Energy Shaping on Systems with Two Degrees of Underactuation and More than Three Degrees of Freedom

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Abstract

We study the stabilizability, via the method of energy shaping, of a given Lagrangian system with two degrees of underactuation and with $n \geq 4$ degrees of freedom. By making use of the formal theory of PDEs, we derive an involutive system of PDEs which governs energy shapability, and thus deduce, for the first time, easily verifiable conditions under which energy shaping is guaranteed. We illustrate our results with an example of a three-cart-one-inverted pendulum system.

1 Introduction

In this paper we study the stabilizability, via the method of energy shaping, of a given Lagrangian system with two degrees of underactuation and with $n \geq 4$ degree of freedom. The energy shaping method employs a feedback control so that the transformed system has a positive definite energy and a dissipative external force. The resulting closed loop system can then be stabilized by a further feedback using dissipative force. Historically the full use of the concept of energy shaping appears in [2, 3, 4, 5, 11]. The equivalence of the Lagrangian approach and Hamiltonian approach to energy shaping has been established in [8]. General matching conditions for energy shaping are derived in [1, 6, 7], but it is in [7] where the general setting of using gyroscopic force is considered, and where the idea of local force shaping is first introduced.

The results to date only focus on energy shaping problems with underactuation degree at most one, with a systematic treatment for any higher degree of underactuation still lacking. In this paper we focus on the case where we have two degrees of underactuation. To find out a control force under the framework of energy shaping one has to solve a system of nonlinear partial differential equations (PDEs), also known as matching conditions in this context. When the degree of underactuation is one the matching conditions give a system of two independent PDEs whose existence of solution was proved in [7]. However, higher degrees of underactuation result in more complex systems of PDEs, and hence it is not obvious if solutions exist for those systems. The complexity is not only due to the

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higher number of PDEs involved, but also due to the possibility of having integrability conditions within the system of PDEs. These integrability conditions arise when we take it into account that mixed partials of a dependent variable are the same. Generally speaking, each given PDE is differentiated (or "prolonged") a number of times, so that the integrability conditions are obtained by "projections", in which the higher order derivatives are eliminated through a process similar to Gaussian elimination, producing new lower order PDEs. These new PDEs are called integrability conditions. This process is repeated a number of times until no more integrability conditions arise. In this case the resulting system of PDEs is called involutive. This whole process of prolongations and projections can be done systematically through the formal theory of PDEs, as summarized in [12, 13, 14, 15]. A formal approach using the formal theory has been taken in [9] to study the energy shaping problem, However, their method centered on setting up intrinsic formalism rather than finding solutions. On the contrary, our work follows a more concrete direction and applies a coordinate-dependent version of the formal theory of PDEs to the set of matching conditions. This lets us derive an equivalent, involutive system of PDEs, in which we can directly obtain a solution to the energy shaping problem using the Cartan-Kähler theorem. From this we can, for the first time, derive a set of verifiable criteria under which energy shaping is possible for a given mechanical system.

The remainder of this paper is organized as follows. In the next section we review the basic background for energy shaping and results when the degree of underactuation is one. Higher degrees of underactuation requires some tools in the formal theory of PDEs, and these are reviewed in Section 3. In the next section we apply this formal theory to derive conditions under which we can obtain a solution from the matching conditions, starting with the case where the degree of freedom n is 4 and then generalize this to $n > 4$. Section 5 includes an example of three-cart-one-inverted pendulum system, a system of underactuation degree two.

2 Preliminaries

In this section we give the basic setting of energy shaping of control systems. We also state the so-called matching conditions and briefly mention results when the degree of underactuation is one.

2.1 Controlled Lagrangian Systems

We first review the basic scenario for the energy shaping problem. We view a configuration space Q as a n-dimensional differentiable manifold, on which we have the tangent bundle TQ and the cotangent bundle T^*Q.

Definition 2.1 ([7]). *A (simple) controlled Lagrangian system on TQ is a triple (L, F, W), in which*

(a) The Lagrangian $L(q,\dot{q}) = \frac{1}{2}m(\dot{q},\dot{q}) - V(q)$ on TQ, where $m \in \Gamma(S^2(T^*Q))$ [1] *is the positive definite, non-degenerate mass matrix, and where $\frac{1}{2}m(\dot{q},\dot{q})$ and $V(q)$ are the kinetic and potential energy, respectively, of the system;*

(b) $F : TQ \to T^*Q$ is an external force;

(c) W is a control bundle, which is a sub-bundle of T^*Q.

[1] $m \in \Gamma(S^2(T^*Q))$ means it is a section of the symmetric (0, 2)-tensor fields, i.e. $m(x, y) = m(y, x)$ for all $x, y \in TQ$.

In what follows, we call $n := \dim Q$ the degree of freedom, $n_2 := \dim W$ the degree of actuation and $n_1 := n - n_2$ the degree of underactuation.

By adopting the Einstein summation convention, the equations of motion in local coordinates are given by

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} &= F_i + u_i, \\ \Rightarrow \quad m_{ij}\ddot{q}^j + [jk,i]\dot{q}^j\dot{q}^k + \frac{\partial V}{\partial q^i} &= F_i + u_i, \end{aligned} \tag{1}$$

where $i = 1,\dots,n$ and $[ij,l]$ are the Christoffel symbols of the first kind such that

$$[ij,l] = \frac{1}{2}\Big(\frac{\partial m_{il}}{\partial q^j} + \frac{\partial m_{jl}}{\partial q^i} - \frac{\partial m_{ij}}{\partial q^l}\Big),$$

while F_i is the i-th component of the external force F, and u_i is the i-th component of the control force, where $i = 1,\dots,n$.

As in [7], we shape the energy function with the introduction of external force into the system. As such, we include a review of some notions with regard to forces. In particular, we only consider forces which can be decomposed into a sum of homogenous forces.

Definition 2.2. *A homogeneous force $F : TQ \to T^*Q$ of degree r on Q is a map defined as follows:*

$$F(v) = \underbrace{v \lrcorner v \lrcorner \cdots v \lrcorner}_{r\ times} \tilde{F}$$

for some section $\tilde{F}$ of $S^r(T^*Q) \otimes T^*Q$, where $\lrcorner$ denotes the interior product. With an abuse of notation, we sometimes identify F with $\tilde{F}$ such that we write $F(v,\dots,v,w) = \langle F(v), w\rangle$ for any $w \in TQ$, where $\langle , \rangle$ is the canonical pairing between T^*Q and TQ.

Definition 2.3. *A force $F : TQ \to T^*Q$ is said to be dissipative if $\langle F(v), v\rangle \le 0$ for all $v \in TQ$. It is gyroscopic if $\langle F(v), v\rangle = 0$ for all $v \in TQ$.*

In this paper, we will only consider forces which depend on velocity up to degree two.

Theorem 2.4 ([7])**.** *Suppose F has a homogeneous force decomposition: $F = F_1 + F_2$, where F_i is of degree i. Then F is dissipative if and only if F_1 is dissipative and F_2 is gyroscopic.*

2.2 Matching Conditions

Two controlled Lagrangian systems (L, F, W) and $(\widehat{L}, \widehat{F}, \widehat{W})$, where

$$L(q,\dot{q}) = \frac{1}{2}m(\dot{q},\dot{q}) - V(q) \quad \text{and} \quad \widehat{L}(q,\dot{q}) = \frac{1}{2}\widehat{m}(\dot{q},\dot{q}) - \widehat{V}(q),$$

are feedback equivalent if for any control $u \in W$, there exists $\widehat{u} \in \widehat{W}$ such that the closed loop dynamics are the same, and conversely. In this sense it can be proved [7] that this is equivalent to the following matching conditions.

Definition 2.5 ([7])**.** *Two controlled Lagrangian systems (L, F, W) and $(\widehat{L}, \widehat{F}, \widehat{W})$ are feedback equivalent if and only if*

ELM1 $\quad m^{-1}W = \widehat{m}^{-1}\widehat{W}$;

ELM2 $\quad \langle \mathcal{EL}(L) - F - m\widehat{m}^{-1}(\mathcal{EL}(\widehat{L}) - \widehat{F}), W^\circ\rangle = 0$, *where* $W^\circ = \{X \in TQ \mid \langle \alpha, X\rangle = 0, \forall \alpha \in W\}$ *and* $(\mathcal{EL})_i := \frac{d}{dt}\frac{\partial}{\partial \dot{q}^i} - \frac{\partial}{\partial q^i}$ *is the* i*-th component of the Euler-Lagrange operator.*

Suppose we now have two feedback equivalent systems (L, F, W) and $(\widehat{L}, \widehat{F}, \widehat{W})$, where $F = F_1 + F_2$ and $\widehat{F} = \widehat{F}_1 + \widehat{F}_2$ are their homogeneous force decompositions up to second degree. Then, by collecting terms of equal orders in $\dot{q}$ in **ELM2** of Definition 2.5, we can obtain the following matching conditions:

Theorem 2.6 (Matching Conditions [7])**.** (L, F, W) *and* $(\widehat{L}, \widehat{F}, \widehat{W})$ *are feedback equivalent systems if and only if the following equations are satisfied:*

$$
\begin{aligned}
(dV - m\widehat{m}^{-1}d\widehat{V})|_{W^\circ} &= 0\\
\widehat{F}_1(X, \widehat{m}^{-1}mZ) &= F_1(X, Z)\\
\widehat{F}_2(X, Y, \widehat{m}^{-1}mZ) &= \widehat{K}(X, Y, \widehat{m}^{-1}mZ) + F_2(X, Y, Z)\\
\widehat{W} &= \widehat{m}m^{-1}W
\end{aligned}
$$

for all $X, Y \in TQ$, $Z \in W^\circ$*. Here* $\widehat{K} \in \Gamma(S^2(T^*Q) \otimes T^*Q)$ *is a* T^*Q*-valued map defined using mass matrices* m *and* $\widehat{m}$ *and their associated connections* ∇, $\widehat{\nabla}$ *by:*

$$\widehat{K}(X, Y, T) = \widehat{m}(\widehat{\nabla}_X Y - \nabla_X Y, T),$$

for all $X, Y, T \in TQ$*.*

In what follows, we will always assume W is integrable, that is, there exists local coordinates $q^1, \ldots, q^n$ so that we can write

$$W^\circ = \text{Span}\left\{\frac{\partial}{\partial q^\alpha} \;\middle|\; \alpha = 1, \ldots, n_1\right\}, \quad W = \text{Span}\left\{dq^a \mid a = n_1 + 1, \ldots, n\right\}.$$

With the only exception in section 3 where the notions of formal theory of PDEs are reviewed, we will consistently use Greek and Roman alphabetical indices in the following manner:

$$\underbrace{\underbrace{1, \cdots, n_1}_{\alpha,\beta,\gamma,\cdots} \;;\; \underbrace{n_1 + 1, \cdots, n}_{a,b,c,\cdots}}_{i,j,k,\cdots}$$

To simplify our discussion, we will assume $F = 0$ for the given system. Then by some algebraic manipulations [7], we have the following matching conditions in local coordinates.

Theorem 2.7 ([7])**.** $(L, 0, W)$ *is feedback equivalent to* $(\widehat{L}, \widehat{F}, \widehat{W})$ *with a gyroscopic force* $\widehat{F}$ *of degree* 2 *if and only if there exists a non-degenerate mass matrix* $\widehat{m}$ *and a potential function* $\widehat{V}$ *such that the following equations are satisfied:*

$$\frac{\partial V}{\partial q^\alpha} - \widehat{T}_{j\alpha}m^{ij}\frac{\partial \widehat{V}}{\partial q^i} = 0 \tag{2}$$

$$\widehat{J}_{\alpha\beta\gamma} + \widehat{J}_{\beta\gamma\alpha} + \widehat{J}_{\gamma\alpha\beta} = 0 \tag{3}$$

where m_{ij} *(resp.* m^{ij}*) is the* (i, j)*-entry of* m *(resp.* m^{-1}*),* $\widehat{T} = m\widehat{m}^{-1}m$,[2] Γ^r_{ij} *are Christoffel symbols of the second kind,*[3] *and*

$$\widehat{J}_{\alpha\beta\gamma} = \frac{1}{2}\widehat{T}_{\gamma s}m^{sk}\left(\frac{\partial \widehat{T}_{\alpha\beta}}{\partial q^k} - \Gamma^r_{\beta k}\widehat{T}_{\alpha r} - \Gamma^r_{\alpha k}\widehat{T}_{\beta r}\right).$$

[2]We choose to use $\widehat{T}$ instead of $\widehat{m}$ so as to reduce the number of unknowns to be solved, [7].

[3]These are defined by $\Gamma^r_{ij} = m^{rl}[ij, l] = \frac{1}{2}m^{rl}\left(\frac{\partial m_{il}}{\partial q^j} + \frac{\partial m_{jl}}{\partial q^i} - \frac{\partial m_{ij}}{\partial q^l}\right)$.

2.3 Construction of Control Forces u and $\widehat{u}$

Suppose we have obtained a feasible solution $\widehat{T}$ (and hence $\widehat{m}$, since $\widehat{m} = m\widehat{T}m^{-1}$) and $\widehat{V}$ for the matching conditions. Then we can write down the Lagrangian $\widehat{L}$ for the feedback equivalent system, and also, by **ELM1**, the corresponding control bundle $\widehat{W}$ which is given by

$$\widehat{W} = \widehat{m}m^{-1}W.$$

In order to compute the gyroscopic force $\widehat{F}$, we need to find $\widehat{C}_{ijk}$ such that the k-th components $\widehat{F}_k$ of $\widehat{F}$ are given by $\widehat{F}_k = \widehat{C}_{ijk}\dot{q}^i\dot{q}^j$ with

$$\widehat{C}_{ijk} = \widehat{C}_{jik}; \qquad \widehat{C}_{ijk} + \widehat{C}_{jki} + \widehat{C}_{kij} = 0. \tag{4}$$

Following [6], we introduce

$$\widehat{S}_{ijk} = m_{ip}m_{jq}\widehat{m}^{pl}\widehat{m}^{qs}(m_{kr}\widehat{m}^{rt}\widehat{[ls,t]} - [ls,k]), \tag{5}$$

$$\widehat{A}_{ijk} = m_{ip}m_{jq}m_{kr}\widehat{m}^{pl}\widehat{m}^{qs}\widehat{m}^{rt}\widehat{C}_{lst}. \tag{6}$$

Notice that $\widehat{S}_{ijk} = \widehat{S}_{jik}$ and $\widehat{A}_{ijk} = \widehat{A}_{jik}$ for all i, j, k.

Once $\widehat{m}$ is determined, we can compute $\widehat{S}_{ijk}$. Then, we can determine $\widehat{A}_{ijk}$ in terms of $\widehat{S}_{ijk}$ using the following scheme:

(a) $\widehat{A}_{ij\alpha} = \widehat{S}_{ij\alpha}$.

(b) $\widehat{A}_{\beta\gamma a} = -\widehat{S}_{a\beta\gamma} - \widehat{S}_{\gamma a\beta}$.

(c) $\widehat{A}_{\gamma ab} = \widehat{A}_{b\gamma a} = -\frac{1}{2}\widehat{S}_{ab\gamma}$.

(d) Finally, we choose any $\widehat{A}_{abc}$ such that $\widehat{A}_{abc} + \widehat{A}_{bca} + \widehat{A}_{cab} = 0$. For simplicity, we can take $\widehat{A}_{abc} = 0$.

Notice that under this scheme, $\widehat{A}_{ijk}$ satisfy the properties in (4). Once $\widehat{A}_{ijk}$ are determined, we can obtain the gyroscopic force terms $\widehat{C}_{ijk}$ by (6), or equivalently,

$$\widehat{C}_{ijk} = \widehat{m}_{xi}\widehat{m}_{yj}\widehat{m}_{zk}m^{xr}m^{ys}m^{zt}\widehat{A}_{rst}. \tag{7}$$

Procedure for solving energy shaping problems. We can now summarize the general procedure for getting a nonlinear control force for a given controlled Lagrangian system with degree of underactuation equal to $n_1 \geq 1$:

S1. Check that the linearization of the given controlled Lagrangian is controllable or its uncontrollable subsystem is oscillatory.[4] If neither holds, then stop; otherwise, proceed to the next step. [7]

S2. Get a solution for $\widehat{V}$ and the (α, i) entries of $\widehat{T}$ which solve the matching PDEs (2) and (3), keeping in mind that the $n_1 \times n_1$ matrix $[T_{\alpha\beta}]$ is positive definite around $q = 0$ and $\widehat{V}$ has a non-degenerate minimum at 0.

S3. Choose the rest of the entries $\widehat{T}_{ab}$ of $\widehat{T}$ so that $\widehat{T}$ is positive definite, at least at $q = 0$.

[4] A linear system $\dot{x} = Ax$ is oscillatory if A is diagonalizable and all eigenvalues of A are nonzero and purely imaginary.

S4. Obtain the mass matrix $\widehat{m}$ of the feedback equivalent system, through the equation: $\widehat{m} = m\widehat{T}^{-1}m$.

S5. Compute the gyroscopic force $\widehat{F}$ by computing $\widehat{S}_{ijk}$, $\widehat{A}_{ijk}$ and then $\widehat{C}_{ijk}$ by (5), (7) and steps (a) – (d) between (6) and (7).

S6. Compute the control bundle $\widehat{W}$, which is given by

$$\widehat{W} = \text{Span}\left\{ \begin{bmatrix} m^{ai}\widehat{m}_{i1} \\ \cdots \\ m^{ai}\widehat{m}_{in} \end{bmatrix} \middle| \; a = n_1 + 1, \cdots, n \right\}$$

S7. Choose a dissipative, $\widehat{W}$-valued linear control force $\widehat{u}$. In particular, for systems with degree of underactuation equal to n_1, one may choose

$$\widehat{u} = -K^T D K \dot{q}, \tag{8}$$

where D is any symmetric positive definite $(n - n_1) \times (n - n_1)$ matrix and K is the $(n - n_1) \times n$ matrix defined by

$$K = \begin{bmatrix} m^{n_1+1i}\widehat{m}_{i1} & \cdots & m^{n_1+1i}\widehat{m}_{in} \\ \vdots & \ddots & \vdots \\ m^{ni}\widehat{m}_{i1} & \cdots & m^{ni}\widehat{m}_{in} \end{bmatrix}.$$

S8. Compute the corresponding control force u:

$$u_a = [jk, a]\dot{q}^j\dot{q}^k + \frac{\partial V}{\partial q^a} - m_{ar}\widehat{m}^{rs}\left(\widehat{[jk, s]}\dot{q}^j\dot{q}^k + \frac{\partial V}{\partial q^s} - \widehat{C}_{jks}\dot{q}^j\dot{q}^k - \widehat{u}_s\right) \tag{9}$$

where $a = n_1 + 1, \cdots, n$. Note that when $\alpha = 1, \ldots, n_1$, u_α, are then zero.

Notice that in the above procedure, we require $\widehat{F}$ to be gyroscopic and $\widehat{u}$ dissipative. This implies $\langle \widehat{F}, \dot{q} \rangle = 0$ and $\langle \widehat{u}, \dot{q} \rangle \le 0$ for each $(q, \dot{q})$. Hence the time derivative of the total energy $\widehat{E}$ of the feedback equivalent system is

$$\frac{d\widehat{E}}{dt} = \langle \widehat{F} + \widehat{u}, \dot{q} \rangle = 0 + \langle \widehat{u}, \dot{q} \rangle \le 0.$$

As a result, Lyapunov stability of the equilibrium $(q, \dot{q}) = (0, 0)$ is guaranteed.

2.4 Systems with One Degree of Underactuation

When a given system has only one degree of underactuation, the matching conditions in Theorem 2.7 reduce to two PDEs, one for $\widehat{V}$ and the other for $\widehat{T}$:

$$\begin{aligned} \frac{\partial V}{\partial q^1} - \widehat{T}_{j1}m^{ij}\frac{\partial \widehat{V}}{\partial q^j} &= 0 \\ \widehat{T}_{1s}m^{sk}\left(\frac{\partial \widehat{T}_{11}}{\partial q^k} - 2\Gamma^r_{1k}\widehat{T}_{1r}\right) &= 0. \end{aligned}$$

From the Frobenius theorem solutions to these 2 PDEs are known to always exist, and the shapability problem can be summarized as follows [7]:

Theorem 2.8 ([7])**.** *Given $(L, 0, W)$ with one degree of underactuation, let $(L^\ell, 0, W^\ell)$ be its linearized system at equilibrium $(q, \dot{q}) = (0, 0)$. Then there exists a feedback equivalent $(\widehat{L}, \widehat{F}, \widehat{W})$ with $\widehat{F}$ gyroscopic of degree 2 and $\widehat{V}$ having a non-degenerate minimum at $(0, 0)$ if and only if the uncontrollable dynamics, if any, of $(L^\ell, 0, W^\ell)$ is oscillatory.*[5] *In addition if $(L^\ell, 0, W^\ell)$ is controllable, then $(\widehat{L}, \widehat{F}, \widehat{W})$ can be exponentially stabilized by any linear dissipative feedback onto $\widehat{W}$.*

This theorem characterizes the energy shapability of a given system with one degree of underactuation.

3 The Formal Theory of PDEs

Unfortunately, the shapability problem becomes considerably more difficult when we have underactuation degree greater than one. In particular, integrability conditions arise when we equate the mixed partials for $\widehat{V}$ and $\widehat{T}_{ij}$. We need to solve the given system of PDEs together with its underlying integrability conditions. The latter can be systematically obtained by applying the formal theory of PDEs.

We will follow closely the approach introduced by Pommaret [12, 13]. First we start from a bundle $\pi : \mathcal{E} \to Q$ with independent variables $q^1, \ldots, q^n$ as coordinates of the base space and the dependent variables $u^1, \ldots, u^m$ as fiber coordinates. Then we construct the r-th jet bundle $J_r\mathcal{E}$ for $r \geq 1$ in which the fiber coordinates consist of $u^1, \ldots, u^m$ together with their derivatives up to order r. The canonical projection is denoted as $\pi_r^{r+s} : J_{r+s}\mathcal{E} \to J_r\mathcal{E}$.

Over each bundle we can define a section and its prolongation. A section is a map $\sigma : Q \to \mathcal{E}$ such that $\pi \circ \sigma = \mathrm{id}_Q$. The r-th prolongation of a section σ can be done locally by adding derivatives up to order r, that is,

$$j_r(\sigma) : q \to \left(q, f(q), \frac{\partial^{|\mu|} f(q)}{(\partial q^1)^{\mu_1} \cdots (\partial q^n)^{\mu_n}} \right),$$

where $\mu = (\mu_1, \ldots, \mu_n), 1 \leq |\mu| := \mu_1 + \ldots + \mu_n \leq r$.

Definition 3.1. *A system of PDEs of order r is a fibered submanifold $\mathcal{R}_r$ of $J_r\mathcal{E}$. A solution to $\mathcal{R}_r$ is a section σ such that $j_r(\sigma)$ lies in $\mathcal{R}_r$.*

The differential equations are usually defined as a map $\Phi : J_r\mathcal{E} \to \mathcal{E}'$ where $\Phi = \Phi^\tau(q^i, u^\alpha, p_\mu^\alpha)$, where

$$p_\mu^\alpha = \frac{\partial^{|\mu|} u^\alpha}{(\partial q^1)^{\mu_1} \cdots (\partial q^n)^{\mu_n}},$$

and $\mathcal{E}'$ is another bundle over Q. For each differential equation we can have two basic operations:

Prolongation: Imitating the usual chain rule of differentiation, we define the formal derivative $D_i\Phi$ for Φ by

$$D_i\Phi(q^i, u^\alpha, p_\mu^\alpha) = \frac{\partial \Phi^\tau}{\partial q^i} + \sum_\alpha \frac{\partial \Phi^\tau}{\partial u^\alpha} p_i^\alpha + \sum_{\alpha,\mu} \frac{\partial \Phi^\tau}{\partial p_\mu^\alpha} p_{\mu+1_i}^\alpha,$$

[5] One can show that any second order system $\ddot{x} = Ax$ is oscillatory if and only if A is diagonalizable and has only negative real eigenvalues [7].

where $\mu + 1_i = (\mu_1, \ldots, \mu_{i-1}, \mu_i + 1, \mu_{i+1}, \ldots, \mu_n)$.[6] We define the prolongation $\mathcal{R}_{r+1} \subseteq J_{r+1}\mathcal{E}$ for $\mathcal{R}_r$ as the set of equations $\Phi^\tau = 0, D_i\Phi^\tau = 0$, $i = 1, \ldots, n$. $\mathcal{R}_{r+1}$ is not necessarily a fibered submanifold. For $s \geq 2$, we can define $\mathcal{R}_{r+s}$ in a similar manner.

Projection: We can also project higher order differential equations into lower order ones. This is done by Gaussian elimination of higher order derivatives by the lower order ones in the equation.

The resulting system of PDEs arising from prolongations of $\mathcal{R}_r$ up to order s followed by projections into $\mathcal{R}_r$, that is, $\pi_r^{r+s}(\mathcal{R}_{r+s})$, is usually denoted as $\mathcal{R}_r^{(s)}$. The process of prolongations followed by projections does not necessarily retrieve the original system, that is, $\mathcal{R}_r^{(s)} \subsetneq \mathcal{R}_r$. The extra independent equations derived from these manipulations are known as integrability conditions. We have to obtain all possible integrability conditions of lower orders before we determine each coefficient of a formal series solution. In this regard, we introduce the idea of formal integrable equations:

Definition 3.2 ([12])**.** *A system $\mathcal{R}_r$ of order r is formally integrable if $\mathcal{R}_{r+s}$ is a fibered manifold for all $s \geq 0$ and $\pi_{r+s}^{r+s+t} : \mathcal{R}_{r+s+t} \to \mathcal{R}_{r+s}$ are epimorphisms for all $s, t \geq 0$.*

3.1 Symbols and Involutive Symbols

A direct verification of formal integrability as defined in Definition 3.2 is difficult computationally, as we have to check infinitely many times whether the projections are epimorphisms. It turns out that, nevertheless, simpler criteria for formal integrability exist and are partly related to an algebraic property of the highest order derivatives involved in the system, known as *involutivity*. We first construct the symbol for a system of PDEs which consists of the highest order derivatives only.

Definition 3.3. *The* symbol G_r *of a system $\mathcal{R}_r$ is defined to be a family of vector spaces whose local representation* [7] *is*

$$G_r : \qquad \sum_{|\mu|=r} \frac{\partial \Phi^\tau}{\partial p_\mu^\alpha}(q^i, u^\beta, p_\mu^\gamma) v_\mu^\alpha = 0,$$

where $\tau = 1, \ldots, p$; $\alpha, \beta, \gamma = 1, \ldots, m$, when $\mathcal{R}_r$ is locally represented as $\Phi^\tau(q^i, u^\beta, p_\mu^\gamma) = 0$.

By definition, the symbol G_{r+s} for the prolonged system $\mathcal{R}_{r+s}$ is given by

$$\sum_{|\mu|=r,\ |\nu|=s} \frac{\partial \Phi^\tau}{\partial p_\mu^\alpha}(q^i, u^\beta, p_\mu^\gamma) v_{\mu+\nu}^\alpha = 0,$$

with $(q^i, u^\beta, p_\mu^\gamma) \in \mathcal{R}_r$.

The symbol G_r provides a simple criterion to check whether extra integrability condition(s) will occur:

Theorem 3.4 ([15])**.** *If G_{r+1} is a vector bundle, then* $\dim \mathcal{R}_r^{(1)} = \dim \mathcal{R}_{r+1} - \dim G_{r+1}$.

[6] For the sake of brevity, we will denote $p_{1_i}^\alpha$ by p_i^α. This also conforms with the usual shorthand notations for first order partials.

[7] Here we resort to a local representation as definition to avoid much technicality using bundle formalism.

This is an important theorem in our later computations which we will frequently refer to. It can be rephrased as follows: if G_{r+1} is of full rank, then we do not have any integrability conditions. otherwise the difference between $\dim G_{r+1}$ and the number of prolonged equations in $\mathcal{R}_{r+1}$ is the number of integrability conditions (and can be figured out by Gaussian eliminations).

We now define an involutive symbol in a coordinate-based fashion. Notice that involutivity, on the contrary, is independent of the choice of coordinates [12]. The use of coordinates make actual computations easier.

We need a specific way of categorizing and prioritizing derivatives. First, we fix a set of local coordinates $q^1, \ldots, q^n$ on Q. In what follows, T^*Q is abbreviated as T^* for simplicity.

Definition 3.5. *With local coordinates $q^1, \ldots, q^n$, we can define the following:*

1. *A jet coordinate v^k_μ is said to be of class 1 if $\mu_1 \neq 0$. In general, it is of class i if $\mu_1 = \ldots = \mu_{i-1} = 0$ but $\mu_i \neq 0$.*
2. *Given a symbol G_r, we define for any $1 \leq i \leq n$, $(G_r)^i$ to be the set of elements of G_q with zero components of class $1, \cdots, i$.*

Now, we can solve the linear system defining G_r pointwise in a manner similar to finding the row reduced echelon form for a linear algebraic system via row operations. We first solve G_r with respect to the maximum number of components of class n, and replace these in the remaining equations. By so doing, only components of class i, where i is at most $n-1$ are left. Then we solve the remaining equations with respect to the maximum number of components of class $n-1$, leaving only components of class i with $i \leq n-2$. We repeat the above steps until we come to class 1 components. We say that the linear system for G_r is *solved*. In each class i equation in its solved form, where $1 \leq i \leq n$, the component of class i which is a linear combination of other components of class $\leq i$, is called the principal derivative, and the rest of other components are called parametric:

$$\begin{bmatrix} \text{principal} \\ \text{component} \\ \text{of class } i \end{bmatrix} + A(q^i, u^\beta, p^\gamma_\mu) \begin{bmatrix} \text{parametric} \\ \text{components} \\ \text{of class } \leq i \end{bmatrix} = 0.$$

We can then easily determine the size of $(G_r)^i$:

$$\dim(G_r)^i = \dim(S^r T^* \otimes E)^i - (\beta_r^{i+1} + \ldots + \beta_r^n), \qquad 1 \leq i \leq n,$$

where β^i_r is the number of equations of class i.

Theorem 3.6. *For any fixed local coordinates, we have*

$$\dim G_{r+1} \leq \alpha^1_r + 2\alpha^2_r + \ldots + n\alpha^n_r, \tag{10}$$

where $\alpha^i_r = \dim(G_r)^{i-1} - \dim(G_r)^i$. We say that G_r is involutive if there exist local coordinates such that the equality holds. Such local coordinates are called δ-regular.

When G_r is in its solved form, we can define the multiplicative variables for each equation of class i to be $q^1, \ldots, q^i$, and the non-multiplicative variables, or the dots, to be $q^{i+1}, \ldots, q^n$.

Theorem 3.7. *The symbol G_r is involutive if and only if there exists a system of local coordinates under which any prolongation with respect to the non-multiplicative variables does not introduce any new equations.*

3.2 Involutive Systems and Cartan-Kuranishi Theorem

With all these preparations we can now come to

Definition 3.8. *A system $\mathcal{R}_r \subseteq J_r(\mathcal{E})$ of order r on $\mathcal{E}$ is involutive if it is formally integrable and its symbol G_r is involutive.*

Further analysis[8] of the action of prolongations and projections leads to the following important and useful theorem:

Theorem 3.9 (Criterion of involutivity [12, 13])**.** *Let $\mathcal{R}_r \subseteq J_r(\mathcal{E})$ be a system of order r over $\mathcal{E}$ such that $\mathcal{R}_{r+1}$ is a fibered submanifold of $J_{r+1}(\mathcal{E})$. If G_r is involutive and if the map $\pi_r^{r+1} : \mathcal{R}_{r+1} \to \mathcal{R}_r$ is an epimorphism, then $\mathcal{R}_r$ is involutive.*

In other words, it is easier to obtain an involutive system if we start with an involutive symbol. We now state the following crucial theorem:

Theorem 3.10 (Cartan-Kuranishi theorem,[10, 12, 13, 15])**.** *For every strongly regular system[9] $\mathcal{R}_r$ of order r, there exist two integers s and t such that $\mathcal{R}_{r+s}^{(t)}$ is involutive and has the same solution space as $\mathcal{R}_r$.*

The general procedure for constructing this $\mathcal{R}_{r+s}^{(t)}$ works as follows: We begin with the symbol G_r of $\mathcal{R}_r$. We assume G_r is involutive, or else we prolong G_r finitely many times to get an involutive symbol.[10] Then we compare $\mathcal{R}_r$ with $\mathcal{R}_r^{(1)}$. If they are not the same, replace $\mathcal{R}_1$ by $\mathcal{R}_r^{(1)}$ and repeat the above procedure by checking involutivity of $R_r^{(1)}$. Notice that at any projection step, it might be possible to obtain inconsistent integrability conditions, in which we will not obtain an equivalent involutive system of PDEs. When an involutive system is obtained, we can conclude the existence of solution by the following theorem:

Theorem 3.11 (Cartan-Kähler theorem,[12])**.** *If $\mathcal{R}_r$ is an involutive and analytic system of order r, then there exists one and only one analytic solution $u^k = f^k(q)$ such that*

1. *$(q_0, \partial_\mu f^k(q_0))$ with $|\mu| \leq r-1$ is a point of $\pi_{r-1}^r(\mathcal{R}_r)$;*
2. *For $i = 1, \cdots, n$, the α_r^i parametric derivatives $\partial_\mu f^k(q)$ of class i are equal for $q^{i+1} = q_0^{i+1}, \cdots, q^n = q_0^n$ given analytic functions of $q^1, \cdots, q^i$.*

4 Energy Shaping with Two Degrees of Underactuation

In the previous section we described the set of tools that we will need to solve the PDEs for our energy shaping problem. In this section we describe a method for solving the resulting PDEs that occur when we have two degrees of underactuation. We first look at the case when of dimension $n = 4$ with the general case following in a similar fashion.

[8] For details, see [12].

[9] A system $\mathcal{R}_r$ is called strongly regular if $\mathcal{R}_{r+s}^{(t)}$ is a fibered manifold and the symbol $G_{r+s}^{(t)}$ of $\mathcal{R}_{r+s}^{(t)}$ is a vector bundle over Q for all $s, t \geq 0$ [13].

[10] The fact that we can obtain an involutive symbol by finitely many times of prolongations is highly nontrivial. A proof of this can be found in [16].

When the degree of underactuation $n_1 = 2$, the matching conditions result in 2 PDEs for $\widehat{V}$ and 4 for $\widehat{T}$:

$$\begin{aligned}
\widehat{T}_{1s} m^{sk} \tfrac{\partial \widehat{V}}{\partial q^k} &= \tfrac{\partial V}{\partial q^1} \\
\widehat{T}_{2s} m^{sk} \tfrac{\partial \widehat{V}}{\partial q^k} &= \tfrac{\partial V}{\partial q^2} \\
\widehat{T}_{1s} m^{sk} \left(\tfrac{\partial \widehat{T}_{11}}{\partial q^k} - 2\Gamma^r_{1k} \widehat{T}_{1r} \right) &= 0 \\
\widehat{T}_{2s} m^{sk} \left(\tfrac{\partial \widehat{T}_{11}}{\partial q^k} - 2\Gamma^r_{1k} \widehat{T}_{1r} \right) & + 2\widehat{T}_{1s} m^{sk} \left(\tfrac{\partial \widehat{T}_{12}}{\partial q^k} - \Gamma^r_{1k} \widehat{T}_{2r} - \Gamma^r_{2k} \widehat{T}_{1r} \right) = 0 \\
\widehat{T}_{1s} m^{sk} \left(\tfrac{\partial \widehat{T}_{22}}{\partial q^k} - 2\Gamma^r_{2k} \widehat{T}_{2r} \right) & + 2\widehat{T}_{2s} m^{sk} \left(\tfrac{\partial \widehat{T}_{12}}{\partial q^k} - \Gamma^r_{1k} \widehat{T}_{2r} - \Gamma^r_{2k} \widehat{T}_{1r} \right) = 0 \\
\widehat{T}_{2s} m^{sk} \left(\tfrac{\partial \widehat{T}_{22}}{\partial q^k} - 2\Gamma^r_{2k} \widehat{T}_{2r} \right) &= 0.
\end{aligned}$$

To simplify our argument, we introduce two auxiliary functions g_1 and g_2 so that the above system of PDEs is equivalent to

$$\mathcal{R}_1 : \left\{ \begin{array}{lrl}
\Phi_1 : & \widehat{T}_{1s} m^{sk} \tfrac{\partial \widehat{V}}{\partial q^k} & = \tfrac{\partial V}{\partial q^1} \\
\Phi_2 : & \widehat{T}_{2s} m^{sk} \tfrac{\partial \widehat{V}}{\partial q^k} & = \tfrac{\partial V}{\partial q^2} \\
\Phi_3 : & \widehat{T}_{1s} m^{sk} \left(\tfrac{\partial \widehat{T}_{11}}{\partial q^k} - 2\Gamma^r_{1k} \widehat{T}_{1r} \right) & = 0 \\
\Phi_4 : & \widehat{T}_{2s} m^{sk} \left(\tfrac{\partial \widehat{T}_{11}}{\partial q^k} - 2\Gamma^r_{1k} \widehat{T}_{1r} \right) & = -2g_1 \\
\Phi_5 : & \widehat{T}_{1s} m^{sk} \left(\tfrac{\partial \widehat{T}_{12}}{\partial q^k} - \Gamma^r_{1k} \widehat{T}_{2r} - \Gamma^r_{2k} \widehat{T}_{1r} \right) & = g_1 \\
\Phi_6 : & \widehat{T}_{2s} m^{sk} \left(\tfrac{\partial \widehat{T}_{12}}{\partial q^k} - \Gamma^r_{1k} \widehat{T}_{2r} - \Gamma^r_{2k} \widehat{T}_{1r} \right) & = g_2 \\
\Phi_7 : & \widehat{T}_{1s} m^{sk} \left(\tfrac{\partial \widehat{T}_{22}}{\partial q^k} - 2\Gamma^r_{2k} \widehat{T}_{2r} \right) & = -2g_2 \\
\Phi_8 : & \widehat{T}_{2s} m^{sk} \left(\tfrac{\partial \widehat{T}_{22}}{\partial q^k} - 2\Gamma^r_{2k} \widehat{T}_{2r} \right) & = 0 \ .
\end{array} \right.$$

In what follows, we define the following differential operators:

$$\begin{aligned}
X_1 &= X_1^k \frac{\partial}{\partial q^k} = \widehat{T}_{1s} m^{sk} \frac{\partial}{\partial q^k} \\
X_2 &= X_2^k \frac{\partial}{\partial q^k} = \widehat{T}_{2s} m^{sk} \frac{\partial}{\partial q^k} \\
X_3 &= X_3^k \frac{\partial}{\partial q^k} = \delta_{3s} m^{sk} \frac{\partial}{\partial q^k} \\
X_4 &= X_4^k \frac{\partial}{\partial q^k} = \delta_{4s} m^{sk} \frac{\partial}{\partial q^k} \ .
\end{aligned}$$

We assume that these four differential operators are linearly independent, say,

$$\widehat{T}_{11} \widehat{T}_{22} - (\widehat{T}_{12})^2 \neq 0. \tag{11}$$

Without loss of generality, one can further assume that $X_1^3 X_2^4 - X_2^3 X_1^4 \neq 0$. The latter inequation is used in the proof of subsequent lemmas.

4.1 Involutive Distribution Assumption

To minimize the number of integrability conditions at later stages, we further assume that the distribution spanned by X_1 and X_2 is involutive, that is, the Lie bracket $[X_1, X_2]$ should satisfy

$$[X_1, X_2] = f_1 X_1 + f_2 X_2, \tag{12}$$

for some analytic functions f_1 and f_2. Rewriting (12) as

$$[X_1, X_2] = f_1 X_1 + f_2 X_2 + 0 \cdot X_3 + 0 \cdot X_4$$

implies that this extra assumption brings about two new equations to the original system of PDEs, namely

$$\det(X_1, X_2, [X_1, X_2], X_4) = 0$$
$$\det(X_1, X_2, X_3, [X_1, X_2]) = 0.$$

We first derive some preliminary results for this assumption on X_1 and X_2.

Lemma 4.1. *On the system $\mathcal{R}_1$, the functions f_1 and f_2 in* (12) *are purely algebraic expression of $\widehat{T}_{ij}$, g_1 and g_2.*

Proof. By Cramer's rule, we know that

$$\begin{aligned} f_1 &= \frac{\det([X_1, X_2], X_2, X_3, X_4)}{\det(X_1, X_2, X_3, X_4)} = \frac{\det(m)\det([X_1, X_2], X_2, X_3, X_4)}{\det(m)\det(X_1, X_2, X_3, X_4)} \\ &= \frac{\det(Expr_k, \widehat{T}_{2k}, \delta_{3k}, \delta_{4k})}{\det(\widehat{T}_{1k}, \widehat{T}_{2k}, \delta_{3k}, \delta_{4k})} = \frac{Expr_1\ \widehat{T}_{22} - Expr_2\ \widehat{T}_{12}}{\widehat{T}_{11}\widehat{T}_{22} - (\widehat{T}_{12})^2}, \end{aligned}$$

where $Expr_k$, $k = 1, \dots, 4$ are defined by

$$Expr_k = m_{jk}\left(\widehat{T}_{1s}m^{si}\frac{\partial}{\partial q^i}(\widehat{T}_{2t}m^{tj}) - \widehat{T}_{2t}m^{ti}\frac{\partial}{\partial q^i}(\widehat{T}_{1s}m^{sj})\right).$$

Similarly, we have

$$f_2 = \frac{Expr_2\ \widehat{T}_{11} - Expr_1\ \widehat{T}_{12}}{\widehat{T}_{11}\widehat{T}_{22} - (\widehat{T}_{12})^2}.$$

It suffices to obtain an explicit formula for $Expr_k$. In this regard we have

$$\begin{aligned} Expr_k &= \widehat{T}_{1s}m^{si}\left(m_{jk}\frac{\partial}{\partial q^i}(\widehat{T}_{2t}m^{tj})\right) - \widehat{T}_{2t}m^{ti}\left(m_{jk}\frac{\partial}{\partial q^i}(\widehat{T}_{1s}m^{sj})\right) \\ &= \widehat{T}_{1s}m^{si}\left(\frac{\partial(\widehat{T}_{2t}m^{tj}m_{jk})}{\partial q^i} - \widehat{T}_{2t}m^{tj}\frac{\partial m_{jk}}{\partial q^i}\right) - \widehat{T}_{2t}m^{ti}\left(\frac{\partial(\widehat{T}_{1s}m^{sj}m_{jk})}{\partial q^i} - \widehat{T}_{1s}m^{sj}\frac{\partial m_{jk}}{\partial q^i}\right) \\ &= \widehat{T}_{1s}m^{si}\left(\frac{\partial \widehat{T}_{2k}}{\partial q^i} - \widehat{T}_{2t}m^{tj}\frac{\partial m_{jk}}{\partial q^i}\right) - \widehat{T}_{2t}m^{ti}\left(\frac{\partial \widehat{T}_{1k}}{\partial q^i} - \widehat{T}_{1s}m^{sj}\frac{\partial m_{jk}}{\partial q^i}\right) \\ &= \widehat{T}_{1s}m^{si}\frac{\partial \widehat{T}_{2k}}{\partial q^i} - \widehat{T}_{2t}m^{ti}\frac{\partial \widehat{T}_{1k}}{\partial q^i} - \widehat{T}_{1s}\widehat{T}_{2t}(m^{si}m^{tj} - m^{ti}m^{sj})\frac{\partial m_{jk}}{\partial q^i} \\ &= X_1\widehat{T}_{2k} - X_2\widehat{T}_{1k} - \widehat{T}_{1s}\widehat{T}_{2t}m^{si}m^{tj}\left(\frac{\partial m_{jk}}{\partial q^i} - \frac{\partial m_{ik}}{\partial q^j}\right). \end{aligned}$$

Using the definition of Christoffel symbols Γ^i_{jk}, we can further simplify the $m^{si}m^{tj}\left(\frac{\partial m_{jk}}{\partial q^i} - \frac{\partial m_{ik}}{\partial q^j}\right)$ term to obtain $m^{si}m^{tj}\left(\frac{\partial m_{jk}}{\partial q^i} - \frac{\partial m_{ik}}{\partial q^j}\right) = m^{si}\Gamma^t_{ik} - m^{tj}\Gamma^s_{kj}$ and hence

$$Expr_k = X_1\widehat{T}_{2k} - X_2\widehat{T}_{1k} - \widehat{T}_{1s}\widehat{T}_{2t}(m^{si}\Gamma^t_{ik} - m^{tj}\Gamma^s_{kj}).$$

We can conclude our proof by verifying that $Expr_1$ and $Expr_2$, after elimination of $X_\gamma\widehat{T}_{\alpha\beta}$, are purely algebraic. Such an elimination is possible by using the fact that $\widehat{T}_{ij}$ satisfy the four PDEs (Φ_4, Φ_5, Φ_6, Φ_7). Hence

$$\begin{aligned} Expr_1 &= X_1\widehat{T}_{12} - X_2\widehat{T}_{11} - \widehat{T}_{1s}\widehat{T}_{2t}(m^{si}\Gamma^t_{1i} - m^{tj}\Gamma^s_{1j}) \\ &= [g_1 + \widehat{T}_{1s}m^{si}(\Gamma^t_{1i}\widehat{T}_{2t} + \Gamma^t_{2i}\widehat{T}_{1t})] - [-2g_1 + 2\widehat{T}_{2s}m^{sj}\Gamma^t_{1j}\widehat{T}_{1t}] \\ &\qquad - \widehat{T}_{1s}m^{si}\Gamma^t_{1i}\widehat{T}_{2t} + \widehat{T}_{2s}m^{sj}\Gamma^t_{1j}\widehat{T}_{1t} \\ &= 3g_1 + \widehat{T}_{1s}m^{si}\Gamma^t_{2i}\widehat{T}_{1t} - \widehat{T}_{2t}m^{ti}\Gamma^s_{1i}\widehat{T}_{1s}. \end{aligned}$$

Similarly, we have

$$Expr_2 = -3g_2 + \widehat{T}_{1s} m^{si} \Gamma^t_{2i} \widehat{T}_{2t} - \widehat{T}_{2s} m^{si} \Gamma^t_{1i} \widehat{T}_{2t} \ .$$

□

With the extra assumption of involutive distribution, we now need to consider the solution for the following system of PDEs:

$$\overline{\mathcal{R}}_1 : \begin{cases} \Phi_1 : & \widehat{T}_{1s} m^{sk} \frac{\partial \widehat{V}}{\partial q^k} & = \frac{\partial V}{\partial q^1} \\ \Phi_2 : & \widehat{T}_{2s} m^{sk} \frac{\partial \widehat{V}}{\partial q^k} & = \frac{\partial V}{\partial q^2} \\ \Phi_3 : & \widehat{T}_{1s} m^{sk} \left(\frac{\partial \widehat{T}_{11}}{\partial q^k} - 2\Gamma^r_{1k} \widehat{T}_{1r} \right) & = 0 \\ \Phi_4 : & \widehat{T}_{2s} m^{sk} \left(\frac{\partial \widehat{T}_{11}}{\partial q^k} - 2\Gamma^r_{1k} \widehat{T}_{1r} \right) & = -2g_1 \\ \Phi_5 : & \widehat{T}_{1s} m^{sk} \left(\frac{\partial \widehat{T}_{12}}{\partial q^k} - \Gamma^r_{1k} \widehat{T}_{2r} - \Gamma^r_{2k} \widehat{T}_{1r} \right) & = g_1 \\ \Phi_6 : & \widehat{T}_{2s} m^{sk} \left(\frac{\partial \widehat{T}_{12}}{\partial q^k} - \Gamma^r_{1k} \widehat{T}_{2r} - \Gamma^r_{2k} \widehat{T}_{1r} \right) & = g_2 \\ \Phi_7 : & \widehat{T}_{1s} m^{sk} \left(\frac{\partial \widehat{T}_{22}}{\partial q^k} - 2\Gamma^r_{2k} \widehat{T}_{2r} \right) & = -2g_2 \\ \Phi_8 : & \widehat{T}_{2s} m^{sk} \left(\frac{\partial \widehat{T}_{22}}{\partial q^k} - 2\Gamma^r_{2k} \widehat{T}_{2r} \right) & = 0 \\ \Phi_9 : & \det(X_1, X_2, [X_1, X_2], X_4) & = 0 \\ \Phi_{10} : & \det(X_1, X_2, X_3, [X_1, X_2]) & = 0 \ . \end{cases}$$

We first observe that Φ_1 to Φ_8 in $\mathcal{R}_1$ can be grouped into four decoupled pairs (Φ_1 with Φ_2; Φ_3 with Φ_4, etc.), in which the differential operator, either X_1 or X_2, acts on $\widehat{V}$ and $\widehat{T}_{\alpha\beta}$. Such pairs are convenient in terms of symbol involutivity,

Lemma 4.2. *The system*

$$\begin{cases} X_1^k \dfrac{\partial \widehat{H}}{\partial q^k} & = h_1 \\ X_2^k \dfrac{\partial \widehat{H}}{\partial q^k} & = h_2 \end{cases}$$

where $\widehat{H} = \widehat{H}(q)$ is the unknown to be found, and h_1, h_2 are analytic functions which do not appear in the equations of the symbol of the system, has an involutive symbol. This system has an integrability condition given by $[X_1, X_2]\widehat{H} = X_1 h_2 - X_2 h_1$.

Proof. The proof of this lemma is given in the Appendix. □

Corollary 4.3. *The symbol G_1 for the system $\mathcal{R}_1$ (the one defined by Φ_1 to Φ_8 only) is involutive.*

Proof. By Lemma 4.2, each decoupled pair of PDEs forms an involutive system. Each pair is exclusively for the partials of one of the unknowns: $\widehat{V}$, $\widehat{T}_{11}$, $\widehat{T}_{12}$ or $\widehat{T}_{22}$. Hence, the whole system $\mathcal{R}_1$ defined by these four pairs has an involutive symbol. □

Lemma 4.2 states that we should have one integrability condition for each of $\widehat{V}$, $\widehat{T}_{11}$, $\widehat{T}_{12}$ and $\widehat{T}_{22}$. In particular, we can exploit some properties of the integrability condition for $\widehat{V}$.

Lemma 4.4. *The integrability condition for $\widehat{V}$ is purely algebraic in $\mathcal{R}_1$. We can use this equation to define $\widehat{T}_{13}$ algebraically provided that*

$$m^{s3}\frac{\partial^2 V}{\partial q^s \partial q^2} \neq 0. \tag{13}$$

In particular, $\widehat{T}_{13}$ can be algebraically defined only if (13) *holds at $q = 0$.*

Proof. By Lemma 4.2, the integrability condition for $\widehat{V}$ is given by

$$[X_1, X_2]\widehat{V} = X_1\left(\frac{\partial V}{\partial q^2}\right) - X_2\left(\frac{\partial V}{\partial q^1}\right).$$

Since $[X_1, X_2] = f_1 X_1 + f_2 X_2$, we have

$$f_1\frac{\partial V}{\partial q^1} + f_2\frac{\partial V}{\partial q^2} = X_1\left(\frac{\partial V}{\partial q^2}\right) - X_2\left(\frac{\partial V}{\partial q^1}\right). \tag{14}$$

The left hand side of (14) is purely algebraic, since we know f_1 and f_2 are purely algebraic from Lemma 4.1. The right hand side of (14) also does not contain any derivatives of unknown variables, since V is given. Hence, (14) is purely algebraic. We now show that this can algebraically define $\widehat{T}_{13}$. First, we note that the left hand side of (14) is equal to

$$\begin{aligned}
&\frac{1}{\widehat{T}_{11}\widehat{T}_{22} - (\widehat{T}_{12})^2}\left[\frac{\partial V}{\partial q^1}(Expr_1\widehat{T}_{22} - Expr_2\widehat{T}_{12}) + \frac{\partial V}{\partial q^2}(Expr_2\widehat{T}_{11} - Expr_1\widehat{T}_{12})\right] \\
&= \frac{1}{\widehat{T}_{11}\widehat{T}_{22} - (\widehat{T}_{12})^2}\left[\left(\frac{\partial V}{\partial q^1}\widehat{T}_{22} - \frac{\partial V}{\partial q^2}\widehat{T}_{12}\right)Expr_1 + \left(\frac{\partial V}{\partial q^2}\widehat{T}_{11} - \frac{\partial V}{\partial q^1}\widehat{T}_{12}\right)Expr_2\right] \\
&= \frac{1}{\widehat{T}_{11}\widehat{T}_{22} - (\widehat{T}_{12})^2}\left[\left(\frac{\partial V}{\partial q^1}\widehat{T}_{22} - \frac{\partial V}{\partial q^2}\widehat{T}_{12}\right)(3g_1 + \widehat{T}_{1s}m^{si}\Gamma^t_{2i}\widehat{T}_{1t} - \widehat{T}_{2t}m^{ti}\Gamma^s_{1i}\widehat{T}_{1s})\right. \\
&\qquad\left. + \left(\frac{\partial V}{\partial q^2}\widehat{T}_{11} - \frac{\partial V}{\partial q^1}\widehat{T}_{12}\right)(-3g_2 + \widehat{T}_{1s}m^{si}\Gamma^t_{2i}\widehat{T}_{2t} - \widehat{T}_{2s}m^{si}\Gamma^t_{1i}\widehat{T}_{2t})\right],
\end{aligned}$$

while the right hand side of (14) is equal to

$$\widehat{T}_{1s}m^{sk}\frac{\partial^2 V}{\partial q^k \partial q^2} - \widehat{T}_{2s}m^{sk}\frac{\partial^2 V}{\partial q^k \partial q^1}\ .$$

Now notice that g_1 first appears in Φ_4 and Φ_5. If we replace g_1 by

$$g_1 = \overline{g}_1 - \frac{1}{3}m^{3i}\Gamma^3_{2i}(\widehat{T}_{13})^2$$

and trace down the calculations, we conclude that all results obtained so far do not change by such replacement and, in addition, we can remove all quadratic terms of $\widehat{T}_{13}$ in (14).

Finally, since we assume $\frac{\partial V}{\partial q^i} = 0$ at $q = 0$ for $i = 1, \ldots, 4$, the left hand side of (14) vanishes at $q = 0$. Hence, in order to define $\widehat{T}_{13}$ using (14), we require the $\widehat{T}_{13}$ to be non-vanishing on the right hand side of (14), that is,

$$m^{s3}\frac{\partial^2 V}{\partial q^s \partial q^2} \neq 0$$

□

Remark. When (13) holds, then $\widehat{T}_{13}$ is defined by

$$\widehat{T}_{13} = \frac{(\widehat{T}_{11}\widehat{T}_{22} - (\widehat{T}_{12})^2)\left(-\widehat{T}_{1\widehat{s}}m^{\widehat{s}k}\frac{\partial^2 V}{\partial q^k \partial q^2} + \widehat{T}_{2s}m^{sk}\frac{\partial^2 V}{\partial q^k \partial q^1}\right) + P_1}{(\widehat{T}_{11}\widehat{T}_{22} - (\widehat{T}_{12})^2)m^{3k}\frac{\partial^2 V}{\partial q^k \partial q^2} - P_2}, \tag{15}$$

where $\widehat{s}$ runs through 1, 2 and 4 only, with P_1, P_2 defined by

$$\begin{aligned} P_1 = &\left(\frac{\partial V}{\partial q^1}\widehat{T}_{22} - \frac{\partial V}{\partial q^2}\widehat{T}_{12}\right)(3\overline{g}_1 + \widehat{T}_{1\widehat{s}}m^{\widehat{s}i}\Gamma^{\widehat{t}}_{2i}\widehat{T}_{1\widehat{t}} - \widehat{T}_{2t}m^{ti}\Gamma^{\widehat{s}}_{1i}\widehat{T}_{1\widehat{s}}) \\ &+ \left(\frac{\partial V}{\partial q^2}\widehat{T}_{11} - \frac{\partial V}{\partial q^1}\widehat{T}_{12}\right)(-3g_2 + \widehat{T}_{1\widehat{s}}m^{\widehat{s}i}\Gamma^{t}_{2i}\widehat{T}_{2t} - \widehat{T}_{2s}m^{si}\Gamma^{t}_{1i}\widehat{T}_{2t}) \\ P_2 = &\left(\frac{\partial V}{\partial q^1}\widehat{T}_{22} - \frac{\partial V}{\partial q^2}\widehat{T}_{12}\right)(\widehat{T}_{1\widehat{s}}m^{\widehat{s}i}\Gamma^{3}_{2i} + \widehat{T}_{1\widehat{t}}m^{3i}\Gamma^{\widehat{t}}_{2i} - \widehat{T}_{2s}m^{si}\Gamma^{3}_{1i}) \\ &+ \left(\frac{\partial V}{\partial q^2}\widehat{T}_{11} - \frac{\partial V}{\partial q^1}\widehat{T}_{12}\right)m^{3i}\Gamma^{t}_{2i}\widehat{T}_{2t}, \end{aligned}$$

where $\widehat{s}, \widehat{t}$ runs for 1, 2 and 4 only. Notice that due to the presence of partial derivatives of V, both P_1 and P_2 are zero at $q = 0$. We will make use of this fact in later proofs.

We now need to consider the solution for the following system of PDEs:

$$\overline{\mathcal{R}}_1 : \left\{ \begin{array}{lrl} \Phi_1 : & \widehat{T}_{1s}m^{sk}\frac{\partial \widehat{V}}{\partial q^k} & = \frac{\partial V}{\partial q^1} \\ \Phi_2 : & \widehat{T}_{2s}m^{sk}\frac{\partial \widehat{V}}{\partial q^k} & = \frac{\partial V}{\partial q^2} \\ \Phi_3 : & \widehat{T}_{1s}m^{sk}\left(\frac{\partial \widehat{T}_{11}}{\partial q^k} - 2\Gamma^r_{1k}\widehat{T}_{1r}\right) & = 0 \\ \Phi_4 : & \widehat{T}_{2s}m^{sk}\left(\frac{\partial \widehat{T}_{11}}{\partial q^k} - 2\Gamma^r_{1k}\widehat{T}_{1r}\right) & = -2(\overline{g}_1 - \frac{1}{3}m^{3i}\Gamma^3_{2i}(\widehat{T}_{13})^2) \\ \Phi_5 : & \widehat{T}_{1s}m^{sk}\left(\frac{\partial \widehat{T}_{12}}{\partial q^k} - \Gamma^r_{1k}\widehat{T}_{2r} - \Gamma^r_{2k}\widehat{T}_{1r}\right) & = \overline{g}_1 - \frac{1}{3}m^{3i}\Gamma^3_{2i}(\widehat{T}_{13})^2 \\ \Phi_6 : & \widehat{T}_{2s}m^{sk}\left(\frac{\partial \widehat{T}_{12}}{\partial q^k} - \Gamma^r_{1k}\widehat{T}_{2r} - \Gamma^r_{2k}\widehat{T}_{1r}\right) & = g_2 \\ \Phi_7 : & \widehat{T}_{1s}m^{sk}\left(\frac{\partial \widehat{T}_{22}}{\partial q^k} - 2\Gamma^r_{2k}\widehat{T}_{2r}\right) & = -2g_2 \\ \Phi_8 : & \widehat{T}_{2s}m^{sk}\left(\frac{\partial \widehat{T}_{22}}{\partial q^k} - 2\Gamma^r_{2k}\widehat{T}_{2r}\right) & = 0 \\ \Phi_9 : & \det(X_1, X_2, [X_1, X_2], X_4) & = 0 \\ \Phi_{10} : & \det(X_1, X_2, X_3, [X_1, X_2]) & = 0 \end{array} \right.$$

where g_1 is replaced by $\overline{g}_1 - \frac{1}{3}m^{3i}\Gamma^3_{2i}(\widehat{T}_{13})^2$ so that $\widehat{T}_{13}$ is well-defined by using the integrability condition for $\widehat{V}$. Here, we do not explicitly eliminate $\widehat{T}_{13}$ for the sake of clarity, but from now on, we should eliminate $\widehat{T}_{13}$ in the system of PDEs whenever it appears.

Lemma 4.5. *The symbol $\overline{G}_1$ of $\overline{\mathcal{R}}_1$, after eliminating $\widehat{T}_{13}$ using the integrability condition for $\widehat{V}$, is involutive if*

$$\widehat{T}_{1s}m^{s4} \neq 0 \tag{16}$$

$$\widehat{T}_{1s}m^{s4} - \frac{m^{3k}\frac{\partial^2 V}{\partial q^k \partial q^1}}{m^{3s}\frac{\partial^2 V}{\partial q^s \partial q^2}}\widehat{T}_{2t}m^{t4} \neq 0. \tag{17}$$

Proof. By Corollary 4.3, we know that the first eight PDEs (Φ_1 to Φ_8) constitute a system of PDEs with an involutive symbol. We now show that the whole system $\overline{\mathcal{R}}_1$, after eliminating $\widehat{T}_{13}$, has an involutive symbol. This is done by observing that Φ_9 and Φ_{10} can be treated as class 4 equations for $\widehat{T}_{23}$ and $\widehat{T}_{24}$. We first consider Φ_{10}, which is equivalent to $\det(\widehat{T}_{1k}, \widehat{T}_{2k}, \delta_{3k}, Expr_k) = 0$ or, more explicitly,

$$(\widehat{T}_{11}\widehat{T}_{22} - (\widehat{T}_{12})^2)(X_1\widehat{T}_{24} - X_2\widehat{T}_{14}) = 0,$$

in the equations of the symbol $\overline{G}_1$ of the system. Thus, this PDE can be used to solve $\frac{\partial \widehat{T}_{24}}{\partial q^4}$ provided that its coefficient in the PDE is nonzero, i.e. if (16) holds.

We now come to Φ_9, which is $\det(\widehat{T}_{1k}, \widehat{T}_{2k}, Expr_k, \delta_{4k}) = 0$ or more explicitly,

$$(\widehat{T}_{11}\widehat{T}_{22} - (\widehat{T}_{12})^2)(X_1\widehat{T}_{23} - X_2\widehat{T}_{13}) = 0$$

in the equations of the symbol $\overline{G}_1$. Making use of (15) to eliminate $\widehat{T}_{13}$, the above PDE in $\overline{G}_1$ around $q = 0$ is

$$(\widehat{T}_{11}\widehat{T}_{22} - (\widehat{T}_{12})^2)\left(X_1\widehat{T}_{23} - X_2\left(\frac{\widehat{T}_{2s}m^{sk}\frac{\partial^2 V}{\partial q^k \partial q^1} - \widehat{T}_{1\overline{s}}m^{\overline{s}k}\frac{\partial^2 V}{\partial q^k \partial q^2}}{m^{3s}\frac{\partial^2 V}{\partial q^s \partial q^2}}\right)\right) = 0 \ .$$

Hence, Φ_9 can be used to define $\frac{\partial \widehat{T}_{23}}{\partial q^4}$ provided that its coefficient is nonzero, or equivalently, if (17) holds. Since Φ_9 and Φ_{10} are both PDEs of class 4 and the rest of the system $\overline{\mathcal{R}}_1$ has an involutive symbol, we can conclude that the symbol $\overline{G}_1$ of the whole system is involutive. □

Since $\overline{\mathcal{R}}_1$ differs from $\mathcal{R}_1$ by having two extra equations of class 4, the number of integrability conditions in $\overline{\mathcal{R}}_1$ is still four. The one for $\widehat{V}$ has been used to define and eliminate $\widehat{T}_{13}$. Hence, we are left with the integrability conditions for $\widehat{T}_{11}$, $\widehat{T}_{12}$ and $\widehat{T}_{22}$. If we can show that these equations are also of class 4, then we can conclude that $\overline{\mathcal{R}}_1^{(1)}$ is involutive and the whole prolongation-projection algorithm ends.

Lemma 4.6. *The integrability conditions for $\widehat{T}_{11}$, $\widehat{T}_{12}$ and $\widehat{T}_{22}$ in their solved forms on the system $\mathcal{R}_1^{(1)}$ are of class 4 if*

$$\widehat{T}_{2s}m^{s4} \neq 0 \tag{18}$$

$$\widehat{T}_{1s}m^{s4}\widehat{T}_{1t}m^{tk}\Gamma^4_{2k} \neq \widehat{T}_{2s}m^{s4}\widehat{T}_{1t}m^{tk}\Gamma^4_{1k}. \tag{19}$$

Proof. We first derive, in the equations of the symbol of the system, the three integrability conditions explicitly. By Lemma 4.2 and the involutive assumption on the differential operators X_1 and X_2, the integrability condition for $\widehat{T}_{11}$ is

$$[X_1, X_2]\widehat{T}_{11} = (f_1X_1 + f_2X_2)\widehat{T}_{11} \ .$$

By Lemma 4.1, f_1 and f_2 are purely algebraic, and we can eliminate $X_1\widehat{T}_{11}$ and $X_2\widehat{T}_{11}$, as they satisfy Φ_3 and Φ_4, by purely algebraic expressions. Thus the right hand side of the above equation, after such elimination, does not appear in the equations of the symbol of the system. In other words, we can simply consider the left hand side of the above integrability condition:

$$\begin{aligned}[X_1, X_2]\widehat{T}_{11} &= X_1(X_2\widehat{T}_{11}) - X_2(X_1\widehat{T}_{11}) \\ &= X_1\left(2\widehat{T}_{2s}m^{sk}\Gamma^r_{1k}\widehat{T}_{1r} - 2\overline{g}_1 + \frac{2}{3}m^{3i}\Gamma^3_{2i}(\widehat{T}_{13})^2\right) - X_2(2\widehat{T}_{1s}m^{sk}\Gamma^r_{1k}\widehat{T}_{1r}),\end{aligned}$$

by using Φ_3 and Φ_4. Now, note that $X_1\widehat{T}_{11} = 0$ and $X_2\widehat{T}_{11} = 0$ in the symbol $\overline{G}_1$. Thus, in the equations of the symbol of the system, the integrability condition for $\widehat{T}_{11}$ reduces to

$$\begin{aligned}2\big(-X_1\overline{g}_1 + \widehat{T}_{2s}m^{sk}\Gamma^{\overline{r}}_{1k}X_1\widehat{T}_{1\overline{r}} + \widehat{T}_{1r}m^{\overline{s}k}\Gamma^r_{1k}X_1\widehat{T}_{2\overline{s}} \\ - \widehat{T}_{1s}m^{sk}\Gamma^{\overline{r}}_{1k}X_2\widehat{T}_{1\overline{r}} - \widehat{T}_{1r}m^{\overline{s}k}\Gamma^r_{1k}X_2\widehat{T}_{1\overline{s}}\big) + \cdots,\end{aligned}$$

where $\overline{r}$ and $\overline{s}$ run from 3 to 4 only, and the terms not containing derivatives of $\widehat{T}_{14}$ are omitted. From Φ_9 and Φ_{10} we know that $X_1\widehat{T}_{23} = X_2\widehat{T}_{13}$ and $X_1\widehat{T}_{24} = X_2\widehat{T}_{14}$ in the equations of the symbol G_1 of the system (as mentioned in the proof of Lemma 4.5). Hence, the integrability condition for $\widehat{T}_{11}$ can further reduce to

$$2(-X_1\overline{g}_1 + \widehat{T}_{2s}m^{sk}\Gamma^{\overline{r}}_{1k}X_1\widehat{T}_{1\overline{r}} - \widehat{T}_{1s}m^{sk}\Gamma^{\overline{r}}_{1k}X_2\widehat{T}_{1\overline{r}}) + \cdots = 0,$$

in the equations of the symbol $\overline{G}_1$, and we omit again terms not containing derivatives of $\widehat{T}_{14}$. In a similar fashion one can derive the other integrability conditions

$$X_2g_2 + \widehat{T}_{2s}m^{sk}\Gamma^{\overline{r}}_{2k}X_2\widehat{T}_{1\overline{r}} - \widehat{T}_{1s}m^{sk}\Gamma^{\overline{r}}_{2k}X_2\widehat{T}_{2\overline{r}} = 0$$
$$X_1g_2 - X_2\overline{g}_1 + \widehat{T}_{2s}m^{sk}(\Gamma^{\overline{r}}_{1k}X_2\widehat{T}_{1\overline{r}} + \Gamma^{\overline{r}}_{2k}X_1\widehat{T}_{1\overline{r}}) - \widehat{T}_{1s}m^{sk}(\Gamma^{\overline{r}}_{1k}X_2\widehat{T}_{2\overline{r}} + \Gamma^{\overline{r}}_{2k}X_2\widehat{T}_{1\overline{r}}) + \cdots = 0$$

in the equations of the symbol $\overline{G}_1$. We now show that these PDEs can solve $\frac{\partial \overline{g}_1}{\partial q^4}$, $\frac{\partial g_2}{\partial q^4}$ and $\frac{\partial \widehat{T}_{14}}{\partial q^4}$ respectively provided that (18) and (19) are satisfied. This is done by computing the determinant of the coefficient matrix of these three derivatives:

$$\begin{vmatrix} -\widehat{T}_{1s}m^{s4} & 0 & \widehat{T}_{2s}m^{sk}\Gamma^4_{1k}\widehat{T}_{1t}m^{t4} - \widehat{T}_{1s}m^{sk}\Gamma^4_{1k}\widehat{T}_{2t}m^{t4} \\ 0 & \widehat{T}_{2s}m^{s4} & \widehat{T}_{2s}m^{sk}\Gamma^4_{2k}\widehat{T}_{2t}m^{t4} \\ -\widehat{T}_{2s}m^{s4} & \widehat{T}_{1s}m^{s4} & \widehat{T}_{2s}m^{sk}\Gamma^4_{2k}\widehat{T}_{1t}m^{t4} + \widehat{T}_{2s}m^{sk}\Gamma^4_{1k}\widehat{T}_{2t}m^{t4} - \widehat{T}_{1s}m^{sk}\Gamma^4_{2k}\widehat{T}_{2t}m^{t4} \end{vmatrix}$$

which simplifies to give $(\widehat{T}_{2s}m^{s4})^2(\widehat{T}_{1s}m^{s4}\widehat{T}_{1t}m^{tk}\Gamma^4_{2k} - \widehat{T}_{2s}m^{s4}\widehat{T}_{1t}m^{tk}\Gamma^4_{1k})$. We can solve the three class 4 derivatives uniquely if and only if the coefficient matrix has a nonzero determinant. This concludes the proof. □

Remark. In the proof we are not concerned about derivatives of unknowns other than $\overline{g}_1$, g_2 and $\widehat{T}_{14}$ though they may appear in the symbol as well. This is valid in the proof as we use the three integrability conditions to define derivatives of $\overline{g}_1$, g_2 and $\widehat{T}_{14}$ only.

We can now summarize our results into the following

Theorem 4.7. *If $n = 4$, and if* (11), (13), (16), (17), (18) *and* (19) *hold, at least at $q = 0$, then the system $\overline{\mathcal{R}}^{(1)}_1$ is involutive.*

Proof. $\overline{\mathcal{R}}^{(1)}_1$ is defined by Φ_1 to Φ_{10}, together with 4 equations, derived from the integrability conditions for $\widehat{V}$, $\widehat{T}_{11}$, $\widehat{T}_{12}$ and $\widehat{T}_{22}$. The one for $\widehat{V}$, as proved in Lemma 4.4, solves $\widehat{T}_{13}$ if (13) holds. The resulting system of PDEs, after eliminating $\widehat{T}_{13}$, still has an involutive symbol. The reason for this is two-fold. First, Φ_1 to Φ_{10} constitute a system of PDEs with involutive symbol, as proved in Lemma 4.5. Secondly, by Lemma 4.6, the extra integrability conditions from $\widehat{T}_{\alpha\beta}$ are of class 4, if (18) and (19) hold.

Now, by Theorem 3.9, if we can show that $\overline{\mathcal{R}}^{(1)}_1 = \pi^2_1((\overline{\mathcal{R}}^{(1)}_1)_{+1})$, then we can conclude that $\overline{\mathcal{R}}^{(1)}_1$ is involutive. But such an equality is true since, with the exception of the integrability condition for $\widehat{V}$, all integrability conditions for $\overline{\mathcal{R}}_1$ are of class 4, and hence we cannot generate further integrability conditions. □

It should be noted that the above procedure of obtaining an involutive system of PDEs is coordinate-dependent. Here we abide by the choice of coordinates as depicted in [12], [13], where $\frac{\partial}{\partial q^i}$ are classified as class i, and we place higher priority for those derivatives in higher classes. One can choose to prioritize coordinates in several different manners, for example, we can define $\frac{\partial}{\partial q^1}$ as class 4 (i.e. highest priority) etc., and obtain an involutive system with a similar set of inequality constraints. In other words, we have the following.

Theorem 4.8. *If $n = 4$, and the following inequalities*

$$\widehat{T}_{11}\widehat{T}_{22} - (\widehat{T}_{12})^2 \neq 0 \; (X_1^1 X_2^2 - X_1^2 X_2^1 \neq 0) \tag{20}$$

$$m^{3s}\frac{\partial^2 V}{\partial q^s \partial q^2} \neq 0 \tag{21}$$

$$\widehat{T}_{1s}m^{s1} \neq 0 \tag{22}$$

$$\widehat{T}_{1s}m^{s1} - \frac{m^{3k}\frac{\partial^2 V}{\partial q^k \partial q^1}}{m^{3s}\frac{\partial^2 V}{\partial q^s \partial q^2}}\widehat{T}_{2t}m^{t1} \neq 0 \tag{23}$$

$$\widehat{T}_{2s}m^{s1} \neq 0 \tag{24}$$

$$\widehat{T}_{1s}m^{s1}\widehat{T}_{1t}m^{tk}\Gamma^4_{2k} \neq \widehat{T}_{2s}m^{s1}\widehat{T}_{1t}m^{tk}\Gamma^4_{1k} \tag{25}$$

hold (at least at $q = 0$), then the system $\overline{\mathcal{R}}_1^{(1)}$ is involutive.

4.2 The Case when $n \geq 4$

The generalization to the case $n \geq 4$ is in fact rather straightforward. First of all, Φ_1 to Φ_8 remain the same except that the indices $r, s, t, \ldots$ runs from 1 to n instead of 1 to 4. We need n linearly independent differential operators X_i, that is,

$$\begin{aligned}
X_1 &= \widehat{T}_{1s}m^{sk}\frac{\partial}{\partial q^k} \\
X_2 &= \widehat{T}_{2s}m^{sk}\frac{\partial}{\partial q^k} \\
X_i &= \delta_{is}m^{sk}\frac{\partial}{\partial q^k}, \qquad i \geq 3 \ .
\end{aligned}$$

As before, we can make the assumption that the differential operators X_1 and X_2 span an involutive distribution, that is, assumption (12). The way we choose to define X_i allows f_1 and f_2 in (12) remains purely algebraic, as in Lemma 4.1. The only difference for $n > 4$ is the number of extra equations due to this involutivity assumption. Previously when $n = 4$, we have two extra PDEs (Φ_9 and Φ_{10}). When $n > 4$, we would have $n - 2$ extra PDEs:

$$\begin{aligned}
\det(X_1, X_2, [X_1, X_2], X_4, X_5, \ldots, X_{n-1}, X_n) &= 0 \\
\det(X_1, X_2, [X_1, X_2], X_3, X_5, \ldots, X_{n-1}, X_n) &= 0 \\
&\cdots\cdots\cdots \\
\det(X_1, X_2, [X_1, X_2], X_3, X_4, \ldots, X_{n-2}, X_{n-1}) &= 0 \ .
\end{aligned}$$

In other words, every time n increases by 1, we have one additional PDE. Nevertheless, we have two more entries in $\widehat{T}$ in the meantime. Indeed, we can assign each of these extra PDEs to solve the class 4 derivatives of $\widehat{T}_{23}, \widehat{T}_{24}, \ldots, \widehat{T}_{2n}$, and still have some free entries in the first row of $\widehat{T}$. Notice that (16) and (17) will guarantee that we can solve these class n derivatives. Finally, the proof of Lemma 4.6 (i.e. the integrability conditions for $\widehat{T}_{\alpha\beta}$ are all of class n) is essentially the same for $n > 4$. Hence, if we define $\frac{\partial}{\partial q^n}$ as class n derivatives etc., then we will have the following generalization of Theorem 4.7.

$\widehat{T}_{13}$ defined by (15) — free entries

$\widehat{T}_{11}$	$\widehat{T}_{12}$	$\widehat{T}_{13}$	$\widehat{T}_{14}$	....	$\widehat{T}_{1n}$
	$\widehat{T}_{22}$	$\widehat{T}_{23}$	$\widehat{T}_{24}$	....	$\widehat{T}_{2n}$

partials defined by determinant equations

Table 1: $\widehat{T}$ matrix with solved and free entries. Note that $\widehat{T}_{\alpha\beta}$ are solved by Φ_3 through Φ_8; and class n derivative of $\widehat{T}_{14}$ is solved by one of the integrability conditions from $\widehat{T}_{\alpha\beta}$.

Theorem 4.9. *$\overline{\mathcal{R}}_1^{(1)}$ is involutive if the following holds (at least at $q = 0$)*

$$\widehat{T}_{11}\widehat{T}_{22} - (\widehat{T}_{12})^2 \neq 0 \; (X_1^{n-1}X_2^n - X_2^{n-1}X_1^n \neq 0) \tag{26}$$

$$m^{s3}\frac{\partial^2 V}{\partial q^s \partial q^2} \neq 0 \tag{27}$$

$$\widehat{T}_{1s}m^{sn} \neq 0 \tag{28}$$

$$\widehat{T}_{1s}m^{sn} - \frac{m^{3k}\frac{\partial^2 V}{\partial q^k \partial q^1}}{m^{3s}\frac{\partial^2 V}{\partial q^s \partial q^2}}\widehat{T}_{2t}m^{tn} \neq 0 \tag{29}$$

$$\widehat{T}_{2s}m^{sn} \neq 0 \tag{30}$$

$$\widehat{T}_{1s}m^{sn}\widehat{T}_{1t}m^{tk}\Gamma^4_{2k} \neq \widehat{T}_{2s}m^{sn}\widehat{T}_{1t}m^{tk}\Gamma^4_{1k} \; . \tag{31}$$

As before, similar conditions can be derived if we prioritize partials in various different manners. In particular, when we rank $\frac{\partial}{\partial q^1}$ as class n derivatives, etc., then we will have the following alternate generalization of Theorem 4.7.

Corollary 4.10. *$\overline{\mathcal{R}}_1^{(1)}$ is involutive if* (20) *to* (25) *hold (at least at $q = 0$).*

Since $\overline{\mathcal{R}}_1^{(1)}$ is involutive, it is natural to ask if we have an analytic solution. The answer is affirmative by the following theorem of stabilizability.

Theorem 4.11. *Let $(L, 0, W)$ be a controlled Lagrangian system with $n \geq 4$ degrees of freedom having a linearized system $(L^\ell, 0, W^\ell)$. Suppose the uncontrollable dynamics of $(L^\ell, 0, W^\ell)$, if any, is oscillatory, and that there exists a linear controlled Lagrangian system $(\overline{L}, 0, \overline{W})$ feedback equivalent to $(L^\ell, 0, W^\ell)$ such that the inequations* (11), (13), (16), (17), (18) *and* (19) *are satisfied.*[11]

Then there exists a controlled Lagrangian system $(\widehat{L}, \widehat{F}, \widehat{W})$ that is feedback equivalent to $(L, 0, W)$, with a positive definite mass matrix $\widehat{m}$, a gyroscopic force $\widehat{F}$ of degree 2, and a potential function $\widehat{V}$ having a non-degenerate minimum at $q = 0$. In particular, we can obtain a nonlinear controlled Lagrangian system $(\widehat{L}, \widehat{F}, \widehat{W})$ whose linearization is equal to $(\overline{L}, 0, \overline{W})$. Furthermore, if $(L^\ell, 0, W^\ell)$ is controllable, then any linear dissipative feedback force onto $\widehat{W}$ exponentially stabilizes the system $(\widehat{L}, \widehat{F}, \widehat{W})$.

Proof. We first need to check that defining $\widehat{T}_{13}$ by (15) does not bring any extra restriction

[11] Here it is understood that $\widehat{T}_{ij}$ are replaced by $\overline{T}_{ij}$ in those equations.

to the linearized system. Indeed, at $q = 0$, (15) reduces to

$$\widehat{T}_{2s} m^{sk} \frac{\partial^2 V}{\partial q^k \partial q^1}\bigg|_{q=0} = \widehat{T}_{1s} m^{sk} \frac{\partial^2 V}{\partial q^k \partial q^1}\bigg|_{q=0}$$

$$\Rightarrow \qquad \widehat{T}_{2s} m^{sk} \frac{\partial}{\partial q^k}\left(\widehat{T}_{1t} m^{tl} \frac{\partial \widehat{V}}{\partial q^l}\right)\bigg|_{q=0} = \widehat{T}_{1s} m^{sk} \frac{\partial}{\partial q^k}\left(\widehat{T}_{2t} m^{tl} \frac{\partial \widehat{V}}{\partial q^l}\right)\bigg|_{q=0}.$$

Since $\frac{\partial \widehat{V}}{\partial q^i}(0) = 0$, the above equation reduces further to

$$\widehat{T}_{2s} m^{sk} \widehat{T}_{1t} m^{tl} \frac{\partial^2 \widehat{V}}{\partial q^k \partial q^l}\bigg|_{q=0} = \widehat{T}_{1s} m^{sk} \widehat{T}_{2t} m^{tl} \frac{\partial^2 \widehat{V}}{\partial q^l \partial q^k}\bigg|_{q=0},$$

which is obviously true.

Hence, we conclude that there are analytic solutions for $\widehat{T}$ and $\widehat{V}$ once we impose suitable initial conditions. We look for initial conditions from the linearized system $(L^\ell, 0, W^\ell)$ of the given controlled Lagrangian system $(L, 0, W)$. It can be proven (c.f. [7]) that there exists a linear controlled Lagrangian system $(\overline{L}, 0, \overline{W})$ which is feedback equivalent to $(L^\ell, 0, W^\ell)$, and which has a positive definite symmetric mass matrix $\overline{M}$ and a potential energy $\overline{U} = \frac{1}{2} q^T \overline{S} q$, where $\overline{S}$ is positive definite and symmetric, if and only if the uncontrollable dynamics of $(L, 0, W)$, if any, is oscillatory. Then, $\overline{U}$ and the corresponding $\overline{T} = m(0)\overline{M}^{-1} m(0)$ can serve as the initial condition for the PDEs governing the unknown nonlinear $\widehat{V}$ and $\widehat{T}$. Thus, we can now apply the Cartan-Kähler theorem on the first order system to conclude the existence of a solution. Using a continuity argument, we can ensure that the nonlinear solutions $\widehat{m}$ and $\widehat{V}$ to this initial value problem are positive definite (at least locally around $q = 0$).

For exponential stability, it can be proved (cf. [7]) that any linear controlled Lagrangian system, with positive definite mass matrix m and positive definite potential energy V, is controllable if and only if it can be exponentially stabilized by a linear dissipative feedback. Then the Lyapunov linearization method can be used to conclude that any linear dissipative feedback force onto $\widehat{W}$ will exponentially stabilize the system $(\widehat{L}, \widehat{F}, \widehat{W})$. □

We end this section by making some comments on the linearized system. In the case of one degree of underaction, we know from Theorem 2.8, one of the main results in [7], that the linearized system provides the boundary conditions for those matching conditions, and as a result, the control designer can freely place the poles of the controllable subsystem of the linearized system. What we proved here is, for the case where the degree of underactuation is two while the degree of freedom is at least four, that the control designer can still achieve the same pole placement of the controllable subsystem of the linearized system, provided the given system satisfies the extra inequations as stated in (26)-(31) (or any equivalent set of inequations depending on the choice of coordinates). The set of elements not satisfying these inequations is comparatively small or of measure zero, and furthermore, these inequations do not add extra equality constraints on the energy shaping of the linearized dynamics. As a result, we can conclude that it is generically the case that one can choose the eigenvalues of the controllable subsystem of the linearization of the system so as to achieve energy shaping when the uncontrollable subsystem, if any, of the linearized system is oscillatory.

5 Example: Three Linked Carts with Inverted Pendulum

We illustrate the use of the theorems developed in this paper through an example of three linked carts with an inverted pendulum; see Figure 1.

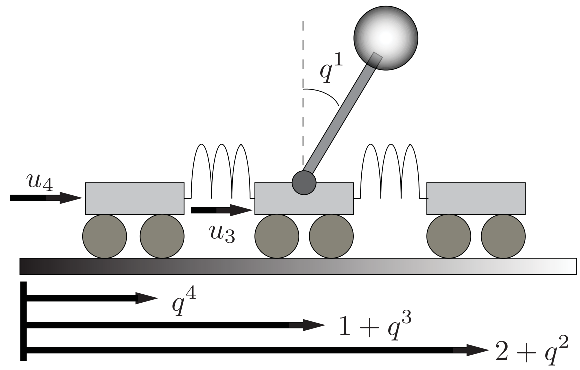

Figure 1: A system of three linked carts with an inverted pendulum.

For simplicity, we assume point masses for the carts and the inverted pendulum, each with a mass of 1 kg. The pendulum has a length of 1 m and each string has a natural length of 1 m. We take g to be the symbol representing the gravitational constant. In this case the mass matrix for the system is given by

$$m = \begin{bmatrix} 1 & 0 & \cos q^1 & 0 \\ 0 & 1 & 0 & 0 \\ \cos q^1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the potential energy is

$$V = \frac{1}{2}\left((q^2 - q^3)^2 + (q^3 - q^4)^2\right) + g\cos q^1.$$

The control bundle W is spanned by dq^3 and dq^4. Now, notice that the Christoffel symbols Γ^i_{jk} are zero at $q = 0$. Hence, to ensure that (19) is still satisfied (at least at $q = 0$), we do the following change of coordinates: $q^i = z^i$ for $i = 1, 2, 3$ and $q^4 = z^1 z^4 + z^4$. By so doing, only $\Gamma^4_{14} = \Gamma^4_{41}$ are nonzero at $z = 0$. Under the new coordinates,

$$m = \begin{bmatrix} 1 + (z^4)^2 & 0 & \cos(z^1) & z^4(z^1 + 1) \\ 0 & 1 & 0 & 0 \\ \cos(z^1) & 0 & 2 & 0 \\ z^4(z^1 + 1) & 0 & 0 & (z^1 + 1)^2 \end{bmatrix},$$

and the potential energy is

$$V = \frac{1}{2}((z^2 - z^3)^2 + (z^3 - z^1 z^4 - z^4)^2) + g\cos z^1.$$

We now need to impose suitable initial conditions for $\widehat{T}$ and $\widehat{V}$ in the new coordinates. Following [7], we can set up these initial conditions by considering the linearization of the given system. The linearized system has a mass matrix given by

$$m^\ell = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It can be proved that the linearized system is controllable. A feedback equivalent system $(\overline{L}, 0, \overline{W})$ is given by

$$\overline{T} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 10 & 4 & 1 \\ 3 & 4 & 100 & 0 \\ 1 & 1 & 0 & 100 \end{bmatrix}, \qquad \overline{S} = \begin{bmatrix} \frac{367}{50} & -\frac{7}{25} & \frac{3}{2} & -\frac{1}{5} \\ -\frac{7}{25} & \frac{11}{100} & -\frac{1}{4} & -\frac{1}{10} \\ \frac{3}{2} & -\frac{1}{4} & 1 & 0 \\ -\frac{1}{5} & -\frac{1}{10} & 0 & 1 \end{bmatrix},$$

both of which are positive definite. Furthermore, we can check that $\widehat{T}$ and $\widehat{V}$ satisfy the inequalities (11), (13), (16), (17), (18) and (19) around $z = 0$. Hence, a solution exists by Theorem 4.11. We can now incorporate these initial conditions to the system of PDEs, leading to the following solutions

$$\begin{aligned}
\widehat{T}_{11} &= 2\cos^2 z^1 - 1 + 2z^4 + 100(z^4)^2 \\
\widehat{T}_{12} &= 2\cos z^1 + z^4 \\
\widehat{T}_{13} &= 3\cos z^1 \\
\widehat{T}_{14} &= (z^1 + 1)(100z^4 + 1) \\
\widehat{T}_{22} &= 10 \\
\widehat{T}_{23} &= 4 \\
\widehat{T}_{24} &= z^1 + 1 \\
\widehat{V} &= (F(z^1, z^2, z^3))^2 + (G(z^1, z^2, z^4))^2 + \frac{4}{25}\cos^2 z^1 - g\cos z^1 \\
&\quad - \frac{6}{25} + \frac{1}{50}(2z^2 - 10z^3)\sin z^1 + \frac{3}{50}(z^2)^2 - \frac{1}{10}z^2 z^3,
\end{aligned}$$

where $F(z^1, z^2, z^3) = \frac{8}{5}\sin z^1 - \frac{z^2}{5} + z^3$ and $G(z^1, z^2, z^4) = -\frac{1}{5}\sin z^1 + \frac{(z^1)^2}{2} - \frac{z^2}{10} + z^1 z^4 + z^4$. It is easy to verify that $\widehat{V}$ is positive definite at $z = 0$. The same is true for $\widehat{T}$, when we assign $\widehat{T}_{33}$ and $\widehat{T}_{44}$ in such a way that they are 100 when $z = 0$. Hence, we have shaped the energy of the given system, and by its linear controllability, we can conclude that the resulting feedback equivalent system can be asymptotically stabilized by an additional dissipative feedback.

6 Conclusion and Future Work

In this paper we have investigated the energy shapability of controlled Lagrangian systems with at least four degrees of freedom and exactly two degrees of underactuation, using the formal theory of PDEs. The criteria of energy shapability was illustrated with a three-cart-one-inverted pendulum example. Our method is practical with the criteria easily verifiable on any given mechanical system with the correct degrees of underactuation and freedom. When the linearized system is controllable, the linear pole placement method will work for local exponential stabilization of the original nonlinear system. However it is well known that shaping a nonlinear system with nonlinear controls has the advantage that it typically gives a signficantly larger region of attraction.

For future work we are interested in the case where the degree of underactuation n_1 goes beyond 2. However, while we have n_1 PDEs for $\widehat{V}$ the number of PDEs for $\widehat{T}$ increases faster than the order of n_1 as n_1 increases. For example, when $n_1 = 3$ and $n_1 = 4$, we have 10 and 20 PDEs, respectively, in these cases. As such making use of the formal theory of PDEs to the problem of higher degrees of underactuation becomes a significantly more challenging task. In addition, we are also interested in the case where the number of degrees of freedom is 3 and the degree of underactuation is 2. In this particular case our methods break down, primarily because all the free components of $\widehat{T}$ get exhausted early. As such our approach of using the formal theory as presented in this paper would need to be modified accordingly in order to handle this case.

7 Appendix

In this appendix we give the proof of Lemma 4.2.

Proof. By Cramer's rule, we can solve $\frac{\partial \widehat{H}}{\partial q^3}$ and $\frac{\partial \widehat{H}}{\partial q^4}$ as follows:

$$(X_1^3 X_2^4 - X_1^4 X_2^3)\frac{\partial \widehat{H}}{\partial q^3} = (X_1^4 X_2^\alpha - X_2^4 X_1^\alpha)\frac{\partial \widehat{H}}{\partial q^\alpha} + X_2^4 h_1 - X_1^4 h_2$$

$$(X_1^3 X_2^4 - X_1^4 X_2^3)\frac{\partial \widehat{H}}{\partial q^4} = (X_2^3 X_1^\alpha - X_1^3 X_2^\alpha)\frac{\partial \widehat{H}}{\partial q^\alpha} + X_1^3 h_2 - X_2^3 h_1$$

where α runs from 1 to 2 (or 1 to $n-2$ for general $n \geq 4$). Thus, this system has a symbol given by

$$(X_1^3 X_2^4 - X_1^4 X_2^3)\frac{\partial \widehat{H}}{\partial q^4} = (X_2^3 X_1^\alpha - X_1^3 X_2^\alpha)\frac{\partial \widehat{H}}{\partial q^\alpha}$$

$$(X_1^3 X_2^4 - X_1^4 X_2^3)\frac{\partial \widehat{H}}{\partial q^3} = (X_1^4 X_2^\alpha - X_2^4 X_1^\alpha)\frac{\partial \widehat{H}}{\partial q^\alpha}$$

1	2	3	4
1	2	3	•

The "dot" board is a bookkeeping way of indicating that the first and second equation are of class 4 and 3 respectively. The prolongation of the second equation $(X_1^3 X_2^4 - X_1^4 X_2^3)\frac{\partial \widehat{H}}{\partial q^3} = (X_1^4 X_2^\alpha - X_2^4 X_1^\alpha)\frac{\partial \widehat{H}}{\partial q^\alpha}$ with respect to the "dot" (i.e. q^4) is a linear combination of other prolongations with respect to the multiplicative variables. Indeed, this linear combination can be derived from the fact that the Lie bracket $[\overline{X}_1, \overline{X}_2]$ is a differential operator of order 1 only, where

$$\overline{X}_1 := (X_1^3 X_2^4 - X_1^4 X_2^3)\frac{\partial}{\partial q^3} - (X_1^4 X_2^\alpha - X_2^4 X_1^\alpha)\frac{\partial}{\partial q^\alpha} = (X_2^4 X_1^k - X_1^4 X_2^k)\frac{\partial}{\partial q^k}$$

$$\overline{X}_2 := (X_1^3 X_2^4 - X_1^4 X_2^3)\frac{\partial}{\partial q^4} - (X_2^3 X_1^\alpha - X_1^3 X_2^\alpha)\frac{\partial}{\partial q^\alpha} = (X_1^3 X_2^k - X_2^3 X_1^k)\frac{\partial}{\partial q^k}\ .$$

Hence, by Theorem 3.7, the symbol for the system of these two PDEs is involutive. Moreover, the integrability condition is

$$[\overline{X}_1, \overline{X}_2]\widehat{H} = \overline{X}_1(X_1^3 h_2 - X_2^3 h_1) - \overline{X}_2(X_2^4 h_1 - X_1^4 h_2)\ ,$$

which can also be derived by Frobenius theorem. We now prove that this is the same as $[X_1, X_2]\widehat{H} = X_1 h_2 - X_2 h_1$. We have

$$\begin{aligned}[\overline{X}_1, \overline{X}_2]\widehat{H} &= \overline{X}_1(X_1^3 X_2^k - X_2^3 X_1^k)\frac{\partial \widehat{H}}{\partial q^k} - \overline{X}_2(X_2^4 X_1^k - X_1^4 X_2^k)\frac{\partial \widehat{H}}{\partial q^k} \\ &= (\overline{X}_1 X_1^3 + \overline{X}_2 X_1^4)\left(X_2^k \frac{\partial \widehat{H}}{\partial q^k}\right) - (\overline{X}_1 X_2^3 + \overline{X}_2 X_2^4)\left(X_1^k \frac{\partial \widehat{H}}{\partial q^k}\right) \\ &\quad + ((X_1^3\overline{X}_1 + X_1^4\overline{X}_2)X_2^k - (X_2^3\overline{X}_1 + X_2^4\overline{X}_2)X_1^k)\frac{\partial \widehat{H}}{\partial q^k} .\end{aligned}$$

But since $X_i^k \frac{\partial \widehat{H}}{\partial q^k} = h_i$ for $i = 1, 2$, and $X_1^3\overline{X}_1 + X_1^4\overline{X}_2 = (X_1^3 X_2^4 - X_1^4 X_2^3)X_1^m \frac{\partial}{\partial q^m}$, $X_2^3\overline{X}_1 + X_2^4\overline{X}_2 = (X_1^3 X_2^4 - X_1^4 X_2^3)X_2^m \frac{\partial}{\partial q^m}$, we have

$$[\overline{X}_1, \overline{X}_2]\widehat{H} = (\overline{X}_1 X_1^3 + \overline{X}_2 X_1^4)h_2 - (\overline{X}_1 X_2^3 + \overline{X}_2 X_2^4)h_1 + (X_1^3 X_2^4 - X_1^4 X_2^3)(X_1 X_2^k - X_2 X_1^k)\frac{\partial \widehat{H}}{\partial q^k} \quad (32)$$

Meanwhile, similar computation gives

$$\begin{aligned}&\overline{X}_1(X_1^3 h_2 - X_2^3 h_1) - \overline{X}_2(X_2^4 h_1 - X_1^4 h_2) \\ &= (\overline{X}_1 X_1^3 + \overline{X}_2 X_1^4)h_2 - (\overline{X}_1 X_2^3 + \overline{X}_2 X_2^4)h_1 + (X_1^3 X_2^4 - X_1^4 X_2^3)(X_1 h_2 - X_2 h_1) \quad (33)\end{aligned}$$

Hence, canceling common terms in (32) and (33), we arrive at $[X_1, X_2]\widehat{H} = X_1 h_2 - X_2 h_1$ as desired. □

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