sqrt{7}}})$.

3: Simple Final Answers

The computation of the Legendre normal form is only unique up to a Landen transform. As such there are many choices for the final answer. Clearly it is best to try for the simplest possible answer that can be found. This is also very important for design goal one, since simpler expressions are less prone to numerical inaccuracies when individually approximated and then combined. In addition, simply using or obtaining intuition from the forms is best done with the simplest form possible. In this case one can use the measure that elementary answers are simpler than non-elementary answer, non-elementary answers should have either the fewest possible Legendre integrals or else Legendre functions having the simplest arguments if possible. As an example, using the algorithm from section 5 as implemented in MAPLE gives the answer

$$\int_0^{\frac{1}{2}} \frac{\sqrt{1+x^4}}{1-x^4} dx = -\frac{\sqrt{2}}{4} \arctan(\frac{\sqrt{17}\sqrt{2}}{4}) + \frac{\sqrt{2}}{8} \ln(\frac{4\sqrt{17}\sqrt{2}}{9} + \frac{25}{9}) + \frac{\pi\sqrt{2}}{8}.$$

The classical algorithm of the next section and the algorithm from section 4 produce an answer having on the order of approximately 100 lines of output. In both cases the answers are equivalent to that given above, but any recognizable structure or form is lost.

4: Answers for Symbolic Input

Texts on elliptic integration often give a large set of tables for results of normal form computations of elliptic integrals. These texts rarely use individual values for the coefficients of y^2. Rather the tables have input entries of the form

$$\int_a^u \frac{x^2}{\sqrt{(x-a)(x-b)(x-c)(x-d)}} dx$$

with qualifications such as for example $a < b < u \leq c < d$, that is, integrands with symbolic data as input. One of the uses of software for reduction of elliptic integrals to normal form is to provide examples for input entries of a form similar to the above but which cannot be found in existing tables. Since there are computer algebra systems which allow for adding assumptions to variables, any practical algorithm should also manipulate elliptic integrals having symbolic entries.

The algorithms presented in the next three sections are all mathematically correct. However, only the last algorithm successful satisfies goals 1 to 4.

3 Classical Algorithm

Every integral of the form (1) can be rationalized into the form

$$\int R(x,y)dx = \int \frac{r_1(x) + r_2(x)y}{r_3(x) + r_4(x)y} = \int R_0(x)dx + \int \frac{R_1(x)}{y} dx \tag{2}$$

where $R_0(x)$ and $R_1(x)$ are rational functions. Since the integral of a rational function always has a closed form solution in terms of elementary functions [12], it only remains to determine closed form solutions of the form $\int \frac{R_1(x)}{y} dx$ with $R_1(x) \in K(x)$ a rational function.

If

$$\begin{aligned} y^2 &= a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 \\ &= b_0(x-c)^4 + b_1(x-c)^3 + b_2(x-c)^2 + b_3(x-c) + b_4 \end{aligned}$$

where c is any constant, then the integrals

$$I_s = \int \frac{x^s}{y} dx \text{ and } J_s = \int \frac{1}{(x-c)^s y} dx$$

satisfy the recurrence relations

$$x^s y = (s+2)a_0 I_{s+3} + \frac{1}{2}a_1(2s+3)I_{s+2} + a_2(s+1)I_{s+1} + \frac{1}{2}a_3(2s+1)I_s + sa_4 I_{s-1} \quad (3)$$

and

$$\frac{1}{(x-c)^{(s-1)}y} = (3-s)b_0 J_{s-4} + \frac{b_1}{2}(5-2s)J_{s-3} + b_2(2-s)J_{s-2} + \frac{b_3}{2}(1-2s)J_{s-1} - (s-1)b_4 J_s \quad (4)$$

for $s = 0, 1, \ldots$. Decomposing the rational function $R_1(x)$ into a full partial fraction expansion (that is, with linear factors) and using the above two recurrence relations shows that such integrals can always be written in terms of a combination of rational expressions of x and y along with the integrals

$$\int \frac{1}{y}dx, \int \frac{x}{y}dx, \int \frac{x^2}{y}dx \text{ and } \int \frac{1}{(x-c)y}dx.$$

Classical algorithms all follow this central idea in one form or another - combine full partial fraction with the recurrences (3) and (4) to reduce an elliptic integral to normal form. One form of the classical algorithm that is described in numerous texts [1, 7, 10, 13, 15] is given as follows:

Classical Algorithm

Input: An expression of the form $\int_a^b \frac{R_1(x)}{y}dx$ with y^2 of degree 3 or 4.

Output: A Legendre Normal Form for the elliptic integral.

1: Remove odd terms in radicand (c.f. [1])

Determine an invertible transformation of the form

$$x = \frac{rt+s}{ut+v} \text{ such that } \frac{dx}{y} = g\frac{dt}{z} \quad (5)$$

where $z^2 = a_0(1 \pm mt^2)(1 \pm mt^2)$ and a_0, m, n are all real quantities. The particular transformation components are determined from the root structure of y^2.

This reduces the problem to converting

$$\int \frac{R_1(x)}{y}dx = \int \frac{R_2(t)}{z}dt,$$

with $R_2(t)$ a rational function of t, to normal form.

2: Remove Odd Terms

Writing $R_2(t) = tR_3(t^2) + R_4(t^2)$ with $R_3(t^2)$ and $R_4(t^2)$ rational functions of t^2, gives

$$\int \frac{R_2(t)}{z}dt = \frac{1}{2}\int \frac{R_3(u)}{\sqrt{a_0(1 \pm mu)(1 \pm nu)}}du + \int \frac{R_4(t^2)}{z}dt$$

The second integral uses the substitution $u = t^2$ and can further be given in terms of elementary functions via trigonometric substitutions.

3: Final Forms for Square Root (c.f. [7])

Apply an invertible transformation of the form

$$t^2 = \frac{rx^2+s}{ux^2+v} \tag{6}$$

which converts the component

$$\int \frac{R_4(t^2)}{\sqrt{a_0(1 \pm mt^2)\cdot(1\pm nt^2)}}dt$$

into a sum of one or more integrals of the form

$$\int \frac{R_5(x^2)}{\sqrt{(1-x^2)(1-k^2x^2)}}dx.$$

with k^2 real and $0 < k^2 < 1$ for each such summand. In each case, the transformation (6) depends both on the form of the terms under the square root $a_0(1 \pm mt^2)\cdot(1\pm nt^2)$ and the limits of integration. A complete description of these transforms can be found in numerous texts (c.f.[7]).

4: Full Partial Fraction Decomposition

For each integral of the form

$$\int \frac{R_5(x^2)}{\sqrt{(1-x^2)\cdot(1-k^2x^2)}}dx.$$

let

$$R_5(x) = p(x) + \sum_{i=1}^{n}\sum_{j=1}^{i} \frac{c_{ij}}{(x-r_i)^j}$$

be a full partial fraction decomposition of $R_5(x)$. Then (with $w^2 = (1-x^2)\cdot(1-k^2x^2)$)

$$\int \frac{R_5(x^2)}{w}dx = \int \frac{p(x^2)}{w}dx + \sum_{i=1}^{n}\sum_{j=1}^{i} c_{ij} \int \frac{1}{(x^2-r_i)^j w}dx$$

5: Final Forms

Using recurrence (3), reduces $\int \frac{p(x^2)}{w}dx$ to Legendre elliptic integrals of the first and second kinds. Using recurrence (4), reduces each $\int \frac{1}{(x^2-r_i)^j w}dx$ to a combination of Legendre's integrals of the first, second and third kinds along with some elementary terms. This gives a normal form for elliptic integrals.

Note that one can alter the order of the steps of the classical algorithm and still compute a normal form. For example many algorithms first compute the reduction to the four basic forms I_0, I_1, I_2 and J_α, then eliminate the odd parts, the transform to modulus and characteristic form.

4 Algorithm 1

In this section we present a slightly altered algorithm for reduction to normal form. This algorithm improves the algorithm of the previous section by first using Hermite reduction to reduce the problem to the case where the denominator of the rational function is square-free, that is, to the case where there are no multiple poles. In addition, the new algorithm also addresses an obvious limitation of the classical algorithm, namely the need (in step 4) for a a full partial fraction decomposition of a rational function. This problem is handled by the use of *implicit* full partial fraction decompositions which avoids the need for splitting fields. It has the effect of delaying any need for explicit factorizations.

4.1 Hermite Reduction

If $R(x, y) \in K(x, y)$, then Hermite reduction computes

$$\int R(x,y)dx = A(x,y) + \int \frac{p(x)}{y}dx + \int \frac{R_1(x)}{y}dx$$

where $A(x, y) \in K(x, y)$, $p(x) \in K[x]$ is a polynomial and $R_1(x) \in K(x)$ is a rational function having a numerator of smaller degree than its denominator and having a denominator without any repeated factors, that is, having a square-free denominator. The important point is that Hermite reduction is done using only polynomial operations over the original domain of coefficients K (such as polynomial division or gcds in $K[x]$). In the case of

$$\int \frac{R(x)}{\sqrt{x^4+3x^2+2}}dx \tag{7}$$

where

$$R(x) = \frac{14x^{12} + 4x^{10} + 149x^8 - 228x^6 + 102x^4 - 93x^2}{(x^6 + x^4 - 3x^2)^2} \tag{8}$$

Hermite reduction gives

$$\int \frac{R(x)}{y}dx = \frac{(-\frac{2287}{13}x^3 + \frac{2971}{13}x)y}{x^4+x^2-3} + \int \frac{\frac{2287}{13}x^2 + \frac{5258}{13}}{y}dx + \int \frac{-\frac{5912}{13}x^2 + \frac{9299}{13}}{(x^4+x^2-3)y}dx. \tag{9}$$

There is no polynomial factorization involved in the above computation.

By (2) we can assume that our integrand is of the form $\frac{R(x)}{y}$ with $R(x) \in K(x)$. Let $R(x) = \frac{a(x)}{b(x)}$ with $a(x), b(x) \in K[x]$ and $\gcd(a(x), b(x)) = 1$ and let $b(x) = b_1(x)b_2(x)^2 \cdots b_k(x)^k$ be the square-free decomposition of $b(x)$. A square-free factorization separates repeated components of a polynomial. It does not require polynomial factorization, rather uses only polynomial gcd operations (c.f. [12]). We can further separate our factors, again using only polynomial operations, into $b_i(x) = c_i(x) \cdot \hat{c}_i(x)$ where $gcd(c_i(x), z(x)) = 1$ and $gcd(\hat{c}_i(x), z(x)) = \hat{c}_i(x)$ with $z(x) = y^2 \in K[x]$. A partial fraction decomposition then gives

$$R(x) = \sum_{i=1}^{k}\sum_{j=1}^{i} \frac{a_{ij}}{c_i(x)^j} + \sum_{i=1}^{k}\sum_{j=1}^{i} \frac{\hat{a}_{ij}}{\hat{c}_i(x)^j}$$

so

$$\int \frac{R(x)}{y} dx = \sum_{i=1}^{k} \sum_{j=1}^{i} \int \frac{a_{ij}}{c_i(x)^j y} dx + \sum_{i=1}^{k} \sum_{j=1}^{i} \int \frac{\hat{a}_{ij}}{\hat{c}_i(x)^j y} dx.$$

In all cases the above computations, from square-free factorization to partial fraction decomposition, are done using the Extended Euclidean Algorithm in $K[x]$. Hermite reduction then relies on the recurrences

$$\int \frac{a(x)}{c(x)^r y} dx = \frac{-v(x)y}{(r-1)c(x)^{(r-1)}} + \int \frac{\hat{a}(x)}{c(x)^{(r-1)} y} dx$$

with $\hat{a}(x) = u(x) + \frac{1}{2(r+1)}(2v'(x)y^2 + v(x)z'(x))$ and where $u(x)$ and $v(x)$ are solutions to the linear diophantine equation

$$a(x) = c(x)u(x) + z(x)c'(x)v(x)$$

which always exists since $gcd(c(x), z(x)c'(x)) = 1$ and (setting $w(x) = z(x)/\hat{c}(x)$)

$$\int \frac{\hat{a}(x)}{\hat{c}(x)^r y} dx = \frac{\hat{v}(x)y}{\hat{c}(x)^r} + \int \frac{\hat{u}(x) - \hat{v}'(x)w(x)}{\hat{c}(x)^{(r-1)} y} dx$$

where $\hat{u}(x)$ and $\hat{v}(x)$ are solutions to the linear diophantine equation

$$\hat{a}(x) = \hat{c}(x)\hat{u}(x) + (z'(x)/2 - rw(x)\hat{c}'(x))\hat{v}(x)$$

Again this always exists since $gcd(\hat{c}(x), z'(x)/2 - rw(x)\hat{c}'(x))) = 1$.

4.2 Implicit Full Partial Fraction Decompositions

Classical algorithms, as presented in the previous section, have a number of limitations when actually implemented in a computer algebra environment. The most obvious problem is that of step 4, the reduction of the rational function to a full partial fraction decomposition. At first glance there are two options. We can factor the denominator of the rational function into numerically approximate linear terms. This has two significant problems. The first problem is that there will still be a number of steps (for example, using recurrence (4)) which will require further arithmetic with the approximate roots. This introduces unwanted numerical inaccuracies into the reduction to normal form computation. A second problem is that the numerical root approximation will be done at the present numerical precision setting. If a second numerical precision is specified, say for complete numerical evaluation of an elliptic integral, then the entire reduction to normal form will need to be redone.

A second option is to factor the denominators into linear factors. If this is not possible in the base field K then we use splitting fields of polynomials. In most cases this becomes highly impractical. For example, the integral

$$\int \frac{-90\,x^{16} - 96\,x^{12} + 96\,x^{8} + 90\,x^{4}}{(9\,x^{20} + 51\,x^{16} + 109\,x^{12} - 3\,x^{10} + 109\,x^{8} + 51\,x^{4} + 9)\sqrt{x^4+1}} dx$$

would require determining the splitting field of a polynomial of degree 20, a field that could possibly require $20! = 2,432,902,008,176,640,000$ variables to represent. Even when it is possible to obtain a full linear factorization the results are often unwieldy, difficult and inefficient to use. For example, the rational function $R(x)$ appearing in example (7) has a denominator with linear roots given by

$$\left\{0, \frac{\sqrt{-2+2\sqrt{13}}}{2}, -\frac{\sqrt{-2+2\sqrt{13}}}{2}, \frac{\sqrt{-2-2\sqrt{13}}}{2}, -\frac{\sqrt{-2-2\sqrt{13}}}{2}\right\}$$

all lying in the field $Q(u,v)$ with $u^2 = 13, v^2 = 2-2u$. The full partial fraction decomposition of $R(x)$ is then computed over this field extension. It has 10 terms, including 8 of a form such as

$$-\frac{1}{1014}\,\frac{\sqrt{-2+2\sqrt{13}}\left(-10829+6823\sqrt{13}\right)}{2\,x+\sqrt{-2+2\sqrt{13}}}.$$

Applying the remaining steps of the classical algorithm to each component involves significant computation with nested square roots.

By the previous subsection we can consider the problem to have already been reduced to the case where there are no repeated poles, that is, we now only have the problem of reducing $\int \frac{R_1(x)}{y}dx$ where the denominator of $R_1(x)$ is square-free. In this case we need a full partial fraction decomposition, something which appears to destroy all the computational gains made by Hermite reduction. We overcome this problem by using an *implicit* rather than an explicit full partial fraction expansion [2]. An implicit partial fraction decomposition for a rational function $R_1(x)$ having no multiple poles returns an expression of the form

$$R_1(x) = \frac{A(x)}{B(x)} = \sum_{B(\alpha)=0} \frac{a(\alpha)}{x-\alpha}$$

where $a(\alpha)$ is a polynomial in the indeterminant α. In the square-free case, which is our main interest here, the $a(\alpha)$ are easily found by $a(\alpha) = A(\alpha)/B'(\alpha)$. This is computed by solving the linear diophantine equation $A(\alpha) = B(\alpha)U(\alpha) + B'(\alpha)V(\alpha)$ using the Extended Euclidean Algorithm (a solution always exists since $B(\alpha)$ is square-free) and then setting

$$V(\alpha) = A(\alpha)\cdot(B'(\alpha))^{-1} \bmod B(\alpha).$$

For example, the remaining rational function from (9) reduces to implicit form via

$$\frac{-\frac{5912}{13}x^2 + \frac{9299}{13}}{x^4+x^2-3} = \sum_{\alpha|\alpha^4+\alpha^2-3=0} \frac{-\frac{26173}{1014}\alpha^3 + \frac{47357}{1014}\alpha}{x-\alpha}. \tag{10}$$

4.3 The Algorithm

Using the implicit full partial reduction reduces the problem to one of computing the normal form of expressions of the form

$$\int_a^b \frac{1}{y}dx, \int_a^b \frac{x}{y}dx, \int_a^b \frac{x^2}{y}dx, \int_a^b \frac{1}{(x-\alpha)y}dx$$

where α is either an indeterminant or a rational number. The algorithm then keeps track of the remaining reductions for these 4 cases. The reductions follow the same transforms as used in the classical algorithm.

Algorithm 1:

Input: $\int_a^b R(x,y)dx$ with $R(x,y) \in K(x,y)$ and $y^2 \in K[x]$ of degree 3 or 4.

Output: A Legendre Normal Form for $\int_a^b R(x,y)dx$.

1: Use (2) to rationalize the integrand and Hermite reduction to reduce the problem to determining the normal form for an integral of the form $\int_a^b \frac{R_1(x)}{y}dx$ with $R_1(x)$ a rational function having a square-free denominator.

2: Implicit Full Partial Fraction Decomposition

Determine a full partial fraction decomposition of $R_1(x)$. This decomposition has explicit roots (case where $\alpha \in K$) and implicit roots.

3: Reduce Polynomial Parts

Use recurrence (3) to reduce all polynomial expressions to ones of degree at most 2,

At this stage it remains to convert

$$I_0 = \int_a^b \frac{1}{y}dx, \quad I_1 = \int_a^b \frac{x}{y}dx, \quad I_2 = \int_a^b \frac{x^2}{y}dx, \text{ and } J_\alpha = \int_a^b \frac{1}{(x-\alpha)y}dx,$$

into their normal forms. We treat α as an indeterminant. The remaining conversion to normal form is then done as follows:

4: Remove odd terms in radicand

Determine an invertible transformation of the form (5) such that

$$\frac{1}{y}dx = \frac{c}{z}dt$$

with c a constant, $z^2 = a_0(1 \pm mt^2)(1 \pm mt^2)$ and a_0, m, n all real quantities. This converts

$$I_0 = c\int \frac{1}{z}\,dt, \quad I_1 = c\int \frac{(rt+s)}{(ut+v)z}\,dt,$$

$$I_2 = c\int \frac{(rt+s)^2}{(ut+v)^2 z}\,dt, \qquad J_\alpha = c\int \frac{(\bar{r}(\alpha)t + \bar{s}(\alpha))}{(\bar{u}(\alpha)t + \bar{v}(\alpha))z}\,dt.$$

5: Remove Odd Terms

For each of I_1, I_2 and J_α separate into odd and even parts. The odd parts are handled via the $u = t^2$ substitution and result in elementary answers. The remaining even rational functions in I_1, I_2 and J_α have numerator and denominator degrees at most $(1,1)$, $(2,2)$ and $(1,1)$, respectively, in t^2.

6: Final Forms for Square Root

Apply an invertible transformation of the form (6) which converts the even components of I_0, I_1, I_2 and J_α into ones of the form

$$\int_a^b \frac{R(x^2)}{w}dx \text{ or } \int_a^b \frac{S(x^2,\alpha)}{w}dx$$

with $w^2 = (1-x^2)(1-k^2x^2)$, $R(x^2)$ a rational function in x^2 of with numerator and denominator degrees at most 2 in x^2 and $S(x^2,\alpha)$ a rational function in x^2 of with numerator and denominator degrees at most 1 in x^2.

7: Final Normal Form

Convert the integrals of the form

$$\int \frac{(a_4x^4 + a_2x^2 + a_0)}{(b_4x^4 + b_2x^2 + b_0)w}dx \text{ and } \int \frac{(a_2(\alpha)x^2 + a_0(\alpha))}{(b_2(\alpha)x^2 + b_0(\alpha))w}dx$$

into their normal forms. Special cases occur when 0, 1 or $1/k$ is a root of the denominator. The remaining cases are handled with a full partial factorization, a simple problem in these cases because of the low degrees of the rational functions.

As an example, consider $R(x)$ and y given previously by (7) and (8). Hermite reduction gives (9) while step 4 is the trivial substitution $x = t$. We separate odd and even parts of J_α by

$$J_\alpha = \int_0^1 \frac{A(\alpha)x}{(x^2-\alpha^2)y}dx + \int_0^1 \frac{A(\alpha)\alpha}{(x-\alpha^2)y}dx$$

and determine that

$$\int_0^1 \frac{A(\alpha)x}{(x^2-\alpha^2)y}dx = \frac{A(\alpha)}{\sqrt{\alpha^4+1}} \cdot (-arctanh(\frac{(1+\alpha^2}{\sqrt{2}\sqrt{\alpha^4+1}}) + arctanh(\frac{1}{\sqrt{\alpha^4+1}}))$$

with $A(\alpha) = -26173\alpha^3 + 47357\alpha$. Therefore the complete odd part of the integral is the sum of such terms

$$\sum_{\alpha|\alpha^4+\alpha^2-3=0} \frac{A(\alpha)}{\sqrt{\alpha^4+1}} \cdot (-arctanh(\frac{1+\alpha^2}{\sqrt{2}\sqrt{\alpha^4+1}}) + arctanh(\frac{1}{\sqrt{\alpha^4+1}}))$$

Step 5 is the substitution

$$t^2 = \frac{x^2}{x^2+1}$$

which transforms I_2 according to

The final answer is given by

$$\begin{aligned}\int_0^1 \tfrac{R(x)}{y}dx \quad &= \quad -\tfrac{684}{13}\sqrt{6} - \tfrac{3890}{39}\sqrt{2} \cdot F(\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}) + \tfrac{2469}{26}\sqrt{2} \cdot \Pi(\tfrac{1}{\sqrt{2}}, 1, \tfrac{1}{\sqrt{2}}) \\ &\quad + \textstyle\sum_{\alpha|\alpha^4+\alpha^2-3=0}(-\tfrac{4085}{338}\sqrt{2}\alpha^2 + \tfrac{26173}{2028}\sqrt{2}) \cdot \Pi(\tfrac{1}{\sqrt{2}}, \tfrac{\alpha^2}{3} + \tfrac{4}{3}, \tfrac{1}{\sqrt{2}}).\end{aligned}$$

5 Algorithm 2

The altered algorithm of the previous section is limited in its effectiveness when implemented in a computer algebra environment. In this case there are again two significant problems. The first occurs as a result of removing the odd terms in the radical. The second is the need for explicitly determining the roots of the degree 3 or 4 polynomial underneath the radical sign.

The problem with removing the odd terms in the radical (step 4 of the previous algorithm) is that it introduces a new radical into the coefficient domain. This in turn often results in a significant problem when the polynomial under the radical has symbolic coefficients. For example, when this polynomial is of the form $\sqrt{(x-r_1)(x-r_2)(x-r_3)(x-r_4)}$ where the symbols have the added assumptions that $r_1 < r_2 < r_3 < r_4$, then the transformation (5) is of the form $t = (rx+s)/(x+1)$ with r and s determined by solutions of the equations

$$r + s = -2\frac{r_1 r_2 - r_3 r_4}{r_1 + r_2 - r_3 - r_4}, \quad r \cdot s = \frac{r_1 r_2 (r_3 + r_4) - r_3 r_4 (r_1 + r_2)}{r_1 + r_2 - r_3 - r_4}$$

as long as $r_1 + r_2 \neq r_3 + r_4$ and $t = x - \frac{r_1+r_2}{2}$, otherwise [7, pp96–97]. In the first case the values of r and s involve radicals of the form $\sqrt{(r_1-r_3)(r_1-r_4)(r_2-r_3)(r_2-r_4)}$. All remaining computations need then be done over the domain $K(u)$ where $u^2 = (r_1 - r_3)(r_1 - r_4)(r_2 - r_3)(r_2 - r_4)$. This causes significant problems in the transformation from the even terms $a_0(1 \pm mt^2)(1 \pm nt^2)$ under the radical to the normal form itself since these transformations require sign information for expressions involving a_0, m, n along with the limits of integration.

5.1 Direct Transformations

We overcome the problems induced from adding new radicals by removing odd terms by first separating intervals and then determining the *direct* transformations that will provide the normal forms. These direct transformations are then separated into the cases where

1. $y^2 = a(x^2 - b^2)(x^2 - c^2)$ with a, b, c real,
2. $y^2 = a(x^2 - b)(x^2 - \bar{b})$ with b complex but not real
3. y^2 quadratic with four real roots
4. y^2 quadratic with two real roots
5. y^2 quadratic with zero real roots
6. y^2 cubic with three real roots
7. y^2 cubic with one real root

In each case, intervals of integration $[a_i, b_i]$ are determined from the polynomial (in cases 1 and 2) or the root structure of the polynomial under the radical (in cases 3 - 7). The direct transforms used in cases 3 to 7 are given in tables 1 to 4 of the appendix.

Using these direct transformations it is possible to reduce the expressions $\int_a^b \frac{1}{y} dx$, $\int_a^b \frac{x}{y} dx$, $\int_a^b \frac{x^2}{y} dx$ and $\int_a^b \frac{1}{(x-\alpha)y} dx$. into terms that are easily placed into their normal form.

$$
\begin{aligned}
\int_a^b \frac{1}{y} dx &= \sum_{i=1} c_i \int_{a_i}^{b_i} \frac{dt}{\sqrt{(1-t^2)(1-k_i^2t^2)}} \\
\int_a^b \frac{x}{y} dx &= \sum_{i=1} d_i \int_{a_i}^{b_i} \frac{r_i(t^2)dt}{\sqrt{(1-t^2)(1-k_i^2t^2)}} + \sum_{i=1} s_i(t)|_{a_i}^{b_i} \\
\int_a^b \frac{x^2}{y} dx &= \sum_{i=1} \bar{d}_i \int_{a_i}^{b_i} \frac{\bar{r}_i(t^2)dt}{\sqrt{(1-t^2)(1-k_i^2t^2)}} + \sum_{i=1} \bar{s}_i(t)|_{a_i}^{b_i} \\
\int_a^b \frac{1}{(x-\alpha)y} dx &= \sum_{i=1} \int_{a_i}^{b_i} \frac{u_{1,i}(t^2,\alpha)dt}{\sqrt{(1-t^2)(1-k_i^2t^2)}} + \sum_{i=1} \int_{a_i^2}^{b_i^2} \frac{u_{2,i}(t^2,\alpha)dt}{\sqrt{(1-t)(1-k_i^2t)}} \\
&\quad + \sum_{i=1} \int_{a_i}^{b_i} \frac{u_{3,i}(t^2,\alpha)dt}{\sqrt{(1-k_i^2t^2)}} + \sum_{i=1} \int_{a_i^2}^{b_i^2} \frac{u_{4,i}(t^2,\alpha)dt}{\sqrt{(1-k_i^2t)}} \\
&\quad + \sum_{i=1} \int_{a_i}^{b_i} \frac{u_{5,i}(t^2,\alpha)dt}{\sqrt{(1-t^2)}} + \sum_{i=1} w_i(t,\alpha)|_{t=a_i}^{b_i}.
\end{aligned}
$$

The even functions that appear in the 4 cases all have degrees at most $(2,2)$ in t^2. In the first three cases the reductions simply separate the even components from the components that result in elementary answers. In the last case the integrals depend on whether or not the α is explicitly or implicitly given. In this case the last three integrals all compute to elementary answers while the first reduces to Legendre's elliptic integrals of the first, second or third kinds.

As an example, if $y^2 = x^4 - 3x^2 + 2$, with four real roots then the (in this case very simple) change of variables will result in the reductions

$$
\begin{aligned}
\int_0^{1/2} \frac{1}{\sqrt{y}} dx &= \frac{1}{2}\sqrt{2} \int_0^{1/2} \frac{1}{\sqrt{(1-x^2)\left(1-\frac{1}{2}x^2\right)}} dx \\
\int_0^{1/2} \frac{x}{\sqrt{y}} dx &= \frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}\, arccosh(3-2x^2) \\
\int_0^{1/2} \frac{x^2}{\sqrt{y}} dx &= \frac{1}{2}\sqrt{2} \int_0^{1/2} \frac{x^2}{\sqrt{(1-x^2)\left(1-\frac{1}{2}x^2\right)}} dx
\end{aligned}
$$

$$
\begin{aligned}
\int_0^{1/2} \frac{1}{(x-a)\sqrt{y}} dx &= -\frac{1}{\sqrt{2}} \int_0^{1/2} \frac{a}{(a^2-x^2)\sqrt{(1-x^2)\left(1-\frac{1}{2}x^2\right)}} dx \\
&\quad + \frac{1}{2\sqrt{2}} \int_0^{1/4} \frac{1}{(a^2-x)\sqrt{(1-x)\left(1-\frac{1}{2}x\right)}} dx
\end{aligned}
$$

5.2 Use of Implicit Roots

Let $y^2 = -85\,x^4 - 55\,x^3 - 37\,x^2 - 35\,x + 97$ (the result of asking MAPLE to generate a random degree 3 polynomial).Then using Algorithm 1 for the reduction to normal form of the simple expression $\int_0^1 \frac{1}{y} dx$ is not possible using the average computing power of todays computers. The reason has to do with the roots and the transforms used by the roots. However, Algorithm 2 also works with roots at some stage and then gets bogged down. In our case we can still work with these roots implicitly to generate an answer given in terms of the roots themselves. Note that such information as the number of real roots can easily be determined by using Sturm sequences. In the case of MAPLE the solve function returns a $RootOf$ (representing an algebraic number) if it determines that the nested depth of the radicals inside the answer is more than 2, a practical boundary for our computations to begin using implicit roots.

5.3 The Algorithm

Algorithm 2:

Input: An expression of the form $\int_a^b R(x, y)dx$ with $R(x, y) \in K(x, y)$ and $y^2 \in K[x]$ of degree 3 or 4.

Output: A Legendre Normal Form for $\int_a^b R(x, y)dx$.

1: Rationalize

Find rational functions $R_0(x), R_1(x) \in K(x)$ such that

$$\int_a^b R(x, y)dx = \int_a^b R_0(x)dx + \int_a^b \frac{R_1(x)}{y} dx.$$

The first integral has an answer in terms of elementary functions and it remains to compute the second integral.

2 (a): Odd and Even Parts – Heuristic

If y^2 has some odd terms then go to step 4. Otherwise, if y^2 consists only of even terms $ax^4 + bx^2 + c$ then find rational functions $R_2(x)$ and $R_3(x)$ such that

$$\int_a^b \frac{R_1(x)}{y} dx = \int_a^b \frac{x \cdot R_2(x^2)}{y} dx + \int_a^b \frac{R_3(x^2)}{y} dx.$$

Then

$$\int_a^b \frac{x \cdot R_2(x^2)}{y} dx = \frac{1}{2} \int_{a^2}^{b^2} \frac{R_2(x)}{\sqrt{ax^2 + bx + c}} dx$$

an elementary function. Therefore it only remains to compute the last even integral.

2 (b): Hermite Transformation

If $R_3(x^2)$ satisfies one of

then there is a transformation that converts $\int \frac{R_3(x^2)}{y} dx$ into an elementary integral. Otherwise go to step 4 (with $R_1(x)$ now set to $R_3(x^2)$).

3: Hermite Reduction

Determine a rational function $A(x, y) \in K(x, y)$, a polynomial $p_1(x) \in K[x]$ and a rational function $R_2(x) \in K(x)$ having a square-free denominator such that

$$\int \frac{R_1(x)}{y} dx = A(x, y) + \int \frac{p_1(x)}{y} dx + \int \frac{R_2(x)}{y} dx.$$

4: Reduce Polynomial Part

Determine a rational function $B(x, y) \in K(x, y)$, along with constants c_0, c_1 and c_2 from K such that

$$\int \frac{p_1(x)}{y} dx = B(x, y) + c_0 \int \frac{1}{y} dx + c_1 \int \frac{x}{y} dx + c_2 \int \frac{x^2}{y} dx.$$

5: Symbolic Full Partial Fractions

Determine a symbolic partial fraction decomposition of the rational function $R_2(x) = \frac{c(x)}{d_1(x) \cdots d_m(x)}$

$$R_2(x) = \sum_{i=1}^{k} \frac{a_i}{x - b_i} + \sum_{i=1}^{m} \sum_{\alpha | d_i(\alpha)=0} \frac{e(\alpha)}{x - \alpha}$$

using the algorithm of Bronstein and Salvy [2]. The algorithm uses only rational operations. The first sum represents explicit linear factors while the second sum involves the implicit linear factors.

6: Determine Transforms

Determine the subintervals $[a_i, b_i]$ and the transformations. Apply the transformations to each term of the full partial fraction decomposition along with each polynomial term.

7: Convert to Normal Form

Transform terms of the form to normal form. Since the rational functions in these cases are alll even with degree at most 2 for both numerator and denominator these can be handled via a direct full partial fraction decomposition in the variable x^2. The values 0, 1 and $1/k$ represent special cases.

6 Conclusion

In this paper we have considered the problem of computing a Legendre Normal Form for an elliptic integral. The computation should allow for easy and accurate conversion to numeric form, reduce the size of the algebraic extensions needed in such a closed form and produce answers with symbolic quantities, at least in those cases where the symbols have added assumptions. We have studied three different approaches to solving this problem, the first a classical algorithm as described in most texts on elliptic integration and the other two being variations of the classical algorithm. The successful algorithm uses implicit full partial factorization for the rational function and implicit root finding for the cubic or quartic polynomial under the radical as a major tool. These implicit forms avoid computing with nested algebraic extensions, a significant computational problem which tend to dominate the previous algorithms.

There are a number of directions for future research for computing with elliptic integrals. The computation to normal form is known to be non-unique. It is an open problem to find a normal form having a "minimal" algebraic extension, that is, a set of field extensions $c_1, \dots, c_k$ such that all the coefficients come from the field $K(c_1, \dots, c_k)$ and such that there is no smaller field of coefficients with this property. Such a property does hold in the case of indefinite integration of rational functions [12].

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7 Appendix

Table 1: $y^2 = lc(t-a)(t-b)(t-c)(t-d)$. Four real roots with $d < c < b < a$.

Interval	t substitution	x substitution	k^2
1. $-\infty \le L < U \le d$	$\frac{c(a-d)x^2-d(a-c)}{(a-d)x^2-(a-c)}$	$\sqrt{\frac{(a-c)(d-t)}{(a-d)(c-t)}}$	$\frac{(a-b)(c-d)}{(a-c)(b-d)}$
2. $d \le L < U \le c$	$\frac{a(c-d)x^2+d(a-c)}{(c-d)x^2+(a-c)}$	$\sqrt{\frac{(a-c)(t-d)}{(c-d)(a-t)}}$	$\frac{(a-b)(c-d)}{(a-c)(b-d)}$
3. $c \le L < U \le b$	$\frac{-d(b-c)x^2+c(b-d)}{-(b-c)x^2+(b-d)}$	$\sqrt{\frac{(b-d)(t-c)}{(b-c)(t-d)}}$	$\frac{(a-d)(b-c)}{(a-c)(b-d)}$
4. $b \le L < U \le a$	$\frac{-c(a-b)x^2+b(a-c)}{-(a-b)x^2+(a-c)}$	$\sqrt{\frac{(a-c)(t-b)}{(a-b)(t-c)}}$	$\frac{(a-b)(c-d)}{(a-c)(b-d)}$
5. $a \le L < U \le \infty$	$\frac{-b(a-d)x^2+a(b-d)}{-(a-d)x^2+(b-d)}$	$\sqrt{\frac{(b-d)(t-a)}{(a-d)(t-b)}}$	$\frac{(a-d)(b-c)}{(a-c)(b-d)}$

Table 2: $y^2 = lc(t-a)(t-b)(t-c)$. Three real roots with $c < b < a$.

Interval	t substitution	x substitution	k^2
1) $-\infty \le L < U \le c$	$\frac{ax^2-(a-c)}{x^2}$	$\sqrt{\frac{a-c}{a-t}}$	$\frac{a-b}{a-c}$
2) $c \le L < U < b$	$(b-c)x^2 + c$	$\sqrt{\frac{t-c}{b-c}}$	$\frac{b-c}{a-c}$
3) $b \le L < U < a$	$\frac{c(a-b)x^2-b(a-c)}{(a-b)x^2-(a-c)}$	$\sqrt{\frac{(a-c)(t-b)}{(a-b)(t-c)}}$	$\frac{b-c}{a-c}$
4) $a \le L < U \le \infty$	$\frac{bx^2-a}{x^2-1}$	$\sqrt{\frac{t-a}{t-b}}$	$\frac{b-c}{a-c}$

Table 3: $y^2 = lc(t-a)(t^2 - 2bt + (b^2+c^2))$. One real root.
Set $A = \sqrt{(b-a)^2 + c^2}$.

Interval	t substitution	x substitution	k^2
1] $-\infty \le L < U \le a - A$	$\frac{(a+A)x^2-2A-2A\sqrt{1-x^2}}{x^2}$	$\frac{2\sqrt{A(a-t)}}{a+A-t}$	$\frac{A-b+a}{2A}$
2] $a - A \le L < U \le a$	$\frac{(a+A)x^2-2A+2A\sqrt{1-x^2}}{x^2}$	$\frac{2\sqrt{A(a-t)}}{a+A-t}$	$\frac{A-b+a}{2A}$
3] $a \le L < U \le a + A$	$\frac{(a-A)x^2+2A-2A\sqrt{1-x^2}}{x^2}$	$\frac{2\sqrt{A(t-a)}}{t-a+A}$	$\frac{A+b-a}{2A}$
4] $a + A \le L < U \le \infty$	$\frac{(a-A)x^2+2A+2A\sqrt{1-x^2}}{x^2}$	$\frac{2\sqrt{A(t-a)}}{t-a+A}$	$\frac{A+b-a}{2A}$

Table 4: $y^2 = lc(t-a)(t-b)(t^2 - 2ct + c^2 + d^2)$. Two real roots, $b < a$.
Set $A = \sqrt{(a-c)^2 + d^2}$, $B = \sqrt{(b-c)^2 + d^2}$, $p_1 = B^2 a + A^2 b + (a+b)AB$,
$p_2 = bA^2 + aB^2 - (a+b)AB$.

Interval	t substitution	x substitution	k^2
1] $-\infty \le L < U \le \frac{bA-aB}{A-B}$	$\frac{p_1x^2-2(a+b)AB-2AB(a-b)\sqrt{1-x^2}}{(A+B)^2x^2-4AB}$	$\frac{2\sqrt{AB}\sqrt{(a-t)(b-t)}}{A(b-t)+B(a-t)}$	$\frac{(A+B)^2-(a-b)^2}{4AB}$
2] $\frac{bA-aB}{A-B} \le L < U \le b$	$\frac{p_1x^2-2(a+b)AB+2AB(a-b)\sqrt{1-x^2}}{(A+B)^2x^2-4AB}$	$\frac{2\sqrt{AB}\sqrt{(a-t)(b-t)}}{A(b-t)+B(a-t)}$	$\frac{(A+B)^2-(a-b)^2}{4AB}$
3] $b \le L < U \le \frac{bA+aB}{A+B}$	$\frac{p_2x^2+2(a+b)AB-2(a-b)AB\sqrt{1-x^2}}{(A-B)^2x^2+4AB}$	$\frac{2\sqrt{AB}\sqrt{(a-t)(t-b)}}{B(a-t)+A(t-b)}$	$\frac{(a-b)^2-(A-B)^2}{4AB}$
4] $\frac{bA+aB}{A+B} \le L < U \le a$	$\frac{p_2x^2+2(a+b)AB+2(a-b)AB\sqrt{1-x^2}}{(A-B)^2x^2+4AB}$	$\frac{2\sqrt{AB}\sqrt{(a-t)(t-b)}}{B(a-t)+A(t-b)}$	$\frac{(a-b)^2-(A-B)^2}{4AB}$
5] $a \le L < U \le \frac{aB-bA}{B-A}$	$\frac{p_1x^2-2(a+b)AB-2(a-b)AB\sqrt{1-x^2}}{(A+B)^2x^2-4AB}$	$\frac{2\sqrt{AB}\sqrt{(t-a)(t-b)}}{A(t-b)+B(t-a)}$	$\frac{(A+B)^2-(a-b)^2}{4AB}$
6] $\frac{aB-bA}{B-A} \le L < U \le \infty$	$\frac{p_1x^2-2(a+b)AB+2(a-b)AB\sqrt{1-x^2}}{(A+B)^2x^2-4AB}$	$\frac{2\sqrt{AB}\sqrt{(t-a)(t-b)}}{A(t-b)+B(t-a)}$	$\frac{(A+B)^2-(a-b)^2}{4AB}$

Note: If $c < \frac{a+b}{2}$ then the right and left endpoints of 1] and 2], respectively, are $-\infty$. Also, the right and left endpoints of 5] and 6] are $+\infty$.

Table 5: $y^2 = lc(t^2 - 2at + a^2 + b^2)(t^2 - 2ct + c^2 + d^2)$ with $a < c$ or $a = c$ and $b < d$.
Set $A = \sqrt{(a-c)^2 + (b+d)^2}$, $B = \sqrt{(a-c)^2 + (b-d)^2}$, $C = \frac{4(c-a)\sqrt{b^2}}{(A+B)^2 - 4b^2}$.

Interval	t substitution	x substitution	k^2
1] $-\infty \leq L < U \leq a + \frac{b}{c}$	$\frac{(a-bC)\sqrt{1-x^2}+(b+aC)x}{\sqrt{1-x^2}+Cx}$	$\frac{t-a+bC}{\sqrt{(1+C^2)((t-a)^2+b^2)}}$	$\frac{4A^2B^2}{2A^2B^2+AB(A^2+B^2)}$
2] $a + \frac{b}{c} \leq L < U \leq \infty$	$\frac{(a-bC)\sqrt{1-x^2}-(b+aC)x}{\sqrt{1-x^2}-Cx}$	$\frac{t-a+bC}{\sqrt{(1+C^2)((t-a)^2+b^2)}}$	$\frac{4A^2B^2}{2A^2B^2+AB(A^2+B^2)}$