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On simultaneous row and column reduction of higher-order linear differential systems

Moulay A. Barkatou [a], Carole El Bacha [a], George Labahn [b,1], Eckhard Pflügel [c]

[a] University of Limoges, CNRS; XLIM UMR 6172, DMI, 87060 Limoges, France
[b] Cheriton School of Computer Science, University of Waterloo, Waterloo, Ontario, Canada
[c] Faculty of SEC, Kingston University, Penrhyn Road, Kingston upon Thames, Surrey KT1 2EE, United Kingdom

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ABSTRACT

In this paper, we define simultaneously row and column reduced forms of higher-order linear differential systems with power series coefficients and give two algorithms, along with their complexities, for their computation. We show how the simultaneously row and column reduced form can be used to transform a given higher-order input system into a first-order system. Finally, we show that the algorithm can be used to compute Two-Sided Block Popov forms as given in Barkatou et al. (2010). These results extend the previous work in Barkatou et al. (2010), on second-order systems, and Harris et al. (1968), on first-order systems, to systems of arbitrary order.

Introduction

Higher-order systems of linear differential equations can be represented as an equation of the form

$$L \cdot \vec{y}(x) = \vec{f}(x), \tag{1}$$

where L is a matrix of differential operators in the variable x and the $\cdot$ denotes operator application. In our case, we are interested in the local analysis of such a system and hence the coefficients of these differential operators (and the components of the right hand side) are considered to be formal power series centered about the point $x = 0$. Such systems arise naturally in many applications of multi-body systems, models of electrical circuits, robotic modeling and mechanical systems (see Mehrmann and Shi (2006), Pantelous et al. (2009) and Schulz (2003) and the references therein).

E-mail addresses: moulay.barkatou@xlim.fr (M.A. Barkatou), carole.el-bacha@xlim.fr (C. El Bacha), glabahn@uwaterloo.ca (G. Labahn), E.Pfluegel@kingston.ac.uk (E. Pflügel).

1 Tel.: +1 519 888 4567x34667; fax: +1 519 885 1208.

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A standard technique for dealing with equations of the form (1) is to transform them to a system having the same properties but which is in a "simpler" form, one where a local analysis is more readily determined.

One method for obtaining systems equivalent to (1) is by using invertible row operations on the matrix L. As an example, the differential row Hermite form (Giesbrecht and Kim, 2009) results in a system having a triangular form and hence is useful for solving such systems. However, the result is often differential equations of order higher than the original system. A second form, the differential row Popov form (Beckermann et al., 2006), always has differential operators of order at most that of the original system. It is useful for transforming the system to one which can easily be transformed into a first-order differential–algebraic system. First-order systems readily lend themselves to singularity analysis using Moser and super-reduction algorithms (Barkatou and Pflügel, 2009). One can also obtain equivalent systems making use of both invertible row and column operations on L. The most extreme example of this is the transformation of our differential matrix to a diagonal form, called the Jacobson Normal Form (Middeke, 2008).

In the case of a first-order system a reduction using both row and column operations was used by Harris et al. (1968) for the purpose of finding analytic solutions. In the first-order case the reductions used row and column transformations only on the coefficient matrices of formal power series. An alternate method was proposed by Barkatou et al. (2010), this time requiring the need for both row and column operations on the entire matrix of differential operators. In the case of first-order systems their transformations resulted in reduction of (1) to one involving a differential system, an algebraic system and a set of conditions on the right hand side functions. They also presented a similar reduction in the case of second-order systems of equations, again using row and column operations on the entire matrix of differential operators.

In this paper we give two new algorithms for reducing systems of the form (1) by transformations on the matrix of differential operators. The methods are applicable for arbitrary order operators and generalize the first and second order methods of Barkatou, El Bacha and Pflügel though using different techniques. The approach found in Barkatou et al. (2010) follows techniques first used in Harris et al. (1968) while in the present paper we make use of reduction techniques used for matrix polynomial operations (Beckermann and Labahn, 1997; Beckermann et al., 2006). Both our algorithms reduce orders in rows and columns by making use of a series of elementary row and column block operations. The methods are also extended to handle Two-Sided Block Popov forms, a special form of a matrix of differential operators which generalizes the Popov normal form. We give a complexity analysis of our algorithms and also illustrate their use in the special case where the operator is of first-order.

While we have chosen to focus our work on matrices of differential operators with coefficients in a domain of Laurent power series $\mathrm{K}((x))$, our methods easily extend to the field of rational functions $\mathrm{K}(x)$. In fact, our algorithms are entirely algebraic involving only basic row and column operations. As such everything that we do can be done for the more general case of matrices of Ore operators having coefficients in the field of rational functions $\mathrm{K}(x)$.

The rest of the paper is organized as follows. Section 1 gives some definitions and basic properties of the matrices of differential operators with the following section detailing the concept of a simultaneously row and column reduced matrix operator and a first algorithm for its computation. In Section 3, we develop a second algorithm for the construction of a simultaneously row and column reduced form. We also introduce the notion of a Two-Sided Block Popov form for a matrix of operators and propose a procedure for its computation. Section 4 gives the complexity of our procedures while Sections 5 and 6 discuss the reduction for the case of first and higher order systems, respectively, as they apply to differential systems. The paper ends with a conclusion along with topics for future research.

Notation. For any vector of integers $\vec{\delta} = (\delta_1, \ldots, \delta_p)$, we denote by $|\vec{\delta}| = \sum_{i=1}^{p} \delta_i$. For any matrix of differential operators L we denote by $L_{i,*}$ the ith row or block row (depending on the context) of L (with $L_{*,j}$ the notation for the jth (block) column).

1. Preliminaries

Let K be an extension of the field of rational numbers ($\mathbb{Q} \subseteq \mathrm{K} \subseteq \mathbb{C}$). We denote by $\mathrm{K}[[x]]$ the ring of formal power series over K in the variable x and by $\mathrm{K}((x))$ its quotient field. Moreover, we denote by ∂

the standard derivation $\frac{d}{dx}$ of $K((x))$ and by $K[[x]][\partial]$ (resp. $K((x))[\partial]$) the ring of differential operators with coefficients in $K[[x]]$ (resp. in $K((x))$). Recall that the multiplication in $K((x))[\partial]$ satisfies the commutation rule:

$$\forall\, a \in K((x)), \quad \partial a = \frac{da}{dx} + a\partial.$$

Definition 1. Any nonzero matrix of differential operators L of size $m \times n$ can be written as

$$L = a_\ell(x)\partial^\ell + a_{\ell-1}(x)\partial^{\ell-1} + \cdots + a_0(x),$$

where $\ell \in \mathbb{N}$, for $i = 0, \ldots, \ell$, $a_i(x) \in K((x))^{m\times n}$ and $a_\ell(x) \neq 0$. The integer ℓ is called the *order* of L and is denoted by $\mathrm{ord}(L)$. The matrix $a_\ell(x)$ is called the *leading coefficient* of L and is denoted by $\ell c(L)$.

When $L = 0$, we set $\mathrm{ord}(L) = -\infty$ and $\ell c(L) = 0$.

Definition 2. Let $L \in K((x))[\partial]^{m\times n}$ and $J \subseteq \{1, \ldots, m\}$. The rows $L_{i,*}$ with index $i \in J$ are said to be $K((x))[\partial]$-*linearly dependent* if there exist differential operators $\{W_i\}_{i\in J} \subseteq K((x))[\partial]$ not all zero such that $\sum_{i\in J} W_i\, L_{i,*} = 0$; otherwise, they are said to be $K((x))[\partial]$-*linearly independent*.

Definition 3. Let $L \in K((x))[\partial]^{m\times n}$ be a matrix of differential operators. Denote by $\mathcal{M}_L$ the submodule of the left $K((x))[\partial]$-module $K((x))[\partial]^{1\times n}$ defined by $\mathcal{M}_L = \{P\,L;\ P \in K((x))[\partial]^{1\times m}\}$. The *row rank* of L is defined to be the rank of the module $\mathcal{M}_L$, *i.e.*, the cardinality of a maximal $K((x))[\partial]$-linearly independent subset of $\mathcal{M}_L$. Analogously, by working with the columns of L, we define the *column rank* of L.

It has been shown in Beckermann et al. (2006, Appendix) that the row rank of L is equal to the maximum number of $K((x))[\partial]$-linearly independent rows of L.

Since $K((x))[\partial]$ is a one-sided Euclidean domain it is a principal ideal domain. Thus we can deduce from Cohn (1971, Chapter 8, Th. 1.1) that if $L \in K((x))[\partial]^{m\times n}$ is a matrix of differential operators, then the row rank and column rank of L are equal.

Definition 4. A square matrix of differential operators $U \in K((x))[\partial]^{m\times m}$ is said to be *unimodular* if it has a two-sided inverse in $K((x))[\partial]^{m\times m}$, that is, if there exists $\widetilde{U} \in K((x))[\partial]^{m\times m}$ such that $\widetilde{U}\,U = U\,\widetilde{U} = I_m$.

In the sequel, we will denote the inverse of a unimodular matrix of differential operators U by U^{-1}.

We are primarily interested in applying elementary row operations (resp. elementary column operations) to a matrix of differential operators L. These operations are of three types:

(E1) interchanging two rows (resp. two columns);
(E2) multiplying a row (resp. a column) on the left (resp. on the right) by a nonzero element of $K((x))$;
(E3) adding to a row (resp. to a column) another one multiplied on the left (resp. on the right) by a scalar differential operator with coefficients in $K((x))$.

Each elementary row operation (resp. elementary column operation) corresponds to a left-multiplication (resp. right-multiplication) by an elementary matrix. Note that it has been shown in Miyake (1980, Theorem III) that a matrix of differential operators U is unimodular if and only if it can be expressed as a product of elementary matrices.

Lemma 1 (*Beckermann et al., 2006, Lemma A.3*)**.** *Let L be a matrix of differential operators of size $m \times n$ and U and V two unimodular matrices of differential operators of sizes $m \times m$ and $n \times n$, respectively. Then the ranks of L, $U\,L$ and $L\,V$ are all equal.*

Definition 5. Two matrices of differential operators $L, \widetilde{L} \in K((x))[\partial]^{m\times n}$ are said to be *equivalent* if there exist unimodular matrices $U \in K((x))[\partial]^{m\times m}$ and $V \in K((x))[\partial]^{n\times n}$ such that $\widetilde{L} = U\,L\,V$.

2. Two-sided reduced matrix differential forms

In the case of a scalar differential operator the leading coefficient (the highest nonzero coefficient) plays an important role in a number of tasks, for example singularity analysis and finding local solutions. In the case of a matrix of differential operators this is complicated by the fact that there are a number of ways to define a leading coefficient. This can include the matrix of highest-order, the matrix of highest row orders or the matrix of highest column orders. Invertibility conditions are then important with a matrix of differential operators having a leading row coefficient matrix invertible called a row-reduced form (a column-reduced form is the corresponding notion for columns). It is often the case that a matrix of differential operators needs to be transformed into reduced form via unimodular matrix operators.

In this section we introduce the concept of a simultaneously row and column reduced matrix of differential operators and give an algorithm for transforming an arbitrary matrix of differential operators into a matrix having such a property. When applied to problems of the form (1) one obtains a conversion into a useful algebraic structure for system simplification. For example, when L is both simultaneously row and column reduced and of first-order then the system (1) can be decoupled into both a purely algebraic and a purely differential system. For higher-order systems one can use such a transformation to extract a purely algebraic part (if it exists) with a second component easily transformable into a square system of first-order.

2.1. Row-reduction

In the case of row-reduction alone we can make use of a procedure used by Beckermann and Labahn (1997) for the commutative case of matrix polynomials and later generalized for Ore matrix polynomials in Beckermann et al. (2006). In this subsection, we will review this method for use with matrices of differential operators.

Definition 6. Let $L \in \mathrm{K}[[x]][\partial]^{m\times n}$ and let $\delta_i = \max(0, \mathrm{ord}(L_{i,*}))$ for $i = 1, \dots, m$.

(a) The row vector $\vec{\delta} = (\delta_1, \dots, \delta_m)$ is called the *row-order* of L.
(b) The leading row coefficient matrix of L is the $m \times n$ matrix with the (i, j) entry being the coefficient of order δ_i of the (i, j) entry of L.
(c) L is *row-reduced* if the nonzero rows of its leading row coefficient matrix are linearly independent over $\mathrm{K}((x))$.

Lemma 2 (*Beckermann et al., 2006, Appendix*)**.** *The rank of a row-reduced matrix of differential operators is equal to the rank of its leading row coefficient matrix.*

The following lemma shows that any matrix of differential operators can be transformed into a row-reduced form by means of elementary row operations.

Lemma 3 (*Beckermann et al., 2006, Th. 2.2*)**.** *Let $L \in \mathrm{K}[[x]][\partial]^{m\times n}$ be a matrix of differential operators of rank $s \le \min(m, n)$. Then, one can always construct a unimodular matrix of differential operators $U \in \mathrm{K}[[x]][\partial]^{m\times m}$ such that $U\,L$ is of the form*

$$U\,L = \left[\begin{array}{c} L^* \\ \hline 0 \end{array}\right]$$

where L^* is a row-reduced matrix of differential operators of size $s \times n$ such that $\mathrm{ord}(L^*) \le \mathrm{ord}(L)$ *and all its rows are nonzero.*

For the sake of completeness, we recall the proof of the lemma here.

Proof. If L is already row-reduced then $U = I_m$ and we are done. Otherwise, we may suppose, without any loss of generality, that L has all its zero rows at the bottom of the matrix. In this case the leading row coefficient matrix of L is of the form

$$\left[\begin{array}{c} L_0 \\ \hline 0 \end{array}\right],$$

where $L_0 \in \mathrm{K}[[x]]^{k\times n}$ is the leading row coefficient matrix of the first k rows of L ($k \ge s$). As L is not row-reduced L_0 is of rank less than k and hence we can find a nonzero row vector $v = (v_1, \dots, v_k) \in \mathrm{K}[[x]]^{1\times k}$ such that $v\,L_0 = 0$. Select an index ν such that $v_\nu \neq 0$ and $\delta_\nu = \max(\delta_i;\, v_i \neq 0)$ where $\vec{\delta} = (\delta_1, \dots, \delta_m)$ denotes the row-order of L. Define $U_1 = \mathrm{diag}(U_{11}, I_{m-k})$ where I_{m-k} denotes the identity matrix of size $m-k$ and

$$U_{11}=\begin{bmatrix}1 &&&&&&&\\ &\ddots&&&&&&\\ &&1&&&&&\\ v_1\,\partial^{\delta_\nu-\delta_1} & \cdots & v_{\nu-1}\,\partial^{\delta_\nu-\delta_{\nu-1}} & v_\nu & v_{\nu+1}\,\partial^{\delta_\nu-\delta_{\nu+1}} & \cdots & v_k\,\partial^{\delta_\nu-\delta_k}\\ &&&&1&&\\ &&&&&\ddots&\\ &&&&&&1\end{bmatrix}.$$

Then U_1 is unimodular and multiplying L on the left by U_1 leaves the rows of index $i\neq\nu$ unchanged and replaces the νth row by

$$\begin{aligned}\sum_{i=1}^{k} v_i\,\partial^{\delta_\nu-\delta_i}L_{i,*} &= \sum_{i=1}^{k} v_i\,\ell c(L_{i,*})\partial^{\delta_\nu} + \text{ terms of order less than } \delta_\nu\\ &= \underbrace{v\,L_0}_{=0}\,\partial^{\delta_\nu} + \text{ terms of order less than } \delta_\nu.\end{aligned}$$

Thus, the νth row of $U_1\,L$ has order less than the one of L. Repeating this process a finite number of times, we get a unimodular matrix of differential operators U such that $U\,L$ is of the form

$$U\,L=\begin{bmatrix}L^*\\ \hline 0\end{bmatrix},$$

where L^* is a row-reduced matrix of differential operators of size $m_1\times n\,(m_1\leq k)$ having only nonzero rows. It remains to show that $m_1=s$. By Lemma 1, we have that $\operatorname{rank} L=\operatorname{rank} U\,L=\operatorname{rank} L^*$. On one hand, the rank of L^* is equal to that of its leading row coefficient matrix which is m_1 (see Lemma 2). On the other hand, we have $\operatorname{rank} L=s$ and thus $m_1=s$. □

Algorithm **Row-Reduction.**

Input: $L\in K[[x]][\partial]^{m\times n}$ a matrix of differential operators of row-order $\vec{\delta}=(\delta_1,\ldots,\delta_m)$ and of order $\ell=\max(\delta_i,\ i=1,\ldots,m)$.
Output: A row-reduced matrix of differential operators $L'\in K[[x]][\partial]^{m\times n}$ and a unimodular matrix of differential operators $U\in K[[x]][\partial]^{m\times m}$ such that $L'=UL$.

Initialization: Let $L'=L$ and define L'_0 as the leading row coefficient matrix of L'.
Let $\vec{\delta}'=(\delta'_1,\ldots,\delta'_m)=\vec{\delta}$ and $U=I_m$.

While the nonzero rows of L'_0 are linearly dependent **do**

1. Compute $v=(v_1,\ldots,v_m)\in K[[x]]^{1\times m}\setminus\{0\}$ in the left nullspace of L'_0;
2. Select an integer ν s.t. $v_\nu\neq 0$ and $\delta'_\nu=\max(\delta'_i;\ v_i\neq 0)$;
3. Replace $L'_{\nu,*}\longleftarrow\sum_{i=1}^m v_i\,\partial^{\delta'_\nu-\delta'_i}L'_{i,*}$;
4. Replace $U_{\nu,*}\longleftarrow\sum_{i=1}^m v_i\,\partial^{\delta'_\nu-\delta'_i}U_{i,*}$;
5. Update L'_0 and $\vec{\delta}'$;

end do;

Return L' and U;

Proposition 1. *Algorithm* Row-Reduction *costs at most* $O\left(m\,n\,\left(|\vec{\delta}|+m\right)\left(m+2\,\ell+|\vec{\delta}|\right)\right)$ *operations in* $\mathbb{K}[[x]]$.

Proof. Computing an element of the left nullspace of a matrix $L'_0 \in \mathbb{K}[[x]]^{m\times n}$ costs at most $O\left(m^2\,n\right)$ operations in $\mathbb{K}[[x]]$ and so gives the cost of Step 1. Since L' has order always bounded by ℓ, the cost of Step 3 is then $O\,(m\,n\,\ell)$ operations in $\mathbb{K}[[x]]$. From Beckermann et al. (2006, Theorem 2.2), we can deduce that the order of the multiplier U is always bounded by $\ell+|\vec{\delta}|$, and so for $i=1,\ldots,m$, $\operatorname{ord}\left(\partial^{\delta'_\nu-\delta'_i}U_{i,*}\right)\leq 2\,\ell+|\vec{\delta}|$. Hence Step 4 can be done in at most $O\left(m\,n\left(2\,\ell+|\vec{\delta}|\right)\right)$ operations in $\mathbb{K}[[x]]$. Finally, as the while loop is repeated at most $|\vec{\delta}|+m-1$ times, we obtain a row-reduced matrix of differential operators equivalent to L after at most $O\left(m\,n\,\left(|\vec{\delta}|+m\right)\left(m+2\,\ell+|\vec{\delta}|\right)\right)$ operations in $\mathbb{K}[[x]]$. □

Analogous definitions and results can also be stated for column-reduction (where now the leading column coefficient matrix of the nonzero columns has full column rank). Thus it is possible to construct a unimodular matrix of differential operators $V \in \mathbb{K}[[x]][\partial]^{n\times n}$ such that $L\,V$ is column-reduced.

2.2. Simultaneous row and column reduction

Let $L \in \mathbb{K}[[x]][\partial]^{m\times n}$ of order ℓ and rank $s \leq \min(m,n)$. Unfortunately, constructing a simultaneously row and column reduced form, equivalent to L, is not as simple as just applying row-reduction to L then followed by column-reduction of the resulting row-reduced form (see Example 1 below). In general, such a computation requires several successive iterations of row-reduction and column-reduction. In this case the stopping criterion is no longer based on decreasing the value of the sum of the row orders $|\vec{\delta}|$ nor that of the sum of the column orders $|\vec{\gamma}|$. Indeed, applying row-reduction (resp. column-reduction) to L as in Lemma 3, the value of $|\vec{\delta}|$ (resp. the value of $|\vec{\gamma}|$) decreases but in the same time the value of $|\vec{\gamma}|$ (resp. of $|\vec{\delta}|$) may increase.

Example 1. Consider the matrix of differential operators given by

$$L=\begin{bmatrix}\partial^3+x & 2\partial^2 & x^2+x\\ \partial^2 & x\,\partial^2 & 2x^2+1\\ \partial & x\,\partial & 1\end{bmatrix}, \tag{2}$$

with row and column orders $\vec{\delta}=(3,2,1)$ and $\vec{\gamma}=(3,2,0)$, respectively. L is not row-reduced since its leading row coefficient matrix

$$\begin{bmatrix}1&0&0\\1&x&0\\1&x&0\end{bmatrix},$$

is singular. By the construction in Lemma 3 we multiply L on the left by

$$U_1=\begin{bmatrix}1&0&0\\0&1&-\partial\\0&0&1\end{bmatrix}$$

which gives

$$L^{(1)}=U_1\,L=\begin{bmatrix}\partial^3+x & 2\partial^2 & x^2+x\\ 0 & -\partial & -\partial+2x^2+1\\ \partial & x\,\partial & 1\end{bmatrix}.$$

In this case the resulting operator is row-reduced with row and column orders given by $\vec{\delta}^{(1)}=(3,1,1)$ and $\vec{\gamma}^{(1)}=(3,2,1)$. Row-reduction gives $|\vec{\delta}^{(1)}|<|\vec{\delta}|$. However, in this case we have increased the value of $|\vec{\gamma}|$ and indeed now $|\vec{\gamma}^{(1)}|>|\vec{\gamma}|$. In addition, $L^{(1)}$ is row-reduced but not column-reduced as the resulting leading column coefficient matrix is given by

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Again, using the construction from Lemma 3, we multiply $L^{(1)}$ on the right by

$$V_1 = \begin{bmatrix} 2 & 0 & 0 \\ -\partial & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

to obtain the column-reduced form

$$L^{(2)} = L^{(1)}\, V_1 = \begin{bmatrix} 2x & 2\partial^2 & x^2 + x \\ \partial^2 & -\partial & -\partial + 2x^2 + 1 \\ -x\partial^2 + 2\partial & x\,\partial & 1 \end{bmatrix}. \quad (3)$$

Unfortunately this is now not row-reduced so we are back to our first case.

While a single call to row-reduction and then column-reduction will not necessarily result in a simultaneously row and column reduced form, it turns out that by repeating this process a finite number of times we can always end up with a simultaneously row and column reduced operator. This is shown by the following proposition.

Proposition 2. *Let $L \in \mathrm{K}[[x]][\partial]^{m\times n}$ of order ℓ. It is always possible to construct two unimodular matrices $U \in \mathrm{K}[[x]][\partial]^{m\times m}$ and $V \in \mathrm{K}[[x]][\partial]^{n\times n}$ such that $U\,L\,V$ is a simultaneously row and column reduced matrix of differential operators.*

Proof. Let us show that by iterating successively row-reduction and column-reduction we end up with a simultaneously row and column reduced operator. For this, we consider the tuple

$$(r_\ell, c_\ell, r_{\ell-1}, c_{\ell-1}, \ldots, r_0, c_0),$$

where, for $i = 0, \ldots, \ell$, r_i and c_i denote the number of rows and columns of L of order i, respectively with $r_i = 0$ ($c_i = 0$) if no such rows (columns) exist. At each step of a row-reduction and of a column-reduction, this tuple strictly decreases in the sense of the lexicographic ordering. Indeed, one step of the row-reduction procedure consists of replacing a row of order i either by a zero row or by a nonzero row of order at most $i - 1$. Let $(r_\ell^{(1)}, c_\ell^{(1)}, r_{\ell-1}^{(1)}, c_{\ell-1}^{(1)}, \ldots, r_0^{(1)}, c_0^{(1)})$ and $(r_\ell^{(2)}, c_\ell^{(2)}, r_{\ell-1}^{(2)}, c_{\ell-1}^{(2)}, \ldots, r_0^{(2)}, c_0^{(2)})$ denote the tuples associated with the operators before and after this row operation, respectively. Then, for $k = i + 1, \ldots, \ell$, we have $r_k^{(2)} = r_k^{(1)}$ and $c_k^{(2)} = c_k^{(1)}$ but $r_i^{(2)} < r_i^{(1)}$ and $c_i^{(2)} \le c_i^{(1)}$. This implies that

$$(r_\ell^{(2)}, c_\ell^{(2)}, r_{\ell-1}^{(2)}, c_{\ell-1}^{(2)}, \ldots, r_0^{(2)}, c_0^{(2)}) <_{lex} (r_\ell^{(1)}, c_\ell^{(1)}, r_{\ell-1}^{(1)}, c_{\ell-1}^{(1)}, \ldots, r_0^{(1)}, c_0^{(1)})$$

where $<_{lex}$ denotes lexicographic ordering. A similar statement holds true when doing column-reduction. Therefore, after a finite number of iterations of row-reduction and column-reduction, we get two unimodular matrices of differential operators U and V such that $U\,L\,V$ is a simultaneously row and column reduced. □

Example 2. Consider the matrix of differential operators L of order $\ell = 3$ given by (2). The tuple $(r_3, c_3, r_2, c_2, r_1, c_1, r_0, c_0)$ associated with L is then $(1, 1, 1, 1, 1, 0, 0, 1)$. The tuples associated with the operators $L^{(1)}$ and $L^{(2)}$ obtained after applying row-reduction to L and then column-reduction to $L^{(1)}$ become $(1, 1, 0, 1, 2, 1, 0, 0)$ and $(0, 0, 3, 2, 0, 1, 0, 0)$, respectively. In this case we do observe that

$$(0, 0, 3, 2, 0, 1, 0, 0) <_{lex} (1, 1, 0, 1, 2, 1, 0, 0) <_{lex} (1, 1, 1, 1, 1, 0, 0, 1).$$

3. A second algorithm for row–column reduction

In this section we describe a second algorithm to convert a matrix of differential operators L into one which is simultaneously row and column reduced. For this method it becomes important to see what the end result can be for such a computation.

Proposition 3. *Let $L \in \mathrm{K}[[x]][\partial]^{m\times n}$ be a simultaneously row and column reduced matrix of differential operators of order ℓ. Let r_i, c_i $i = 0, \ldots, \ell$ denote the number of rows and columns of order i, respectively, with $r_i = 0$ ($c_i = 0$) if no such rows (columns) exist. Then one can permute the rows and columns of L so that it has the block form*

$$\left[\begin{array}{ccc|c} L_{11} & \cdots & L_{1k} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ L_{k1} & \cdots & L_{kk} & 0 \\ \hline 0 & \cdots & 0 & 0 \end{array}\right] \tag{4}$$

where L_{ii} are block square matrices satisfying

(a) *L_{ii} is row and column reduced with same row and column order,*
(b) *$\operatorname{ord}(L_{ii}) > \operatorname{ord}(L_{i+1\,i+1})$ for all i,*
(c) *$\operatorname{ord}(L_{ij}) \le \operatorname{ord}(L_{ii})$ for all $j < i$ and $\operatorname{ord}(L_{ij}) < \operatorname{ord}(L_{ii})$ for all $j > i$,*
(d) *$\operatorname{ord}(L_{ij}) \le \operatorname{ord}(L_{jj})$ for all $i < j$ and $\operatorname{ord}(L_{ij}) < \operatorname{ord}(L_{jj})$ for all $i > j$.*

Conversely, a matrix of differential operators of the form (4) is simultaneously row and column reduced.

Proof. We first sort the rows and columns of L so that the row and column orders of L are decreasing, the zero rows are at the bottom and the zero columns are at the end. The nonzero rows and columns then form a square submatrix of full rank (since the leading row and column coefficient matrix has this property). For ease of presentation let us assume that L is a square matrix consisting only of these nonzero rows and columns and having decreasing row and column orders.

For each nonzero r_i let $n_i = r_{i+1} + \cdots + r_\ell$. Note that both r_ℓ and c_ℓ are nonzero since L is of order ℓ. Let L_{11} be the $r_\ell \times c_\ell$ matrix in the first r_ℓ and c_ℓ rows and columns. Then, the leading row and column coefficient matrices of L are given by

$$\begin{bmatrix} \ell c(L_{11}) & 0 \\ * & * \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \ell c(L_{11}) & * \\ 0 & * \end{bmatrix},$$

respectively. These are of full row rank and full column rank, respectively, and so it must be the case that $\ell c(L_{11})$ is square (so $r_\ell = c_\ell$) and nonsingular.

We assume now that $r_j = c_j$ for $j = i + 1, \ldots, \ell$ and we will show that $r_i = c_i$. Assume that $r_i \ne 0$ and that L_i, the first n_i rows and columns of L, are of the form

$$L_i = \begin{bmatrix} L_{11} & \cdots & L_{1u} \\ \vdots & \ddots & \vdots \\ L_{u1} & \cdots & L_{uu} \end{bmatrix}$$

all satisfying conditions (a)–(d) in the proposition. If $c_i = 0$ then the leading row coefficient matrix of L has the form

$$\begin{bmatrix} \ell c_{row}(L_i) & 0 \\ * & 0 \\ * & * \end{bmatrix},$$

where $\ell c_{row}(L_i)$ denotes the leading row coefficient matrix of L_i. This gives a full row rank matrix of size $n_{i-1} \times n_i$ with $n_i < n_{i-1}$ (since L is row-reduced), a contradiction. A similar situation would occur if c_i was nonzero but less than r_i and so $r_i \le c_i$. Repeating the argument using the fact L is column-reduced gives $c_i \le r_i$ and so $r_i = c_i$. Letting L_{i-1} denote the square matrix of size n_{i-1} gives us the next submatrix of L satisfying (a)–(d). The process continues until rows and columns of lowest order are reached. □

Example 3. Let L be the matrix of differential operators from Example 1. Then L is equivalent to the column-reduced operator $L^{(2)}$ given by (3) which is not row-reduced. However continuing with the procedure described in Proposition 2 we use the construction of Lemma 3 to obtain a unimodular matrix

$$U_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & x & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

with

$$L^{(3)} = U_2\, L^{(2)} = \begin{bmatrix} 2x & 2\partial^2 & x^2 + x \\ 2\partial & 0 & -x\partial + 2x^3 + x + 1 \\ -x\partial^2 + 2\partial & x\partial & 1 \end{bmatrix}. \tag{5}$$

Now $L^{(3)}$ is simultaneously row and column reduced of order $2 < \text{ord}(L) = 3$. Here, we have $U\,L\,V = L^{(3)}$ with

$$U = U_2\, U_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & x & -x\partial + 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad V = V_1 = \begin{bmatrix} 2 & 0 & 0 \\ -\partial & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Finally, by swapping the second and third row of $L^{(3)}$ given in (5), we can partition the resulting matrix into four blocks

$$L^* = \left[\begin{array}{cc|c} 2x & 2\partial^2 & x^2 + x \\ -x\partial^2 + 2\partial & x\partial & 1 \\ \hline 2\partial & 0 & -x\partial + 2x^3 + x + 1 \end{array}\right] = \left[\begin{array}{c|c} L_{11} & L_{12} \\ \hline L_{21} & L_{22} \end{array}\right], \tag{6}$$

satisfying properties (a)–(d) of Proposition 3.

In order to describe our second algorithm for converting a matrix of differential operators L into one which is simultaneously row and column reduced let us first take advantage of the previous proposition. Namely, by doing column operations we can ensure that all zero columns are located in the last columns. Furthermore, by Lemma 3 we can ensure that the nonzero columns are in row-reduced form. The remaining nonzero matrix is square and nonsingular.

Proposition 4. *Let $L \in \mathbb{K}[[x]][\partial]^{m\times m}$ be row-reduced, nonsingular (i.e., of rank m) of order ℓ with rows sorted by decreasing order. Then we can construct an invertible matrix $V \in \mathbb{K}[[x]]^{m\times m}$ such that*

$$L\,V = \begin{bmatrix} L_{11} & \cdots & L_{1k} \\ \vdots & \ddots & \vdots \\ L_{k1} & \cdots & L_{kk} \end{bmatrix} \tag{7}$$

where L_{ii} are block square matrices satisfying

(a) *L_{ii} is row and column reduced with the same row and column order,*
(b) *$\text{ord}\,(L_{ii}) > \text{ord}\,(L_{i+1,i+1})$ for all i,*
(c) *$\text{ord}\,(L_{ij}) \le \text{ord}\,(L_{ii})$ for all $j < i$ and $\text{ord}\,(L_{ij}) < \text{ord}\,(L_{ii})$ for all $j > i$.*

Proof. Let r_i, $i = 0, \ldots, \ell$ denote the number of rows of order i, with $r_i = 0$ if no such rows exist, and $n_i = r_{i+1} + \cdots + r_\ell$. For each nonzero r_i, we will then construct a matrix of differential operators of size $n_{i-1} (= n_i + r_i)$ by m of the form

$$L_i = \begin{bmatrix} A_i & * & * \\ * & B_i & * \end{bmatrix}$$

with A_i and B_i square row-reduced matrices of differential operators of size $n_i \times n_i$ and $r_i \times r_i$, respectively. Furthermore, for each i the leading row coefficient matrix of L_i will be

$$\begin{bmatrix} \ell c_{row}(A_i) & 0 & 0 \\ * & \ell c(B_i) & 0 \end{bmatrix}. \tag{8}$$

The matrix L_i is constructed as follows. Let A_i be the matrix composed of the first n_i rows and columns of L. Suppose that A_i is row-reduced. If $r_i \neq 0$ then the fact that L is row-reduced implies that the strip of L composed of rows $n_i + 1$ to n_{i-1} is row-reduced. Since L and A_i are both row-reduced, the matrix formed from columns $n_i + 1$ to m of this strip is row-reduced as well, that is, its leading row coefficient matrix is of full row rank. Using elementary column operations on L we can ensure that the square submatrix in rows and columns $n_i + 1$ to n_{i-1} has a nonsingular leading coefficient matrix. Using elementary column operations on L we can then ensure that the last $m - n_{i-1}$ columns of this strip have order at most $i - 1$. Let B_i be the square matrix in rows and columns $n_i + 1$ to n_{i-1} and set

$$A_{i-1} = \begin{bmatrix} A_i & * \\ * & B_i \end{bmatrix}$$

where the $*$ denotes the remaining elements in the matrix. The iteration, with initial value A_ℓ an empty matrix, ends when $L = L_k$ for a particular k. Finally, we note that in all cases the column operations given above only require elements from $\mathbb{K}[[x]]$ as we are always eliminating only by means of entries from leading coefficient matrices. □

Remark 1. For a given strip of same row order, the procedure requires that we ensure that the $r_i \times r_i$ matrix starting at row and column $n_i + 1$ have nonsingular leading coefficient. This is accomplished by converting the full rank leading row coefficient matrix of this strip from columns $n_i + 1$ to m into column echelon form.

Example 4. We can show how the row orders are reduced with a matrix having block of orders 6, 4, 2 and 1 as in

$$\begin{bmatrix} 6 & 6 & 6 & 6 \\ 4 & 4 & 4 & 4 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Working top to bottom, we can ensure that the diagonal blocks have nonsingular leading coefficient matrices and that the blocks to the right have lower order by computing the column echelon form of the leading row coefficient matrix of each strip from columns $n_i + 1$ to m. In the above case the resulting orders would then become

$$\begin{bmatrix} 6 & 5 & 5 & 5 \\ 4 & 4 & 3 & 3 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Remark 2. Note that in fact we can also continue the process (again working top to bottom) by eliminating the highest coefficients to obtain order bounds such that the blocks to the left of any diagonal block are of smaller order, that is, condition (c) is replaced by

(c′) $\operatorname{ord}(L_{ij}) < \operatorname{ord}(L_{ii})$ for all $j \neq i$.

Example 5. Continuing with the previous example, we can use the leading coefficient in each diagonal block to reduce the orders in all blocks to the left. Thus, there are block column operations which reduce the orders to

$$\begin{bmatrix} 6 & 5 & 5 & 5 \\ 3 & 4 & 3 & 3 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

if so desired.

Of course the construction in Proposition 4 does not necessarily result in a simultaneously row and column reduced matrix. For this to hold we need to reduce the column orders. Fortunately, Proposition 3 provides the required orders for a simultaneously row and column reduced matrix.

Definition 7. Let $L = \left[L_{ij}\right]_{1 \le i,j \le k} \in \mathbb{K}[[x]][\partial]^{m \times m}$ be a matrix of differential operators satisfying conditions (a)–(c) of Proposition 4. The *defect* of each block row $L_{i,*}$ for $i = 1, \ldots, k-1$ is defined by

$$\text{defect}\,(L)_i = \max(\operatorname{ord}(L_{ij}) - \operatorname{ord}(L_{jj}), j = i+1, \ldots, k).$$

Thus, a nonsingular row-reduced operator $L = \left[L_{ij}\right]_{1\le i,j\le k}$ satisfying the conditions (a)–(c) of Proposition 4 is simultaneously row and column reduced if and only if defect $(L)_i \le 0$ for $i = 1, \ldots, k-1$.

Proposition 5. *Let $L = \left[L_{ij}\right]_{1\le i,j\le k} \in \mathrm{K}[[x]][\partial]^{m\times m}$ be a nonsingular row-reduced matrix of differential operators satisfying the conditions (a)–(c) of Proposition 4. Then we can construct a unimodular matrix $U \in \mathrm{K}[[x]][\partial]^{m\times m}$ such that $U\,L$ is simultaneously row and column reduced.*

Proof. Since L already satisfies the conditions (a)–(c) of Proposition 4, it remains then to do row operations in order that $\mathrm{ord}(L_{ij}) \le \mathrm{ord}(L_{jj})$ for all $i < j$. We do this by reducing the positive defects to 0 proceeding from the bottom to the top block rows. Suppose that defect $(L)_i \le 0$ for $i = i_0+1, \ldots, k-1$ and defect $(L)_{i_0} > 0$. Let us explain how one lowers the defect of the i_0th block row of L. Let j_0 be the smallest integer j for which $\mathrm{ord}(L_{i_0j}) - \mathrm{ord}(L_{jj}) =$ defect $(L)_{i_0}$. We first lower the order of the block $L_{i_0j_0}$ in the following way. Compute the adjoint (the transpose of the cofactor matrix) of $\ell c(L_{j_0j_0})$ that we denote by $\mathrm{adj}(\ell c(L_{j_0j_0}))$. Then replace the i_0 th block row $L_{i_0,*}$ of L by

$$\det(\ell c(L_{j_0j_0}))\, L_{i_0,*} - \ell c(L_{i_0j_0})\, \mathrm{adj}(\ell c(L_{j_0j_0}))\partial^{\alpha} L_{j_0,*}, \tag{9}$$

where $\alpha =$ defect $(L)_{i_0}$. This is achieved by multiplying L on the left by a unimodular matrix of differential operators with coefficients in $\mathrm{K}[[x]]$. Let $\widetilde{L} = \left[\widetilde{L}_{ij}\right]_{1\le i,j\le k}$ denote the resulting matrix of differential operators. Then one can check that

$$\mathrm{ord}(\widetilde{L}_{i_0j}) - \mathrm{ord}(\widetilde{L}_{jj}) \begin{cases} < \text{defect } (L)_{i_0} & j \le j_0 \\ \le \text{defect } (L)_{i_0} & j > j_0, \end{cases}$$

and so, defect $(\widetilde{L})_{i_0} \le$ defect $(L)_{i_0}$. Two cases then arise:

(1) defect $(\widetilde{L})_{i_0} <$ defect $(L)_{i_0}$, (in which case we are done) or
(2) defect $(\widetilde{L})_{i_0} =$ defect $(L)_{i_0}$.

For case (2) the smallest integer j for which $\mathrm{ord}(\widetilde{L}_{i_0j}) - \mathrm{ord}(\widetilde{L}_{jj}) =$ defect $(\widetilde{L})_{i_0}$ is greater than j_0. Hence, the "value of j_0" increases, and so after a finite number of iterations the defect will decrease. □

Remark 3. The elimination step in Eq. (9) can be viewed as the block row operations which replace the i_0th block row $L_{i_0,*}$ of L by

$$L_{i_0,*} - \ell c(L_{i_0j_0})\,(\ell c(L_{j_0j_0}))^{-1}\partial^{\alpha} L_{j_0,*}.$$

However, for computational purposes it is better to work in the ring $\mathrm{K}[[x]]$ rather than in its quotient field $\mathrm{K}((x))$. Elimination in this case implies solving the linear system of equations

$$X_{i_0}\ell c(L_{j_0j_0}) = \ell c(L_{i_0j_0})$$

for X_{i_0} a matrix of the same size as $L_{i_0j_0}$. We can solve such a system and remain in the domain $\mathrm{K}[[x]]$ by using fraction-free Gaussian elimination (cf. Geddes et al. (1992, chapter 9)). However this produces a solution for the system

$$\det(\ell c(L_{j_0j_0}))X_{i_0} = \ell c(L_{i_0j_0})\; adj(\ell c(L_{j_0j_0}))$$

and hence replacing the i_0th block row $L_{i_0,*}$ of L is done via Eq. (9). We remark that in order to minimize growth of coefficients for a given row – say row $\hat{i}$ of block i_0 – one still needs to remove common factors of the elimination terms, which in this case means removing the greatest common factor of the terms in row $\hat{i}$.

Example 6. Consider the matrix having order bounds

$$\begin{bmatrix} 6 & 5 & 5 & 5 \\ 4 & 4 & 3 & 3 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

as in Example 4. Suppose that we have already reduced the defects of first the third and then the second block rows to 0 and get the matrix having order bounds

$$\begin{bmatrix} 6 & 5 & 5 & 5 \\ 4 & 4 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Let us now reduce the defect of the first block row $i_0 = 1$. Assuming all orders are attained, then the defect is 4 and is attained for $j_0 = 4$. In order to reduce the order in block $(1, 4)$ we use the (invertible) leading coefficient of the $(4, 4)$ square block multiplied by ∂^4 and obtain a matrix of orders of the form

$$\begin{bmatrix} 6 & 5 & 5 & 4 \\ 4 & 4 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

The defect of the first block row is now 3 with the defect attained for $j = 3, 4$. We repeat the reduction process, first with block entry $(1, 3)$ reduced by block $(3, 3)$ and then block entry $(1, 4)$ reduced by block $(4, 4)$ giving a matrix having order bounds

$$\begin{bmatrix} 6 & 5 & 4 & 3 \\ 4 & 4 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

and with the first block row now having defect 2. Continuing with the first block row, we end up reducing to defect 0 giving order bounds

$$\begin{bmatrix} 6 & 4 & 2 & 1 \\ 4 & 4 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

We summarize our result in the following theorem:

Theorem 1. *Let $L \in \mathrm{K}[[x]][\partial]^{m\times n}$ be a matrix of differential operators. We can construct two unimodular matrices $U \in \mathrm{K}[[x]][\partial]^{m\times m}$ and $V \in \mathrm{K}[[x]][\partial]^{n\times n}$ such that $U\,L\,V$ has the form (4) with conditions* (a)–(d) *of Proposition 3 satisfied.*

Definition 8. Let $L \in \mathrm{K}[[x]][\partial]^{m\times m}$ of rank m. Then L is said to be in Two-Sided Block Popov form if

$$L = \begin{bmatrix} L_{11} & \cdots & L_{1k} \\ \vdots & \ddots & \vdots \\ L_{k1} & \cdots & L_{kk} \end{bmatrix} \tag{10}$$

where L_{ii} are block square matrices satisfying

(a) L_{ii} is row and column reduced with same row and column order,
(b) $\operatorname{ord}(L_{ii}) > \operatorname{ord}(L_{i+1,i+1})$ for all i,
(c) $\operatorname{ord}(L_{ij}) < \operatorname{ord}(L_{ii})$ for all $j \neq i$,
(d) $\operatorname{ord}(L_{ij}) < \operatorname{ord}(L_{jj})$ for all $i \neq j$.

Corollary 1. *Let $L \in \mathrm{K}[[x]][\partial]^{m\times m}$ be a nonsingular matrix of differential operators. Then there exist unimodular matrices $U \in \mathrm{K}[[x]][\partial]^{m\times m}$ and $V \in \mathrm{K}[[x]]^{m\times m}$ such that $U\,L\,V$ is in Two-Sided Block Popov form.*

Proof. From Remark 2 we can ensure that all blocks before a diagonal block have lower order using only column operations. The procedure in the proof of Proposition 5 can be extended to reduce the orders of the entries in the upper triangular part and hence ensure that the column of block rows before a diagonal block have lower orders. □

Example 7. If our starting point was the matrix in Example 5 and we repeated the row operations of Example 6 then we would have a matrix of orders bounded by

$$\begin{bmatrix} 6 & 4 & 2 & 1 \\ 3 & 4 & 2 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

If our procedure produced defects of -1 rather than 0 at every block row then we would get

$$\begin{bmatrix} 6 & 3 & 1 & 0 \\ 3 & 4 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is in Two-Sided Block Popov form.

4. Algorithms and complexity

In this section we give the algorithmic descriptions of the methods described in Propositions 4 and 5 along with their complexities.

ALGORITHM : **Block Reduced Form**

INPUT: $L \in \mathbb{K}[[x]][\partial]^{m\times m}$ a nonsingular row-reduced matrix of differential operators of order ℓ.

OUTPUT: An invertible matrix $V \in \mathbb{K}[[x]]^{m\times m}$ (over $\mathbb{K}((x))$) and a matrix of differential operators $\widetilde{L}$ such that $\widetilde{L} = LV$ and $\widetilde{L}$ can be partitioned into blocks $\widetilde{L}_{ij}$ for $1 \le i, j \le k$ satisfying the conditions (a)–(c) of Proposition 4.

INITIALIZATION: Let $V = I_m$ and $\widetilde{L} = L$.
Let r_i, for $i = 0, \dots, \ell$, denote the number of rows of $\widetilde{L}$ of order i with $r_i = 0$ if no such rows exist.
Define $n_\ell = 0$ and $n_i = \sum_{j=i+1}^{\ell} r_j$ for $i = -1, \dots, \ell - 1$.

1. Sort rows of $\widetilde{L}$ in decreasing order;
2. **For** i from ℓ by -1 to 0 **do**
   - 2.1. Define $\widetilde{L}_0$ as the leading row coefficient matrix of $\widetilde{L}$;
   - 2.2. Let B_i denote the submatrix of $\widetilde{L}$ composed of rows $n_i + 1$ to n_{i-1} and columns $n_i + 1$ to m;
   - 2.3. Compute an invertible matrix $V_i \in \mathbb{K}[[x]]^{(m-n_i)\times(m-n_i)}$ such that B_iV_i is in column echelon form;
   - 2.4. Let $\widetilde{L} = \widetilde{L}\ \mathrm{diag}(I_{n_i}, V_i)$ and $V = V\ \mathrm{diag}(I_{n_i}, V_i)$;

   end do;
3. **Return** V and $\widetilde{L}$;

Proposition 6. *Algorithm* Block Reduced Form *costs at most* $O(m^3\ell^3)$ *operations in* $\mathbb{K}[[x]]$.

Proof. We first consider the cost of one passage of the for loop, say at index $i \in \{0, \dots, \ell\}$. In this case Step 2.3 requires at most $O\left(r_i(m - n_i)^2\right)$ operations in $\mathbb{K}[[x]]$. In Step 2.4, the product $V\ \mathrm{diag}(I_{n_i}, V_i)$

can be done in at most $O\left(m\,(m-n_i)^2\right)$ operations in $\mathrm{K}[[x]]$. In order to determine the cost of the product $\widetilde{L}\ \mathrm{diag}(I_{n_i}, V_i)$ we can write the operator $\widetilde{L}$ as

$$\widetilde{L} = \widetilde{a}_\ell(x)\,\partial^\ell + \widetilde{a}_{\ell-1}(x)\,\partial^{\ell-1} + \cdots + \widetilde{a}_0(x),$$

where $\widetilde{a}_j(x) \in \mathrm{K}[[x]]^{m\times m}$ for $j = 0, \ldots, \ell$, and then observe that

$$\partial^j\,\mathrm{diag}(I_{n_i}, V_i) = \mathrm{diag}\left(I_{n_i}\partial^j, \sum_{s=0}^{j}\binom{j}{s}\partial^s(V_i)\,\partial^{j-s}\right).$$

Thus the cost of one product of the form $\widetilde{a}_j(x)\,\partial^j\,\mathrm{diag}(I_{n_i}, V_i)$ is equal to the cost of $j+1$ products of an $m \times (m-n_i)$ matrix by an $(m-n_i)\times(m-n_i)$ matrix. Therefore, the product $\widetilde{L}\ \mathrm{diag}(I_{n_i}, V_i)$ can be done in at most $O\left(\sum_{j=0}^{\ell}(j+1)\,m\,(m-n_i)^2\right) = O\left(m\,\ell^2\,(m-n_i)^2\right)$ operations in $\mathrm{K}[[x]]$. Hence, one passage of the for loop can be done in at most $O\left(m\,\ell^2\,(m-n_i)^2\right)$ operations in $\mathrm{K}[[x]]$. Since $m-n_i \le m$ and the for loop is repeated $\ell+1$ times, the above algorithm returns V and $\widetilde{L}$ after at most $O(m^3\ell^3)$ operations in $\mathrm{K}[[x]]$. □

ALGORITHM **Simultaneously Row and Column Reduced Form**

INPUT: $L = \left[L_{ij}\right]_{1\le i,j\le k} \in \mathrm{K}[[x]][\partial]^{m\times m}$ a nonsingular matrix of differential operators of order ℓ where the blocks $L_{ij} \in \mathrm{K}[[x]][\partial]^{m_i\times m_j}$ satisfy conditions (a)–(c) of Proposition 4.

OUTPUT: A simultaneously row and column reduced matrix of differential operators $\widetilde{L} \in \mathrm{K}[[x]][\partial]^{m\times m}$ and a unimodular matrix of differential operators $U \in \mathrm{K}[[x]][\partial]^{m\times m}$ such that $\widetilde{L} = UL$.

INITIALIZATION: Let $\widetilde{L} = L$, respectively $U = I_m$, partitioned into blocks $\left[\widetilde{L}_{ij}\right]_{1\le i,j\le k}$, respectively $\left[U_{ij}\right]_{1\le i,j\le k}$, of the same partition as L.

For i from $k-1$ by -1 to 1 **do**

1. Define defect $(\widetilde{L})_i = \max(\mathrm{ord}(\widetilde{L}_{ij}) - \mathrm{ord}(\widetilde{L}_{jj}), j = i+1, \ldots, k)$;
2. Let $W = I_m$ partitioned into blocks $\left[W_{ij}\right]_{1\le i,j\le k}$ of the same partition as $\widetilde{L}$;
3. **While** defect $(\widetilde{L})_i > 0$ **do**
    - 3.1. Define $j_0 = \min\left(j \in \{i+1, \ldots, k\}\,;\ \mathrm{ord}(\widetilde{L}_{ij}) - \mathrm{ord}(\widetilde{L}_{jj}) = \text{ defect } (\widetilde{L})_i\right)$;
    - 3.2. Define α = defect $(\widetilde{L})_i$;
        Comment: We avoid fractions in elimination Steps 3.3 and 3.4
    - 3.3. Replace $\widetilde{L}_{i,*} \longleftarrow \det(\ell c(\widetilde{L}_{j_0j_0}))\,\widetilde{L}_{i,*} - \ell c(\widetilde{L}_{ij_0})\,\mathrm{adj}(\ell c(\widetilde{L}_{j_0j_0}))\partial^\alpha\widetilde{L}_{j_0,*}$;
    - 3.4. Replace $W_{i,*} \longleftarrow \det(\ell c(\widetilde{L}_{j_0j_0}))\,W_{i,*} - \ell c(\widetilde{L}_{ij_0})\,\mathrm{adj}(\ell c(\widetilde{L}_{j_0j_0}))\partial^\alpha W_{j_0,*}$;
    - 3.5. Update defect $(\widetilde{L})_i$;

    end do;
4. Replace $U_{i,*} \longleftarrow \sum_{j=i}^{k} W_{ij}\,U_{j,*}$;

end do;

Return $\widetilde{L}$ and U;

Proposition 7. *Algorithm* Simultaneously Row and Column Reduced Form *costs at most* $O\left(k^3\, m^3\, \ell^2\right)$ *operations in* $\mathbb{K}[[x]]$ *with* $k \le \min(m, \ell+1)$.

Proof. During the algorithm, the leading coefficients of the diagonal blocks $\widetilde{L}_{jj}$ for $j = 1, \ldots, k$ remain unchanged. Therefore, we need only compute once for all the determinants and the adjoints (and hence the inverses) of the matrices $\ell c(\widetilde{L}_{jj})$ for $j = 2, \ldots, k$. The blocks $\widetilde{L}_{jj}$ are of size $m_j \times m_j$ and so computing the determinants and adjoints can be done in at most $O\left(\sum_{j=2}^{k} m_j^3\right) = O\left(m^3\right)$ operations in $\mathbb{K}[[x]]$ since $\sum_{j=2}^{k} m_j \le m$. Let us now study the cost of the while loop, starting with the cost of Step 3.3. Multiplying $\ell c(\widetilde{L}_{ij_0})$ by $\mathrm{adj}(\ell c(\widetilde{L}_{j_0j_0}))$ can be done in at most $O\left(m_i\, m_{j_0}^2\right)$ operations in $\mathbb{K}[[x]]$. Consider now the cost to multiply the operator $\partial^{\alpha}\widetilde{L}_{j_0,*}$ on the left by the matrix $\ell c(\widetilde{L}_{ij_0})\,\mathrm{adj}(\ell c(\widetilde{L}_{j_0j_0}))$. Note that, since $\mathrm{ord}(\widetilde{L}_{j_0,*}) = \mathrm{ord}(\widetilde{L}_{j_0j_0})$, we have

$$\mathrm{ord}(\partial^{\alpha}\widetilde{L}_{j_0,*}) = \alpha + \mathrm{ord}(\widetilde{L}_{j_0,*}) = \mathrm{ord}(\widetilde{L}_{ij_0}) - \mathrm{ord}(\widetilde{L}_{j_0j_0}) + \mathrm{ord}(\widetilde{L}_{j_0,*}) = \mathrm{ord}(\widetilde{L}_{ij_0}).$$

Thus, the order of the operator $\partial^{\alpha}\widetilde{L}_{j_0,*}$ is at most ℓ. Therefore, the cost of multiplying $\partial^{\alpha}\widetilde{L}_{j_0,*}$ on the left by $\ell c(\widetilde{L}_{ij_0})\,\mathrm{adj}(\ell c(\widetilde{L}_{j_0j_0}))$ is equivalent to the cost of at most $\ell + 1$ products of an $m_i \times m_{j_0}$ matrix by an $m_{j_0} \times m$ matrix with entries in $\mathbb{K}[[x]]$, resulting in $O\left(m_i\, m_{j_0}\, m\, \ell\right)$ operations in $\mathbb{K}[[x]]$. Since the order of $\widetilde{L}_{i,*}$ is always bounded by ℓ, multiplying the block row $\widetilde{L}_{i,*}$ by $\det(\ell c(\widetilde{L}_{j_0j_0}))$ in Step 3.3 can be done in at most $O\left(m_i\, m\, \ell\right)$ operations in $\mathbb{K}[[x]]$. Thus, Step 3.3 can be done in at most $O\left(m_i\, m_{j_0}\, m\, \ell\right) = O\left(m^3\, \ell\right)$ operations in $\mathbb{K}[[x]]$. The cost of Step 3.4 is seen to be the cost of multiplying $W_{i,*}$ by $\det(\ell c(\widetilde{L}_{j_0j_0}))$ since the block row $W_{j_0,*}$ is always equal to

$$W_{j_0,*} = \begin{bmatrix} 0 & \cdots & 0 & I_{m_{j_0}} & 0 & \cdots & 0 \end{bmatrix},$$

where $I_{m_{j_0}}$ comes at the j_0th position. Let $\alpha_i^0 \le \ell$ denote the defect of the block row $L_{i,*}$. It is easy to check that the order of $W_{i,*}$ is always bounded by α_i^0. Thus, Step 3.4 costs at most $O\left(m_i\, m\, \ell\right)$ operations in $\mathbb{K}[[x]]$. One passage of the while loop can then be done in at most $O\left(m^3\, \ell\right)$ operations in $\mathbb{K}[[x]]$. To reduce the defect of the ith block row to zero, Steps 3.1–3.5 are repeated at most $(k-i)\,\alpha_i^0$ times. Since $(k-i)\,\alpha_i^0 \le k\,\ell$, Step 3 can be done in at most $O\left(k\, m^3\, \ell^2\right)$ operations in $\mathbb{K}[[x]]$. It remains to determine the cost of Step 4. Note that the kth block row of U is always of the form

$$U_{k,*} = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & I_{m_k} \end{bmatrix},$$

and so by induction we can show that for $j = i+1, \ldots, k-1$, we have $\mathrm{ord}(U_{j,*}) \le \sum_{s=j}^{k-1} \alpha_s^0$ where α_s^0 denotes the defect of the block row $L_{s,*}$. Consider now the cost of multiplying operator W_{ij} by $U_{j,*}$. Since $W_{ij} \in \mathbb{K}[[x]][\partial]^{m_i \times m_j}$ is of order bounded by $\alpha_i^0 \le \ell$ and $U_{j,*} \in \mathbb{K}[[x]][\partial]^{m_j \times m}$ is of order bounded by $\sum_{s=j}^{k-1} \alpha_s^0 \le k\,\ell$, one product of the form $W_{ij}\, U_{j,*}$ costs at most $O\left(k\, m^3\, \ell^2\right)$ operations in $\mathbb{K}[[x]]$. Step 4 can thus be done in at most $O\left(k^2\, m^3\, \ell^2\right)$ operations in $\mathbb{K}[[x]]$. As Steps 3 and 4 are repeated $k-1$ times, the algorithm returns a simultaneously row and column reduced operator equivalent to the input L after at most $O\left(k^3\, m^3\, \ell^2\right)$ operations in $\mathbb{K}[[x]]$. □

5. First-order matrices of differential operators

Linear differential–algebraic equations (DAEs) of first-order are equations of the form

$$L \cdot \vec{y}(x) = a(x)\vec{y}\,'(x) + b(x)\vec{y}(x) = \vec{f}(x), \tag{11}$$

where $a(x), b(x) \in \mathbb{K}[[x]]^{m\times n}$ and $\vec{f}(x) \in \mathbb{K}[[x]]^m$. These have been the subject of interest of many papers. For example, in the square case (where $m = n$) Harris et al. (1968) develop an algorithm by which the existence of solutions of (11) can be decided and constructed through solutions of algebraic systems or solutions of a first-order system of ordinary differential equations (ODEs). This algorithm, reviewed in Barkatou et al. (2010) can be viewed as an application of a series of elementary row operations and elementary column operations on L.

In our case computing a simultaneously row and column reduced operator equivalent to L allows us to reduce system (11) into an algebraic system or a first-order system of ODEs and so one can deduce the existence of the solutions of (11) and compute them when they exist (cf. Barkatou et al. (2010)). The purpose of this section is to review this method and show that, when dealing with a matrix of differential operators of the first-order, we are assured of getting a simultaneously row and column reduced operator after at most the second application of row-reduction.

Let $s \le \min(m, n)$ denote the rank of L given by (11) and $L^{(1)}$ the operator obtained after applying row-reduction to L. Then $L^{(1)}$ can be written as

$$L^{(1)} = \left[\begin{array}{c} A_{11}\partial + B_{11} \\ \hline B_{21} \\ \hline 0 \end{array}\right]$$

where A_{11} and B_{21} are of respective sizes $r_1^{(1)} \times n$ and $(s - r_1^{(1)}) \times n$ and

$$\operatorname{rank} \left[\begin{array}{c} A_{11} \\ \hline B_{21} \end{array}\right] = s.$$

Here $r_1^{(1)} \le s$. If $s = n$ and either $r_1^{(1)} = 0$ or $r_1^{(1)} = s$ then we are in simultaneously row and column reduced form and so we are done. Otherwise, we apply column-reduction to $L^{(1)}$ and get a column-reduced operator of the form

$$L^{(2)} = \left[\begin{array}{c|c|c} A_{11}^*\partial + B_{11}^* & B_{12}^* & 0 \\ \hline B_{21}^* & B_{22}^* & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

where A_{11}^* and B_{22}^* are of respective sizes $r_1^{(2)} \times c_1^{(2)}$ and $(s - r_1^{(2)}) \times (s - c_1^{(2)})$ (with $r_1^{(2)} \le r_1^{(1)}$) and the matrix

$$\left[\begin{array}{c|c} A_{11}^* & B_{12}^* \\ \hline 0 & B_{22}^* \end{array}\right] \tag{12}$$

is invertible since it is square and of full column rank ($L^{(2)}$ is column-reduced). This implies that $c_1^{(2)} \le r_1^{(2)} \le s$ and B_{22}^* is of full row rank.

If $c_1^{(2)} = s$ then $r_1^{(2)} = s$ and $L^{(2)}$ is simultaneously row and column reduced since it is of the form

$$L^{(2)} = \left[\begin{array}{c|c} A_{11}^*\partial + B_{11}^* & 0 \\ \hline 0 & 0 \end{array}\right]$$

with invertible matrix A_{11}^*.

If $c_1^{(2)} = 0$ then $r_1^{(2)} = 0$ (recall that $r_1^{(2)}$ is the number of rows of $L^{(2)}$ of order 1). Thus, $L^{(2)}$ is of order 0 and of the form

$$L^{(2)} = \left[\begin{array}{c|c} B_{22}^* & 0 \\ \hline 0 & 0 \end{array}\right]$$

with invertible matrix B_{22}^*. Then $L^{(2)}$ is simultaneously row and column reduced and we are done.

Otherwise, if $0 \ne c_1^{(2)} = r_1^{(2)} \ne s$ then A_{11}^* is invertible and B_{22}^* is square. Since B_{22}^* is of full row rank then B_{22}^* is also invertible and so $L^{(2)}$ is simultaneously row and column reduced. If now $0 \ne c_1^{(2)} < r_1^{(2)}$, then the rank of the leading row coefficient matrix of $L^{(2)}$

$$\left[\begin{array}{c|c|c} A_{11}^* & 0 & 0 \\ \hline B_{12}^* & B_{22}^* & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

is less than s which means that $L^{(2)}$ is not row-reduced. Consequently, we now apply a row-reduction to $L^{(2)}$.

In this instance we will find that a unimodular multiplier U such that $U L^{(2)}$ is row-reduced is of order 0, that is, $U \in K[[x]]^{m\times m}$. Indeed, let $(v_1, v_2, 0) \in K[[x]]^{m\times 1}$, with v_1 and v_2 of sizes $r_1^{(2)} \times 1$ and $(s - r_1^{(2)}) \times 1$, respectively, such that

$$(v_1, v_2, 0) \left[\begin{array}{c|c|c} A_{11}^* & 0 & 0 \\ \hline B_{12}^* & B_{22}^* & 0 \\ \hline 0 & 0 & 0 \end{array}\right] = 0.$$

This implies that $v_2 B_{22}^* = 0$ which, since B_{22}^* is of full row rank implies $v_2 = 0$. Thus, we construct an invertible matrix $U_1 \in K[[x]]^{m\times m}$ such that the new operator $L^{(3)} = U_1 L^{(2)}$ is of the form

$$L^{(3)} = \left[\begin{array}{c|c|c} \widetilde{A}_{11}\partial + \widetilde{B}_{11} & \widetilde{B}_{12} & 0 \\ \hline \widetilde{B}_{21} & \widetilde{B}_{22} & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

with invertible matrix $\widetilde{A}_{11}$. As $L^{(2)}$ is column-reduced the following lemma implies that $L^{(3)}$ is also column-reduced.

Lemma 4. *Let $L \in K[[x]][\partial]^{m\times n}$ be a matrix of differential operators with A its leading column coefficient matrix. Let $U \in K[[x]]^{m\times m}$ be an invertible matrix over $K((x))$. Then, the leading column coefficient matrix of $U L$ is $U A$. Thus, if L is column-reduced then so is UL.*

Proof. Without any loss of generality we can assume that all the columns of L are nonzero. Recall that the jth column of the leading column coefficient matrix of $U L$ is the leading coefficient of the jth column of UL. Thus, it is sufficient to prove that the leading coefficient of the jth column of $U L$ is equal to U multiplied by the leading coefficient of the jth column of L, that is, $\ell c((U L)_{*,j}) = U\, \ell c(L_{*,j})$. The jth column of $U L$ is indeed equal to U multiplied by the jth column of L, *i.e.*,

$$(U L)_{*,j} = U L_{*,j}.$$

Let $\gamma_j = \mathrm{ord}(L_{*,j}) \geq 0$. Then $L_{*,j}$ can be written as

$$L_{*,j} = \ell c(L_{*,j})\, \partial^{\gamma_j} + \text{ terms of lower order},$$

with $\ell c(L_{*,j}) \neq 0$ (since all columns of L are assumed to be nonzero). Therefore,

$$(UL)_{*,j} = UL_{*,j} = U\, \ell c(L_{*,j})\, \partial^{\gamma_j} + \text{ terms of lower order},$$

with $U\, \ell c(L_{*,j}) \neq 0$ since $\ell c(L_{*,j}) \neq 0$ and U is invertible. Thus $\ell c((U L)_{*,j}) = U\, \ell c(L_{*,j})$. □

Now, $L^{(3)}$ is column-reduced with invertible matrix $\widetilde{A}_{11}$ and consequently, $\widetilde{B}_{22}$ is also invertible. Thus, $L^{(3)}$ is simultaneously row and column reduced and we have used at most 3 row–column–row reduction steps.

6. Reducing higher-order systems

In this section we let L be a simultaneously row and column reduced matrix of differential operators and assume that it is partitioned into blocks L_{ij} for $i, j = 1, \ldots, k$ satisfying the conditions (a)–(d) of Proposition 3. We are interested in the case where L_{kk} has order 0. In particular, we will show that when applied to linear differential systems of the form (1) then the problem can be decoupled into separate purely differential and purely algebraic problems.

Inspired by the definition of simple transformations on matrix polynomials given in Mulders and Storjohann (2003, page 379), we give the following definition:

Definition 9. Let L be a matrix of differential operators partitioned into blocks L_{ij} for $i, j = 1, \ldots, k$ with $k \geq 2$. Suppose that there exists $1 \leq d \leq k$ such that the block L_{dd} is square and has an invertible leading coefficient $\ell c(L_{dd})$.

- For any $l \neq d$ such that $\operatorname{ord}(L_{ld}) \geq \operatorname{ord}(L_{dd})$ the block row operation consisting in replacing $L_{l,*}$ by

$$L_{l,*} - \ell c(L_{ld})(\ell c(L_{dd}))^{-1}\partial^{\alpha} L_{d,*}$$

where $\alpha = \operatorname{ord}(L_{ld}) - \operatorname{ord}(L_{dd})$ is called a *simple row transformation of* $L_{d,*}$ *onto* $L_{l,*}$.
- For any $l \neq d$ such that $\operatorname{ord}(L_{dl}) \geq \operatorname{ord}(L_{dd})$ the block column operation consisting in replacing $L_{*,l}$ by

$$L_{*,l} - L_{*,d}(\ell c(L_{dd}))^{-1}\ell c(L_{dl})\partial^{\beta}$$

where $\beta = \operatorname{ord}(L_{dl}) - \operatorname{ord}(L_{dd})$ is called a *simple column transformation of* $L_{*,d}$ *onto* $L_{*,l}$.

Lemma 5. *Let N be the matrix of differential operators obtained after applying to* $L = [L_{ij}]$ *the simple row transformation of* $L_{d,*}$ *onto* $L_{l,*}$ *(resp. the simple column transformation of* $L_{*,d}$ *onto* $L_{*,l}$*). If N is partitioned as L then* $N_{i,*} = L_{i,*}$ *for* $i \neq l$ *and* $\operatorname{ord}(N_{ld}) < \operatorname{ord}(L_{ld})$ *(resp.* $N_{*,j} = L_{*,j}$ *for* $j \neq l$ *and* $\operatorname{ord}(N_{dl}) < \operatorname{ord}(L_{ld})$*).*

Proof. From Definition 9, the block rows $L_{i,*}$ for $i \neq l$ remain unchanged and hence $N_{i,*} = L_{i,*}$ for $i \neq l$. The inequality $\operatorname{ord}(N_{ld}) < \operatorname{ord}(L_{ld})$ follows from the relation $N_{ld} = L_{ld} - \ell c(L_{ld})(\ell c(L_{dd}))^{-1}\partial^{\alpha} L_{dd}$. A similar argument can be used for the case of simple column transformations. □

Assume now that $\operatorname{ord}(L_{kk}) = 0$. Then, using the next proposition, we can use simple row and column transformations to eliminate all the blocks above and before L_{kk}.

Proposition 8. *Assume that L is a simultaneously row and column reduced operator partitioned into blocks* L_{ij} *for* $i, j = 1, \ldots, k$ *satisfying the conditions (a)–(d) of Proposition 3 with* $\operatorname{ord}(L_{kk}) = 0$*. For any* $l < k$ *such that* $\operatorname{ord}(L_{lk}) = 0$*, let N denote the matrix of differential operators obtained by applying to L the simple transformation of* $L_{k,*}$ *onto* $L_{l,*}$*. Then, N is simultaneously row and column reduced, with a block partitioning into blocks* N_{ij} *with* $N_{lk} = 0$ *and* $\forall\, i = 1, \ldots, k$, $\operatorname{ord}(N_{ii}) = \operatorname{ord}(L_{ii})$ *and* $\ell c(N_{ii}) = \ell c(L_{ii})$*.*

Proof. From Lemma 5, we have that $N_{i,*} = L_{i,*}$ for $i \neq l$ and $\operatorname{ord}(N_{lk}) < \operatorname{ord}(L_{lk}) = 0$. Thus $\operatorname{ord}(N_{ii}) = \operatorname{ord}(L_{ii})$ and $\ell c(N_{ii}) = \ell c(L_{ii})$ for $i \neq l$ and $N_{lk} = 0$. To see that N remains simultaneously row and column reduced, it is sufficient to show that the blocks N_{ij} satisfy conditions (a)–(d) of Proposition 3. Since $N_{i,*} = L_{i,*}$ for $i \neq l$, we thus need to show that

(1) $\operatorname{ord}(N_{ll}) = \operatorname{ord}(L_{ll})$ and $\ell c(N_{ll}) = \ell c(L_{ll})$;
(2) for $j < l$, $\operatorname{ord}(N_{lj}) \leq \operatorname{ord}(N_{ll})$ and for $j > l$, $\operatorname{ord}(N_{lj}) \leq \operatorname{ord}(N_{jj})$.

For any $j < k$ we have

$$N_{lj} = L_{lj} - L_{lk}(L_{kk})^{-1}L_{kj}, \tag{13}$$

with $\operatorname{ord}(L_{lk}(L_{kk})^{-1}L_{kj}) \leq 0$. Therefore when $j = l$ then since $\operatorname{ord}(L_{ll}) > \operatorname{ord}(L_{kk}) = 0$ and $\operatorname{ord}(L_{lk}(L_{kk})^{-1}L_{kl}) \leq 0$ we get $\operatorname{ord}(N_{ll}) = \operatorname{ord}(L_{ll})$ and $\ell c(N_{ll}) = \ell c(L_{ll})$. On the other hand, when $j \neq l$ then

$$\operatorname{ord}(N_{lj}) \leq \max(\operatorname{ord}(L_{lj}), \operatorname{ord}(L_{lk}(L_{kk})^{-1}L_{kj})) \leq \max(\operatorname{ord}(L_{lj}), 0).$$

Therefore, for $j < l$, then $\operatorname{ord}(N_{lj}) \leq \operatorname{ord}(L_{ll}) = \operatorname{ord}(N_{ll})$ since $\operatorname{ord}(L_{ll}) > 0$ and $\operatorname{ord}(L_{lj}) \leq \operatorname{ord}(L_{ll})$. For $j > l$, we have $\operatorname{ord}(L_{jj}) \geq \operatorname{ord}(L_{kk}) = 0$ and $\operatorname{ord}(L_{lj}) \leq \operatorname{ord}(L_{jj})$ and so $\operatorname{ord}(N_{lj}) \leq \operatorname{ord}(L_{jj}) = \operatorname{ord}(N_{jj})$. □

An analogous result can be stated when working with columns. Consequently, we derive the following corollary:

Corollary 2. *Let L be a simultaneously row and column reduced matrix of differential operators partitioned into blocks* L_{ij} *with* $1 \leq i, j \leq k$ *and* $k \geq 2$ *satisfying the conditions* (a)–(d) *of Proposition 3 and* $\operatorname{ord}(L_{kk}) = 0$*. Then we can compute two unimodular matrices U and V with coefficients in* $\mathrm{K}((x))$ *such that U L V is a simultaneously row and column reduced matrix of differential operators of the form*

$$\left[\begin{array}{c|c} L^* & 0 \\ \hline 0 & L_{kk} \end{array}\right] = \left[\begin{array}{ccc|c} L^*_{11} & \cdots & L^*_{1\,k-1} & 0 \\ \vdots & & \vdots & \vdots \\ L^*_{k-1\,1} & \cdots & L^*_{k-1\,k-1} & 0 \\ \hline 0 & \cdots & 0 & L_{k\,k} \end{array}\right],$$

where for $1 \leq i \leq k-1$, $\operatorname{ord}(L^*_{ii}) = \operatorname{ord}(L_{ii}) > 0$ *and* $\ell c(L^*_{ii}) = \ell c(L_{ii})$*.*

In the case of a linear differential system given by

$$L \cdot \vec{y}(x) = \vec{f}(x)$$

we can separate such systems into purely differential and purely algebraic systems. Indeed, using Corollary 2, we see that the system can be transformed as

$$\begin{cases} L^* \cdot \vec{w}_1(x) = \vec{h}_1(x) \\ L_{kk}\, \vec{w}_2(x) = \vec{h}_2(x), \end{cases}$$

where

$$\begin{bmatrix} \vec{w}_1(x) \\ \vec{w}_2(x) \end{bmatrix} = V^{-1}\vec{y}(x) \quad \text{and} \quad \begin{bmatrix} \vec{h}_1(x) \\ \vec{h}_2(x) \end{bmatrix} = U\vec{f}(x).$$

Additionally, system $L^* \cdot \vec{w}_1(x) = \vec{h}_1(x)$ can be converted into a first-order system of ordinary differential equations as it is stated by the following proposition.

Proposition 9. *Let L be a simultaneously row and column reduced matrix of differential operators partitioned into blocks L_{ij} with $1 \le i, j \le k$ and $k \ge 2$ satisfying the conditions* (a)–(d) *of Proposition* 3 *and* $\mathrm{ord}(L_{kk}) > 0$. *For $i = 1, \ldots, k$, let m_i denote the dimension of the block L_{ii}. Then the differential system $L \cdot \vec{y}(x) = \vec{f}(x)$ can be converted into a first-order system of ordinary differential equations of size $\sum_{i=1}^{k} m_i \ \mathrm{ord}(L_{ii}) \times \sum_{i=1}^{k} m_i \ \mathrm{ord}(L_{ii})$ having an invertible leading coefficient.*

Example 8. Suppose that L is a simultaneously row and column reduced matrix of differential operators of the form

$$L = \begin{bmatrix} a_{11}\partial^3 + b_{11}\partial^2 + c_{11}\partial + d_{11} & b_{12}\partial^2 + c_{12}\partial + d_{12} \\ b_{21}\partial^2 + c_{21}\partial + d_{21} & b_{22}\partial^2 + c_{22}\partial + d_{22} \end{bmatrix}$$

where a_{11} and b_{22} are invertible matrices of sizes $m_1 \times m_1$ and $m_2 \times m_2$, respectively. Consider the differential system $L \cdot \vec{y}(x) = \vec{f}(x)$ with $\vec{y}(x)$ and $\vec{f}(x)$ partitioned into blocks of the same partition of L, that is,

$$\vec{y}(x) = \begin{bmatrix} \vec{y}_1(x) \\ \vec{y}_2(x) \end{bmatrix} \quad \text{and} \quad \vec{f}(x) = \begin{bmatrix} \vec{f}_1(x) \\ \vec{f}_2(x) \end{bmatrix}.$$

Then $L \cdot \vec{y}(x) = \vec{f}(x)$ can be converted into a system of ordinary differential equations of first-order and size $3m_1 + 2m_2$ of the form

$$\left(\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & a_{11} & 0 & b_{12} \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & b_{22} \end{bmatrix} \partial + \begin{bmatrix} 0 & -I & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 \\ d_{11} & c_{11} & b_{11} & d_{12} & c_{12} \\ 0 & 0 & 0 & 0 & -I \\ d_{21} & c_{21} & b_{21} & d_{22} & c_{22} \end{bmatrix} \right) \cdot \begin{bmatrix} \vec{y}_1(x) \\ \vec{y}_1{}'(x) \\ \vec{y}_1{}''(x) \\ \vec{y}_2(x) \\ \vec{y}_2{}'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vec{f}_1(x) \\ 0 \\ \vec{f}_2(x) \end{bmatrix}.$$

7. Conclusion

In this paper, we have developed a new two-sided row and column reduction algorithm for a matrix of differential operators. When applied to a linear system of differential–algebraic equations such a reduction decouples this into separate differential and algebraic system. Our algorithm leads to a complete mastering of the first-order differential–algebraic case. Our methods are entirely algebraic and easily extend to matrices of Ore operators having rational function coefficients.

The simultaneously row and column reduced form allows for both conversion to a first-order system and the extraction of algebraic constraints. As such it is an alternative to the Popov-form. We have presented two algorithms for the computation of a simultaneous row and column reduced form. The first, given in Section 2.2, does a series of row, then column, reductions. However, the result of any row reduction and then column operation reduction can undo a row reduction. Similarly a column reduction followed by a row reduction can undo a column reduction. Thus, while this approach seems simpler in concept, it has the drawback that we were unable to determine a polynomial complexity for the algorithm. In practice, this approach does appear to be reasonably efficient for small problems. Our second method, algorithm *Simultaneous Row and Column Reduced Form* given in Section 4, does have polynomial time complexity and is inspired by techniques used for computing Popov-forms. It is of interest to see if the first algorithm has polynomial complexity.

All our algorithms are defined on power series coefficients since we want to use them for the analysis of singular systems. At present the only methods for characterizing singularities are for the case of a first-order system. Ultimately our goal is to be able to do a local analysis for higher-order systems directly without the need for conversion to first-order. In order to accomplish this goal we will need a generalization of the concepts of Moser-reduction and super-reduction to higher-order systems. At present this remains an open problem. We expect that the first step in such a direction would be a generalization of Moser-reduction to differential–algebraic systems of the form $A\vec{y}\,'(x) = B\vec{y}(x)$ where A is not necessarily invertible.

Our methods have been implemented in the Computer Algebra system Maple. However a number of implementation-related details still need to be clarified. As an example, it is likely the case that one would want to use lazy evaluation in such computations.

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