Existence Problem of Telescopers: Beyond the Bivariate Case

Shaoshi Chen1,2, Qing-Hu Hou3, George Labahn2, Rong-Hua Wang1

1KLMM, AMSS, Chinese Academy of Sciences, Beijing, 100190, (China)
2Symbolic Computation Group, University of Waterloo, Ontario, N2L3G1, (Canada)
3Center for Applied Mathematics, Tianjin University, Tianjin, 300072, (China)
4Center for Combinatorics, Nankai University, Tianjin, 300071, (China)

schen@amss.ac.cn, qh_hou@tju.edu.cn

glabahn@uwaterloo.ca, wangwang@mail.nankai.edu.cn

I.1.2 [Computing Methodologies]: Symbolic and Algebraic Manipulation—Algebraic Algorithms

Keywords
Rational function, Telescooper, Summability, Reduction

1. INTRODUCTION

The method of creative telescoping is an algorithmic tool in the symbolic evaluation of parameterized definite sums and integrals. In order to evaluate a multiple sum of a given function \( f(x, y_1, \ldots, y_n) \) with respect to \( y_1, \ldots, y_n \), with \( x \) a discrete parameter, the key step of creative telescoping is to find a nonzero linear recurrence operator \( L \) in \( x \) such that

\[
L(f) = \Delta_{y_1}(g_1) + \cdots + \Delta_{y_n}(g_n),
\]

where \( \Delta_{y_i} \) denotes the difference operator in \( y_i \) and the \( g_i \)'s belong to the same class of functions as \( f \). The operator \( L \) is then called a telescoper for \( f \), and the \( g_i \)'s are called the certificates of \( L \). In order to be useful in applications, one needs to address two problems: (1) determine whether such an operator \( L \) exists for a given function \( f \) and (2) if telescopers exist, then design an algorithm for computing them along with their certificates.

Permission to make digital or hard copies of all or part of this work for personal or classroom use is granted without fee provided that copies are not made or distributed for profit or commercial advantage and that copies bear this notice and the full citation on the first page. Copyrights for components of this work owned by others than the author(s) must be honored. Abstracting with credit is permitted. To copy otherwise, or republish, to post on servers or to redistribute to lists, requires prior specific permission and/or a fee. Request permissions from permissions@acm.org.

ISSAC ’16, July 19 - 22, 2016, Waterloo, ON, Canada
© 2016 Copyright held by the owner/author(s). Publication rights licensed to ACM.
ISBN 978-1-4503-4380-0/16/07...
S15.00

DOI: http://dx.doi.org/10.1145/2930889.2930895

ISSAC ’16, July 19 - 22, 2016, Waterloo, ON, Canada
© 2016 Copyright held by the owner/author(s). Publication rights licensed to ACM.
ISBN 978-1-4503-4380-0/16/07...
S15.00

DOI: http://dx.doi.org/10.1145/2930889.2930895
problem for rational functions of three variables. However, the existence problem in the trivariate case is considerably more involved. As an example, the rational function \(1/(x + y + z^2)\) is not proper (even after the reduction). However, it does have a telescoper (see Example 6.4), a phenomenon which does not happen in the bivariate case.

The remainder of this paper is organized as follows. The basic notations and concepts of telescopers are given in Section 2. In Sections 3 and 4, we review the previous work on solving the summability problem for bivariate rational functions and present special properties of linear recurrence operators. The existence problem for general rational functions is reduced to one with simpler rational functions in Section 5 with the existence criteria for these special rational functions presented in Section 6. The paper ends with a conclusion along with topics for future research.

2. PRELIMINARIES

Let \(K\) be a field of characteristic zero and let \(E = K(x, y, z)\) be the field of rational functions in \(x, y, z\) over \(K\). For \(f \in E\) define the shift operators \(\sigma_x, \sigma_y, \sigma_z\) on \(E\) by \(\sigma_x(f) = f(x + 1, y, z), \sigma_y(f) = f(x, y + 1, z), \) and \(\sigma_z(f) = f(x, y, z + 1)\), respectively. Let \(\mathcal{R} := E[S_x, S_y, S_z]\) denote the ring of linear recurrence operators over \(E\), in which \(S_x, S_y, S_z\) commute and \(S_v \cdot f = \sigma_v(f) \cdot S_v\) for any \(f \in E\) and \(v \in \{x, y, z\}\). The action of an operator \(P = \sum_{i,j,k} p_{i,j,k} S_x^i S_y^j S_z^k\) in \(\mathcal{R}\) on a rational function \(f \in E\) is then given by

\[P(f) = \sum_{i,j,k} p_{i,j,k} f(x + i, y + j, z + k).\]

The difference operators \(\Delta_x, \Delta_y\) and \(\Delta_z\) with respect to \(x, y\) and \(z\) are defined by

\[\Delta_x = S_x - 1, \quad \Delta_y = S_y - 1, \quad \text{and} \quad \Delta_z = S_z - 1.\]

A rational function \(f \in E\) is said to be \((\sigma_y, \sigma_z)\)-summable in \(E\) if \(f = \Delta_y(g) + \Delta_z(h)\) for some \(g, h \in E\). We also just say summable if the meaning is clear. For brevity, we sometimes just write \(f \equiv_{y,z} 0\) if \(f = (\sigma_y, \sigma_z)\)-summable.

Definition 2.1. A nonzero linear recurrence operator \(L \in E[S_x][S_z]\) is called a telescoper for a rational function \(f \in E\) if \(L(f)\) is \((\sigma_y, \sigma_z)\)-summable in \(E\), that is, there exist \(g, h \in E\) such that

\[L(f) = \Delta_y(g) + \Delta_z(h).\]

Then the central problem to be solved in this paper is:

Problem 2.2. Given \(f \in E\), decide whether \(f\) has a telescoper in \(K(x)[S_z]\).

An operator \(L \in K(x)[S_z]\) is called a common left multiple of operators \(L_1, \ldots, L_m \in K(x)[S_z]\) if there exist operators \(L'_1, \ldots, L'_m \in K(x)[S_z]\) such that \(L = L'_1 L_1 = \cdots = L'_m L_m\). Since \(K(x)[S_z]\) is a left Euclidean domain, such an \(L\) always exists. Amongst all of them, the one of smallest degree in \(S_z\) is called the least common left multiple (LCLM). When the field \(K\) is computable, e.g., \(K = \mathbb{Q}\), many efficient algorithms for computing LCLM have been developed [11, 6].

Remark 2.3. Let \(f = f_1 + \cdots + f_m\) with all \(f_i \in E\). If each \(f_i\) has a telescoper \(T_i\) for \(i = 1, \ldots, m\), then the LCLM of the \(T_i\) is a telescoper for \(f\). This fact follows from the definition of LCLM along with the commutativity between operators in \(K(x)[S_z]\) and the difference operators \(\Delta_y, \Delta_z\).

Let \(G = \langle \sigma_x, \sigma_y, \sigma_z \rangle\) be the free Abelian multiplicative group generated by \(\sigma_x, \sigma_y, \sigma_z\). Let \(f \in E\) and \(H\) be a subgroup of \(G\). We call \([f]_H = \{\sigma(f) \mid \sigma \in H\}\) the \(H\)-orbit at \(f\). Two elements \(f, g \in E\) are said to be \(H\)-equivalent if \([f]_H = [g]_H\), denoted by \(f \sim_H g\). The relation \(\sim_H\) is an equivalence relation. Typically, we will take \(H = G\) or \(H = \langle \sigma_y, \sigma_z \rangle\) in the rest of this paper.

Example 2.4. Let \(f = y^2 + x + 2z\) and \(g = y^2 + x - 4y + 2z + 7\). Then \(f\) and \(g\) are \(G\)-equivalent since \(g = \sigma_x^2 \sigma_z^3(f)\). However they are not \((\sigma_y, \sigma_z)\)-equivalent. Indeed, if \(g = \sigma_y^m \sigma_z^n(f)\) for some \(n, k \in \mathbb{Z}\), then the coefficients lead to the linear system \(\{2n = -4, n^2 + 2k = 7\}\). But this implies that \(n = -2\) and \(k = 3/2\), a contradiction.

3. SUMMABILITY

The first necessary step for solving the existence problem of telescopers is to decide whether a given multivariate function \(f(x_1, \ldots, x_n)\) in a specific class of functions is equal to \(\Delta_{x_1}(g_1) + \cdots + \Delta_{x_n}(g_n)\) for some \(g_1, \ldots, g_n\) in the same class as \(f\). For univariate rational functions the summability problem was first solved by Abramov [1, 2] with alternative methods later presented in [24, 25]. The Gosper algorithm [18] solves the problem for univariate hypergeometric terms. This was then used by Zeilberger [28] to design a fast algorithm to construct telescopers for bivariate hypergeometric terms. The Gosper algorithm was extended further to the \(D\)-finite case by Abramov and van Hoeij in [8, 4], and to a more general difference-field setting by Karr [22, 23] and Schneider [26]. A significant step in the path towards the multivariate case was taken by Chen et al. in [15], which gave some necessary conditions for the summability of bivariate hypergeometric terms. Chen and Singer [13] then presented the first necessary and sufficient condition for the summability of bivariate rational functions. Based on the theoretical criterion in [13], Hou and Wang [21] gave a practical algorithm for deciding the summability in the bivariate rational case.

In this section, we will recall the summability criterion for bivariate rational functions from [21]. Let \(F := K(x)\) and \(f \in F(y, z)\). The key idea is to decompose \(f\) into the following form

\[f = \Delta_y(g) + \Delta_z(h) + r,\]

where \(g, h \in F(y, z)\) and \(r\) is of the form

\[r = \sum_{i=1}^{m} \sum_{j=1}^{m} a_{i,j} d_i^{-1} \tag{3.1}\]

with \(a_{i,j} \in F[y][z]\), \(\deg_y(a_{i,j}) < \deg_y(d_i)\), \(d_i \in F[y, z]\) are irreducible polynomials, and \(d_i, d_i'\) are not \((\sigma_y, \sigma_z)\)-equivalent for any \(i \neq i'\). The existence of such decompositions has been shown in [21, Lemma 3.1]. Then \(f = (\sigma_y, \sigma_z)\)-summable if and only if \(r = (\sigma_y, \sigma_z)\)-summable. Since shift operators preserve the multiplicities of the fractions \(a_{i,j}/d_i'\), we have \(r = (\sigma_y, \sigma_z)\)-summable if and only if \(\sum_{i=1}^{m} a_{i,j}/d_i'\) is \((\sigma_y, \sigma_z)\)-summable for each \(j\). Furthermore, Lemma 3.2 in [21] shows that \(\sum_{i=1}^{m} a_{i,j}/d_i'\) is \((\sigma_y, \sigma_z)\)-summable if and only if \(a_{i,j}/d_i'\) is \((\sigma_y, \sigma_z)\)-summable for all \(i\) with \(1 \leq i \leq n\). Thus, the summability problem for general rational functions in \(F(y, z)\) is reduced to the summability problem for simple fractions of the special form \(a/d'\). The following
Theorem 3.1. Let \( f = a/d \in \mathbb{F}(y, z) \) with \( d \in \mathbb{F}[y, z] \) being irreducible, \( a \in \mathbb{F}(y)[z] \setminus \{0\} \) and \( \deg_y(a) < \deg_z(d) \). Then \( f \) is \((\sigma_y, \sigma_z)\)-summable if and only if

1. there exist integers \( t, t \) with \( t \neq 0 \) such that
\[
\sigma_t^y(d) = \sigma_t^z(d),
\]
(3.2)
2. for the smallest positive integer \( t \) such that (3.2) holds, we have \( a = \sigma_t^y \sigma_t^z(p) - p \) for some \( p \in \mathbb{F}(y)[z] \) with \( \deg_y(p) < \deg_z(d) \).

Definition 3.2. For a rational function \( f \in \mathbb{F}(y, z) \), we call the triple \((g, h, r) \in \mathbb{F}(y, z)^3\) an additive decomposition of \( f \) with respect to \( y \) and \( z \) if \( f = \Delta_y(g) + \Delta_z(h) + r \), where \( r \) is of the form (3.1) and none of the fractions \( a_{i,j}/d_i \) is \((\sigma_y, \sigma_z)\)-summable.

Remark 3.3. From the decision procedure for summability given above, additive decompositions always exist for rational functions in \( \mathbb{F}(y, z) \). However, we remark that such decompositions may not be unique.

4. EXPONENT SEPARATION

In this section, we will present some special properties of linear recurrence operators having to do with separating exponents. This separation of exponents of an operator will be used in the next section for separating orbits of shift operators and will help in simplifying the existence problem.

Let \( m \in \mathbb{N} \) and \( L \) be a nonzero operator in \( \mathbb{K}(x)[S_x] \). Then we can always decompose \( L \) into the form
\[
L = L_0 + L_1 + \cdots + L_{m-1},
\]
where \( L_i = \sum_{j=0}^{m-1} \delta_{i,j} S_x^{m+i+1} \) for \( i = 0, 1, \ldots, m - 1 \). We call such a decomposition an \( m \)-exponent separation of \( L \). It is clear that \( L = 0 \) if and only if \( L_i = 0 \) for all \( i \). Denote
\[
L_m = \begin{bmatrix}
L_0 & L_{m-1} & L_{m-2} & \cdots & L_1 \\
L_1 & L_0 & L_{m-1} & \cdots & L_2 \\
L_2 & L_1 & L_0 & \cdots & L_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
L_{m-1} & L_{m-2} & L_{m-3} & \cdots & L_0
\end{bmatrix}
\]
(4.2)
The next lemma and proposition will show that the \( m \) rows of \( L_m \) are linearly independent over the ring \( \mathbb{K}(x)[S_x] \).

Lemma 4.1. Suppose
\[
[T_0, \ldots, T_{m-1}] \cdot L_m = 0
\]
with each \( T_k \in \mathbb{K}(x)[S_x] \). Then \( T_0 + T_1 + \cdots + T_{m-1} = 0 \).

Proof. Note that \( L_m \cdot [1, \ldots, 1]^T = [L_0, \ldots, L_m]^T \). Hence any solution of (4.3) implies that
\[
(T_0 + \cdots + T_{m-1}) \cdot L = 0.
\]
Since \( L \) is nonzero and \( \mathbb{K}(x)[S_x] \) is a left Euclidean domain we have \( T_0 + \cdots + T_{m-1} = 0 \).

In fact our goal is to show that the left kernel of \( L_m \) is trivial, and so need to show that each component \( T_k \) of (4.3) is zero. In order to do this we do an \( m \)-exponent separation of each \( T_k \) and look at the resulting decomposition. Suppose \( T_0, \ldots, T_{m-1} \cdot L_m = [R_0, \ldots, R_{m-1}] \)
and that for each \( k \)
\[
T_k = T_{k,0} + T_{k,1} + \cdots + T_{k,m-1}
\]
\[
R_k = R_{k,0} + R_{k,1} + \cdots + R_{k,m-1}
\]
are the \( m \)-exponents for \( T_k \) and \( R_k \), respectively. Let \( T \) and \( R \) be the \( m \times m \) matrices defined as
\[
T = \begin{bmatrix}
T_{0,0} & T_{1,m-1} & T_{2,m-2} & \cdots & T_{m-1,1} \\
T_{0,1} & T_{1,0} & T_{2,m-1} & \cdots & T_{m-1,2} \\
T_{0,2} & T_{1,1} & T_{2,0} & \cdots & T_{m-1,3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
T_{0,m-1} & T_{1,m-2} & T_{2,m-3} & \cdots & T_{m-1,0}
\end{bmatrix}
\]
(4.4)
and
\[
R = \begin{bmatrix}
R_{0,0} & R_{1,m-1} & R_{2,m-2} & \cdots & R_{m-1,1} \\
R_{0,1} & R_{1,0} & R_{2,m-1} & \cdots & R_{m-1,2} \\
R_{0,2} & R_{1,1} & R_{2,0} & \cdots & R_{m-1,3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
R_{0,m-1} & R_{1,m-2} & R_{2,m-3} & \cdots & R_{m-1,0}
\end{bmatrix}
\]
(4.5)
Then it is straightforward to show that
\[
T \cdot L_m = R.
\]

Proposition 4.2. Suppose
\[
[T_0, \ldots, T_{m-1}] \cdot L_m = 0
\]
with each \( T_k \in \mathbb{K}(x)[S_x] \). Then \( T_k = 0 \) for each \( k \).

Proof. From (4.5) and (4.6) we have that each \( R_k = 0 \) and hence also that each \( R_{k,j} = 0 \). Thus \( T \cdot L_m = 0 \) and so for each \( j = 1, 2, \ldots, m \) we have
\[
[T_{0,j-1}, \ldots, T_{j-1,0}, T_{j, m-1}, \ldots, T_{m-1,j}] \cdot L_m = 0.
\]
From Lemma 4.1 we get for each \( j \)
\[
T_{0,j-1} + \cdots + T_{j-1,0} + T_{j,m-1} + \cdots + T_{m-1,j} = 0.
\]
This implies that each \( T_{k,j} = 0 \) and hence also that \( T_k = 0 \) for all \( k \).

We will also later need to use the following:

Proposition 4.3. There is a matrix \( M \in \mathbb{K}(x)[S_x]^{m \times m} \) such that
\[
M \cdot L_m = \text{diagonal}(T_0, T_1, \ldots, T_{m-1})
\]
with nonzero \( T_k \in \mathbb{K}(x)[S_x] \).

Proof. From the definition of LCLM, we know that for any nonzero \( A, B \in \mathbb{K}(x)[S_x] \), there always exist nonzero \( A', B' \in \mathbb{K}(x)[S_x] \) such that \( A' + B'B = 0 \). Hence similar to the use of the division-free Gaussian elimination over a Euclidean domain, we can find \( M \in \mathbb{K}(x)[S_x]^{m \times m} \) satisfying (4.7) (c.f. [10]). Note that row reductions preserve the linear independency and all rows of \( L_m \) are linearly independent by Proposition 4.2. Then all rows of \( M \cdot L_m \) are linearly independent. In particular, each diagonal element of \( M \cdot L_m \) is nonzero, since \( M \cdot L_m \) is of triangular form.
5. REDUCTION TO SIMPLE FRACTIONS

In this section, we will reduce the existence problem of telescopes for rational functions in $\mathbb{E}$ into the same problem but for simpler rational functions.

Let $f \in \mathbb{E}$ be nonzero with $f = \Delta_y(g) + \Delta_z(h) + r$ with $(g,h,r)$ an additive decomposition of $f$ with respect to $y$ and $z$. Then $f$ has a telescoper in $\mathbb{K}(x)[S_z]$ if and only if $r$ has a telescoper in $\mathbb{K}(x)[S_z]$. As such, we need only to study the existence problem for rational functions of the form in Theorem 3.1.

For any $\sigma \in \langle \sigma_x, \sigma_y, \sigma_z \rangle$ and $a, b \in \mathbb{E}$, we have

$$\frac{a}{\sigma^n(b)} = \sigma(g) - g + \frac{\sigma^{-n}(a)}{b},$$

(5.1)

where $g$ is equal to $\sum_{i=0}^{n-1} \sigma(i-a)(b)$ if $n \geq 0$, and equal to $-\sum_{i=0}^{-n-1} \sigma(i-a)(b)$ if $n < 0$. We now simplify the fractions in the form (3.1) using the the formula (5.1). Suppose that $d_i = \sigma_i^n \sigma_i^k d_i$ for some index $i \neq i'$ and $m, n, k \in \mathbb{Z}$ with $m \geq 0$. Applying the formula (5.1) repeatedly yields

$$\frac{a_i d_i}{\sigma^n_i d_i} = \Delta_y(u) + \Delta_z(v) + \frac{\sigma_i^n \sigma_i^{-k}(a_{i,i'})}{\sigma_i^n d_i}$$

for some $u, v \in \mathbb{E}$. With this reduction, we can always decompose $r$ of the form (3.1) into the form

$$r = \sum_{i=1}^l \sum_{j=1}^{\ell_i} \sum_{\sigma} b_{i,j,\ell} \sigma_{i,j}^n d_i$$

(5.2)

with $b_{i,j,\ell} \in \mathbb{K}(x,y)[z], d_i \in \mathbb{K}(x,y,z)$, $\deg_x(b_{i,j,\ell}) < \deg_x(d_i)$, and $d_i$ are irreducible polynomials with $d_i$ and $d_i'$ being in distinct $\langle \sigma_x, \sigma_y, \sigma_z \rangle$-orbits for any $1 \leq i \neq i' \leq m$.

Let $\mathcal{O} = \{p/q \in \mathbb{E} | \deg_x(p) < \deg_x(q)\}$ and $V_m$ be the set of all rational functions of the form $\sum_{i=1}^m \frac{a_i}{b_i}$, where $a_i, b_i \in \mathbb{K}(x,y)[z]$, $\deg_x(a_i) < \deg_x(b_i)$ and $b_i$'s are distinct irreducible polynomials in the ring $\mathbb{K}(x,y,z)$. By definition, the set $V_m$ forms a subspace of $\mathcal{O}$ viewed as vector spaces over $\mathbb{K}(x,y)$. By the irreducible partial fraction decomposition, any $f \in \mathbb{O}$ can be uniquely decomposed into $f = \sum_{i=1}^m f_i$, where $f_i \in V_i$ and so $\mathbb{O} = \bigoplus_{i=1}^m V_i$. The following lemma shows that the space $V_m$ is invariant under certain linear recurrence operators.

Lemma 5.1. Let $f \in V_m$ and $P \in \mathbb{K}(x,y)[S_x, S_y, S_z]$. Then $P(f) \in V_m$.

Proof. Let $f = \sum_{i=1}^m a_i/b_i$ and $P = \sum_{i,j,k} p_{i,j,k} S_x^i S_y^j S_z^k$. For any $\sigma = \sigma_x^i \sigma_y^j \sigma_z^k$ with $i, j, k \in \mathbb{Z}$, $\sigma(b_i)$ is irreducible and $\deg_x(\sigma(a_i)) < \deg_x(\sigma(b_i))$. Then all of the simple fractions $\sum_{i,j,k} p_{i,j,k} S_x^i S_y^j S_z^k(a_{i,j,k})$ appearing in $P(f)$ are proper in $z$ and have irreducible denominators. If some of denominators are the same, we can simplify them by adding the numerators to get a simple fraction. After this simplification, we see that $P(f)$ can be written in the same form as $f$, so it is in $V_m$. $\square$

Lemma 5.2. Let $r \in \mathbb{E}$ be of the form (5.2). Then $r$ has a telescoper if and only if the summand $\sum_{i,j} b_{i,j} \sigma_i^{n} d_i$ has a telescoper for all $i,j$ with $1 \leq i \leq I$ and $1 \leq j \leq J_i$.

Proof. From Lemma 5.1 we see that any $r$ as in (5.2) has a telescoper if and only if all $\sum_{i,j} b_{i,j} \sigma_i^{n} d_i$ has a telescoper for all different multiplicities $j$. Also, from Lemma 3.2 in [21] we have that $\sum_{i=1}^l \sum_{\sigma} b_{i,\ell} \sigma_i^n d_i$ has a telescoper if and only if $\sum_{i=1}^l b_{i,\ell} \sigma_i^n d_i$ has a telescoper for all $i$ with $1 \leq i \leq I$. $\square$

At this stage we have reduced the existence of telescopes problem for general rational functions to those having the simple form $r = \sum_{i=0}^l b_{i} \sigma_i d_i$. If $\sigma_i d_i = \sigma_i^k \sigma_i^l d_i$ for some $\ell \neq \ell'$ and $n, k \in \mathbb{Z}$, then applying the formula (5.1), we get

$$\frac{b_{i,j,v}}{\sigma_i^n d_i} = \frac{\Delta_y(u_{i,j}) + \Delta_z(v_{i,j})}{\sigma_i^k \sigma_i^l d_i}$$

for some $u_{i,j}, v_{i,j} \in \mathbb{K}(x,y,z)$. Repeating the above transformation gives a decomposition

$$r = \Delta_y(u) + \Delta_z(v) + \sum_{\ell=0}^{l-1} \frac{b_{\ell} \sigma_{\ell} d_{\ell}}{\sigma_{\ell}^{n} d_{\ell}}$$

(5.3)

where $u, v \in \mathbb{K}(x,y,z)$ and $\sigma_i^k(d_i)$ and $\sigma_i^l(d_i)$ are not $\langle \sigma_x, \sigma_y, \sigma_z \rangle$-equivalent for $0 \leq \ell \neq \ell' \leq l$. We claim that $\Delta_y(u) + \Delta_z(v)$ has a telescoper for $0 \leq i \leq I$.

Lemma 5.3. Let $r = \sum_{i=0}^l b_i \sigma_i d_i$ with $b_i \in \mathbb{K}(x,y)[z], d_i \in \mathbb{K}(x,y,z)$. Suppose $b_i, d_i$ are irreducible, $\deg_x(b_i) < \deg_x(d_i)$ with $\sigma_i^k d_i$ and $\sigma_i^l d_i$ in distinct $\langle \sigma_x, \sigma_y, \sigma_z \rangle$-orbits, for $0 \leq i \leq I$. Then $r$ has a telescoper if and only if each simple fraction $\frac{b_i \sigma_i d_i}{\sigma_{i}^k \sigma_{i}^l d_i}$ has a telescoper for $0 \leq i \leq I$.

Proof. Sufficiency follows from Remark 2.3. For the other direction assume that $L = \sum_{i=0}^l \sigma_i^k d_i$ (with $b_0 \neq 0$) is a telescoper for $r$. There are two cases to be considered depending on whether there exists a positive integer $m$ such that $\sigma_i^{m} d_i = \sigma_i^{k} \sigma_i^{l} d_i$ for some integers $n, k$.

Case 1. There is no positive integer $m$ such that $\sigma_i^m d_i = \sigma_i^{k} \sigma_i^{l} d_i$ for some $n, k \in \mathbb{Z}$.

In this case, $\sigma_i^k d_i$ and $\sigma_i^l d_i$ are in distinct $\langle \sigma_x, \sigma_y, \sigma_z \rangle$-orbits for any $i \neq i'$. We claim that $\frac{b_i \sigma_i d_i}{\sigma_{i}^k \sigma_{i}^l d_i}$ is $(\sigma_y, \sigma_z)$-summable for $0 \leq i \leq I$. Since

$$L(r) = \sum_{i=0}^l \sum_{\ell=0}^{l-1} \frac{b_i \sigma_i d_i}{\sigma_{i}^k \sigma_{i}^l d_i} = \sum_{p=0}^{\rho} \sum_{i=0}^{\rho} \frac{b_i \sigma_i d_i}{\sigma_{i}^k \sigma_{i}^l d_i}$$

is $(\sigma_x, \sigma_y, \sigma_z)$-summable, according to Lemma 3.2 in [21], we get that for any $0 \leq p \leq \rho + l$, there exist $u_p, v_p \in \mathbb{K}(x,y)$ such that

$$\frac{b_i \sigma_i d_i}{\sigma_{i}^k \sigma_{i}^l d_i} = \Delta_y(u_p) + \Delta_z(v_p).$$

(5.4)

We prove the claim by induction. The result is true for $p = 0$ in (5.4) since then $\frac{b_0 \sigma_0 d_0}{\sigma_{0}^k \sigma_{0}^l d_0} = \Delta_y(0) + \Delta_z(0)$. Suppose we have shown that $\frac{b_i \sigma_i d_i}{\sigma_{i}^k \sigma_{i}^l d_i}$ is $(\sigma_y, \sigma_z)$-summable for $i = 0, 1, \ldots, k - 1$ with $k \leq l$. Letting $p = k$ in (5.4), we get

$$\sum_{i=0}^k \frac{b_i \sigma_i d_i}{\sigma_{i}^k \sigma_{i}^l d_i} = \Delta_y(u_k) + \Delta_z(v_k).$$
As \( \frac{b_{k-i}}{\sigma_{z-i}^d} \) is \((\sigma_y, \sigma_z)\)-summable for all \( 1 \leq i \leq k \), it is easy to check that \( \sum_{i=1}^{k} \ell_i \sigma_i^d (\frac{b_{k-i}}{\sigma_{z-i}^d}) \) is also \((\sigma_y, \sigma_z)\)-summable.

Thus \( \frac{b_i}{\sigma_{x-i}^d} \) is \((\sigma_y, \sigma_z)\)-summable.

Case 2. Suppose \( \sigma_i^m d = \sigma_i^n \sigma_j^k d \) for \( m \) a positive integer and \( n, k \) some integers. Let \( m_0 \) be the smallest such integer with \( \sigma_i^m d = \sigma_i^n \sigma_j^k d \) for some integers \( n_0, k_0 \). Since \( \sigma_i^j d \) and \( \sigma_i^j d \) are in distinct \((\sigma_y, \sigma_z)\)-orbits, we can assume \( r = \sum_{i=0}^{m_0-1} \frac{b_i}{\sigma_{x-i}^d} \). Suppose the \( m_0 \)-exponent separation of \( L \) is

\[
L = L_0 + L_1 + \cdots + L_{m_0-1}.
\]

According to Lemma 3.1 and Lemma 3.2 in [21], we have

\[
\begin{align*}
L_0 \frac{b_0}{d_i} &+ L_{m_0-1} \frac{b_1}{\sigma_x d_i} + \cdots + L_1 \frac{b_{m_0-1}}{\sigma_{x-1}^d d_i} \equiv_{y,z} 0 \\
L_1 \frac{b_0}{d_i} &+ L_0 \frac{b_1}{\sigma_x d_i} + \cdots + L_2 \frac{b_{m_0-1}}{\sigma_{x-1}^d d_i} \equiv_{y,z} 0 \\
&\vdots \\
L_{m_0-1} \frac{b_0}{d_i} &+ L_{m_0-2} \frac{b_1}{\sigma_x d_i} + \cdots + L_0 \frac{b_{m_0-1}}{\sigma_{x-1}^d d_i} \equiv_{y,z} 0.
\end{align*}
\]

If we let \( V = \left[ \frac{b_0}{d_i}, \frac{b_1}{\sigma_x d_i}, \ldots, \frac{b_{m_0-1}}{\sigma_{x-1}^d d_i} \right] \),

then we can write this as

\[
L_{m_0} \cdot V^T \equiv_{y,z} 0,
\]

with \( L_{m_0} \) from (4.2). From Proposition 4.3 there exists a matrix \( M \) having entries from \( \mathbb{K}(x)[S_z] \) such that

\[
M \cdot L_{m_0} = \text{diagonal}(T_0, \ldots, T_{m_0-1}).
\]

By the commutativity between operators in \( \mathbb{K}(x)[S_z] \) and the difference operators \( \Delta_y, \Delta_z \), we know \( T_i \) is a telescoper for \( \frac{b_i}{\sigma_{x-i}^d} \) for \( 0 \leq i \leq m_0-1 \).

6. EXISTENCE CRITERIA

Lemma 5.3 from the previous section implies that the telescoper existence problem for rational functions is reduced to the case of a rational function of the form

\[
f = \frac{b(x,y,z)}{c(x,y) d(x,y,z)},
\]

where \( \lambda \in \mathbb{N}, c \in \mathbb{K}[x,y], d \in \mathbb{K}[x,y,z] \) with \( \text{deg}_c(b) < \text{deg}_d(c) \). In this section, we will give a criterion for deciding the existence of telescopers for rational functions of the above form. If \( b \) and \( c \) are not primitive, i.e., their content is not 1, then we can write \( b = b_0(x) b_1(x,y,z) \) and \( c = c_0(x) c_1(x,y) \), where \( b_1, c_1 \) are primitive in \( y, z \). Similarly to the proof of Lemma 7.4 in [12], \( \frac{b_i}{\sigma_i d} \) has a telescoper if and only if \( \frac{b_i}{\sigma_i d} \) has a telescoper. As such, we can assume in form (6.1) that \( b, c, d \) are all primitive in \( y, z \).

As we did in the proof of Lemma 5.3 we will proceed by case distinction according to whether or not certain polynomials \( p \in \mathbb{K}[x,y,z] \) and \( q \in \mathbb{K}[x,y] \) satisfy the following conditions:

- there exist a positive integer \( m \) such that
  \[
  \sigma_m^p(p) = \sigma_k^q \sigma_j^l(p)
  \]
  for some \( n, k \in \mathbb{Z} \); (6.2)
- there exist \( n_1, k_1 \in \mathbb{Z} \) with \( n_1 > 0 \) such that
  \[
  \sigma_{m_1}^n(p) = \sigma_{k_1}^l(p);
  \]
- for \( m, n \) as in (6.2), there exists a positive integer \( t \) such that
  \[
  \sigma_{m_1}^t(q) = \sigma_{k_1}^l(q);
  \]
- there exist \( n_2, k_2 \in \mathbb{Z} \) with \( n_2 > 0 \) such that
  \[
  \sigma_{m_1}^n(q) = \sigma_{k_2}^l(q).
  \]

To test the existence of telescopers for a simple fraction, we will need to test the conditions as above for polynomials. This amounts to solving the following problem:

**Problem 6.1** (Integer Shift Equivalence Testing Problem). Let \( \mathbb{K} \) be any computable field of characteristic zero and \( \sigma_i \) be the shift operator w.r.t. \( x_i \) on \( \mathbb{K}[x_1, \ldots, x_n] \). Given \( p \in \mathbb{K}[x_1, \ldots, x_n] \) decide whether there exist integers \( m_1, \ldots, m_n \) with \( m_1 > 0 \) such that \( \sigma_{m_1}^1 \cdots \sigma_{m_1}^n(p) = p \).

This problem is a special case of the problem proposed and solved by Grigoriev in [19, 20] and more recently by Dvir et al. in [17].

First, we consider the case that the polynomial \( d \) in (6.1) does not satisfy the condition (6.2). In this case, the existence problem is reduced to the summability problem.

**Lemma 6.2.** Let \( f = b/(cd^k) \in \mathbb{E} \) be of the form (6.1), with \( d \) not satisfying condition (6.2). Then \( f \) has a telescoper if and only if \( f \) is \((\sigma_y, \sigma_z)\)-summable.

**Proof.** The sufficiency is obvious. For the necessity, we assume that \( L = \sum_{i=0}^{l} \frac{\ell_i}{\sigma_{i}^d} \in \mathbb{E} \) with \( \ell_0, \ell_i \neq 0 \) is a telescoper for \( f \). Then

\[
L(f) = \sum_{i=0}^{l} \frac{\ell_i}{\sigma_{i}^d} = \Delta_y(g) + \Delta_z(h)
\]

for some \( g, h \in \mathbb{E} \). Since \( \sigma_{m_1}^n(d) \neq \sigma_{m_1}^n(d) \) for any positive integer \( m \) and \( n, k \in \mathbb{Z} \) we have \( \sigma_{m_1}^n(d) \) and \( \sigma_{m_1}^n(d) \) are in distinct \((\sigma_y, \sigma_z)\)-orbits for any \( i \neq i' \). By Lemma 3.2 in [21], the summands \( \frac{\ell_i}{\sigma_{i}^d} \) of \( L(f) \) are all \((\sigma_y, \sigma_z)\)-summable. In particular, \( \ell_0 f \) is \((\sigma_y, \sigma_z)\)-summable.

The second case where (6.2) holds for \( d \) is considerably more involved. Let \( \mathbb{K} \) be the algebraic closure of \( \mathbb{K} \). An irreducible polynomial \( q \in \mathbb{K} \) is said to be integer-linear in \( x, y \) and \( z \) over \( \mathbb{K} \) if it is of the form \( \alpha_i x + \beta_i y + \gamma_i z + \delta_i \), where \( \alpha_i, \beta_i, \gamma_i \in \mathbb{Z} \) and \( \delta_i \in \mathbb{K} \). A rational function \( f \in \mathbb{E} \) is said to be proper if it can be written in the form \( f = \frac{p}{\prod_{i=1}^{n} q_i} \),

where \( p, q_i \in \mathbb{K}[x,y,z] \) and all \( q_i \) are integer-linear in \( x, y \) and \( z \) over \( \mathbb{K} \). By the fundamental theorem in [27, p. 590], any proper rational function has a telescoper.

The following lemma describes some necessary conditions for the existence of telescopers.

**Lemma 6.3.** Let \( f = b/(cd^k) \in \mathbb{E} \) be of the form (6.1), and let \( d \) satisfy the condition (6.2). Then \( f \) has a telescoper if one of the following conditions is also satisfied:

(i) \( c \) and \( d \) satisfy the conditions (6.5) and (6.3), resp.;
(ii) \( c \) satisfies the condition (6.4).
Proof. Suppose that the polynomials $c$ and $d$ satisfy the conditions (6.2) and (i). By Lemma 3 in [7], the equalities $\sigma_y^2(c) = \sigma_y^2(d)$ and $\sigma_y^0(d) = \sigma_y^0(c)$ imply that there exist $p \in \mathbb{K}[z]$ and $q \in \mathbb{K}[z_1, z_2]$ such that

$$c = p(y + \frac{k_2}{n_2}z) \quad \text{and} \quad d = q(y + \frac{m}{m_2}z + \frac{k_1}{m_2}),$$

Furthermore, the equality $\sigma_y^0(d) = \sigma_y^0(c)$ implies that there exists $h \in \mathbb{K}[z]$ such that

$$d = h(z + \frac{k}{m}z + \frac{k_1}{m_1}y + \frac{n_1}{m_1}x).$$

Thus both $c$ and $d$ factor into products of integer-linear polynomials in $x, y,$ and $z$ over $\mathbb{K}$. Therefore $f$ is a proper rational function, and hence it has a telescope.

Suppose that $c$ satisfies the condition (ii). Set

$$L = \sum_{\ell=0}^\rho \ell_i S_{x\ell}^{t \ell},$$

where $\rho \in \mathbb{N}$ and $\ell_i \in \mathbb{K}(x)$ are to be determined. Applying the reduction formula (5.1) yields

$$L(f) = \sum_{\ell=0}^\rho \ell_i \sigma_x^{t \ell}(b) = \sum_{\ell=0}^\rho \ell_i \sigma_x^{t \ell}(b_{\ell}) = \sum_{\ell=0}^\rho \ell_i \sigma_x^{t \ell}(\sigma_y^{-t} \sigma_z^{-t} (b_{\ell})).$$

for some $u, v \in \mathbb{K}(x, y)$.

As a result $L = \sum_{\ell=0}^\rho \ell_i S_{x\ell}^{t \ell}$ is a telescope for $f$.

Example 6.4. Let $f = 1/d$ with $d = x + y + z^2$. Since $\sigma_y(d) = \sigma_y(d)$ and $c = 1$, $f$ has a telescope by Lemma 6.3.

Using partial fraction decomposition, we can decompose the rational function $f = \frac{b_1}{d^3}$ into the form

$$f = \frac{1}{d^3} \left( p + \frac{B_1}{C_1} + \frac{B_2}{C_2} + \sum_{i=1}^l \frac{b_i \ell}{c_i} \right),$$

where $p \in \mathbb{K}[x, y, z]$, $B_1, B_2, b_i, \ell \in \mathbb{K}[x, y, z]$, $C_1, C_2, c_i \in \mathbb{K}[x, y, z]$, and $\deg_y(B_2) < \deg_y(C_2)$, $\deg_y(b_i, \ell) < \deg_y(c_i)$, all irreducible factors of $C_1$ satisfy the condition (6.4), but not any factor of $C_2$ and the $c_i$’s, and the condition (6.5) holds for all irreducible factors of $C_2$, but not for any of the $c_i$’s. By Lemma 6.3, $(p + B_1/C_1)/d^3$ has a telescope and so for the existence problem of telescopes we need only to consider

$$r = \frac{1}{d^3} \left( \frac{B_2}{C_2} + \sum_{i=1}^l \frac{b_i \ell}{c_i} \right).$$

From now on, we always assume that $d$ satisfies the condition (6.2). As before we consider two distinct cases according to whether or not $d$ satisfies the condition (6.3).

Theorem 6.5. Let $r = \sum_{i=1}^l \frac{b_i \ell}{c_i}$ such that none of the $c_i$’s satisfies the condition (6.4). Suppose that $d$ satisfies the condition (6.2) but not the condition (6.3). Then $r$ has a telescope if and only if $r = 0$.

Proof. The sufficiency is clear. For the necessity, we assume that $L = \sum_{i=0}^\rho \ell_i S_x^{t \ell} \in \mathbb{K}(x)[S_x]$ with $\ell_0, \ell_1 \neq 0$ is a telescope for $r$. Let $m$ be the smallest positive integer such that $\sigma_y^m(d) = \sigma_y^m(d)$ for some $n, k \in \mathbb{Z}$. Then $\sigma_y^l(d) and $\sigma_y^l(d)$ are distinct $\langle \sigma_y^l, \sigma_y^l \rangle$-orbits for $m + (i - j)$. Let $L = L_0 + \ldots + L_{m-1}$ be the $m$-exponent separation of $L$. Since the denominators of $L(r)$ are in distinct $\langle \sigma_y^l, \sigma_y^l \rangle$-orbits, Lemma 3.2 in [21] implies that $L(r)$ is $(\sigma_y, \sigma_y)$-sumnable for all $i$ with $0 \leq i \leq m - 1$. Then $L_0 \neq 0$ is a telescope for $r$. Write $L_0 = \sum_{i=0}^\rho \ell_i S_x^{t \ell}$. Then

$$L_0(r) = \sum_{i=0}^\rho \ell_i \sigma_x^{t \ell}(b_{\ell} \ell) = \sum_{i=0}^\rho \ell_i \sigma_x^{t \ell}(\sigma_y^{-t} \sigma_z^{-t} (b_{\ell} \ell)) = \Delta_y(u) + \Delta_z(v) + \frac{1}{\lambda} h,$$

for some $u, v \in \mathbb{K}(x, y)$ and

$$h = \sum_{i=0}^\rho \ell_i \sigma_x^{t \ell}(\sigma_y^{-t} \sigma_z^{-t} (c_{\ell})).$$

Since $L_0(r)$ is $(\sigma_y, \sigma_y)$-sumnable but $d$ does not satisfy condition (6.3), Theorem 3.1 implies that $h = 0$. By Lemma 5.1, for each multiplicity $\ell$, we have

$$h_\ell = \sum_{i=0}^\rho \ell_i \sigma_x^{t \ell}(\sigma_y^{-t} \sigma_z^{-t} (c_{\ell})) = 0.$$

We first claim that there exists a polynomial $p \in \mathbb{K} := \{c_i \mid 1 \leq i \leq I\}$ such that $p \neq \sigma_y^{t \ell}(\sigma_y^{-t} \sigma_z^{-t} (q))$ for any $q \in \mathbb{K}$ and $\nu \in \mathbb{N}$. We prove this claim by contradiction. Suppose that for any $p_i \in \mathbb{K}$, there always exists $p_2 \in \mathbb{K}$ such that $p_2 = \sigma_y^{t \ell}(\sigma_y^{-t} \sigma_z^{-t} (p_i))$ for some positive integer $v$. If $p_1 = p_2$, then we get a contradiction with the assumption on the $c_i$’s in (6.7). If $p_1 \neq p_2$, then there exists $p_3 \in \mathbb{K}$ such that $p_2 = \sigma_y^{t \ell}(\sigma_y^{-t} \sigma_z^{-t} (p_3))$ for some positive integer $v_2$. Continuing this process, we get a sequence of polynomials $p_1, p_2, \ldots \in \mathbb{K}$. Since $\Omega$ is a finite set, $p_1 = p_i$ for some $i < j$ in this sequence. Then $p_i = \sigma_y^{t \ell}(\sigma_y^{-t} \sigma_z^{-t} (p_i)) = \alpha_{i \ell}$ for $\nu = \nu_1 + \ldots + \nu_{i-1} > 0$, a contradiction. This completes the proof of the claim.

Suppose now that $c_1$ is an element in $\Omega$ such that $c_1 \neq \sigma_y^{t \ell}(\sigma_y^{-t} \sigma_z^{-t} (q))$ for any $q \in \mathbb{K}$ and $\nu \in \mathbb{N}$. Then the fraction $\alpha_{i \ell}$ has a different irreducible denominator from the other fractions in $h_\ell$ which implies that $a_0 b_{i \ell} = 0$. Since $a_0 \neq 0$ we have that $b_{i \ell} = 0$ for all $\ell$. We can now repeat the argument for $\Omega \setminus \{c_1\}$ to get $b_{i \ell} = 0$ for all $i = 2, \ldots, n$ and all $\ell$. Thus, $r = 0$.

Example 6.6. Let

$$f = \frac{xy + xz + y^2 + yz + 1}{(x + y)((x + y)^2 + z^2)}$$

for some $x, y, z \in \mathbb{K}$.
We first rewrite \( f \) into
\[
f = \left( y + z + \frac{1}{x + y} \right) \frac{1}{(x + y)^2 + z^2}.
\]

Letting \( d = (x + y)^2 + z^2 \) one has \( \sigma_d = \sigma_y^2 \) and hence from Remark 2.3 and Lemma 6.3 we see that \( f \) has a telescoper. In fact, following the proof of Lemma 6.3, we can see that
\[
L_1 = S_y^2 - 2S_y + 1 = (S_y - 1)^2 \quad \text{and} \quad L_2 = S_z - 1
\]
are telescopers for \( \frac{y + z}{x + y} \) and for \( \frac{1}{(x + y)^2} \) respectively. Thus \( L = (S_y - 1)^2 \) is a telescoper for \( f \).

We now study the case when \( d \) satisfies the condition (6.3). Assume that \( n_1 \) is the smallest positive integer such that
\[
\sigma_y^n(d) = \sigma_y^k(d)
\]
for some \( k_1 \in \mathbb{Z} \). By Lemma 6.3, the fraction \( \frac{b_2}{c_2d^2} \) in (6.7) has a telescoper. It remains to study the existence of telescopers for rational functions of the form
\[
r = \sum_{i=1}^j \frac{b_{i,\ell}}{c_i d^i},
\]
where \( b_{i,\ell} \in \mathbb{K}[x,y,z], c_i \in \mathbb{K}[x,y] \), \( \deg_y(b_{i,\ell}) < \deg_y(c_i) \), and the \( c_i \)'s are irreducible polynomials such that condition (6.5) is not satisfied.

**Theorem 6.7.** Let \( r \) be of the form (6.8) with \( d \) satisfying conditions (6.2) and (6.3) and where \( c_i \)'s do not satisfy the condition (6.5). Then \( r \) has a telescoper if and only if
\[
r_{\ell} := \sum_{i=1}^j \frac{b_{i,\ell}}{c_i d^i}
\]
is \((\sigma_y, \sigma_z)\)-summable for all \( \ell \).

**Proof.** The sufficiency follows from Remark 2.3. For the necessity, we assume that \( L \) is a telescoper for \( r \). By the same argument as in the proof of Theorem 6.5, we may always assume that \( L = \sum_{i=0}^T a_i S^{tm} \) with \( a_0 \neq 0 \). The same calculation as in the proof of Theorem 6.5 then yields
\[
L(r) = \Delta_\ell(u + \Delta_\ell(r)) + \frac{1}{a_{\ell}} h_\ell,
\]
where \( u, v \in \mathbb{K}(x, y, z) \) and \( h := Q(\sum_{i=1}^j a_i b_{i,\ell}/c_i^\ell) \) with
\[
Q = \sum_{i=0}^T a_i S^{tm} S^{t-n} S^{t-k} \in \mathbb{K}(x)[S_z, S_y, S_x].
\]
Since \( L(r) \) is \((\sigma_y, \sigma_z)\)-summable but \( d \) satisfies the condition (6.3), Theorem 3.1 implies that \( h = \sigma_y^n \sigma_z^{k_{2,\ell}}(p) \), where \( p \in \mathbb{K}(x, y)[z] \) with \( \deg_y(p) < \deg_y(d) \). By Lemma 5.1, for each multiplicity \( \ell \), we have
\[
h_{\ell} = Q \left( \sum_{i=1}^j \frac{b_{i,\ell}}{c_i^\ell} \right) = \sigma_y^n \sigma_z^{k_{2,\ell}}(p_{\ell}) - p_{\ell}.
\]
Let \( \Delta := \{ c_{\ell} \mid 1 \leq \ell \leq I \} \). As in the argument for the proof of Theorem 6.5, we may assume \( c_{\ell} \in \Delta \) satisfying \( c_{\ell} \neq \sigma_y^n \sigma_z^{c_{\ell}} \) for any \( c_{\ell} \in \Delta \), when \( m, n \in \mathbb{Z} \) with \( m > 0 \). Note that there may exist some \( c_{\ell} \in \Delta \setminus \{ c_{\ell} \} \) such that \( c_{\ell} = \sigma_y^n \sigma_z^{c_{\ell}} \) for some \( n \in \mathbb{Z} \), and we will let
\[
\Delta_{+} := \{ 1 \leq \ell \leq I, c_{\ell} = \sigma_y^n \sigma_z^{c_{\ell}} \}.
\]
Continuing now with \( \Delta \setminus \Delta_{+} \), we will find \( c_{1}, c_{2}, \ldots, c_{M} \in \Delta \) and \( \Delta_{1}, \Delta_{2}, \ldots, \Delta_{M} \) such that for \( 1 \leq i < i' \leq M \), we have
\[
c_{i} \neq \sigma_y^n \sigma_z^{c_{i'}} \text{ when } m, n \in \mathbb{Z}, m > 0 \text{ and } \{ 1, 2, \ldots, I \} = \bigcup_{i=1}^M \Delta_{i}.
\]
We can therefore rewrite \( h_{\ell} \) as
\[
Q \left( \sum_{i=1}^M \frac{b_{i,\ell}}{c_i^\ell} \right) = \sigma_y^n \sigma_z^{k_{2,\ell}}(p_{\ell}) - p_{\ell}.
\]
Since \( p_{\ell} \in \mathbb{K}(x, y)[z] \), we can decompose it into
\[
p_{\ell} = \sum_{j=1}^M \beta_j \frac{u_{j,\ell}}{c_j^\ell} + q_{\ell},
\]
where \( \alpha_j, \beta_j \in \mathbb{Z} \) and \( q_{\ell} \) contains no term of the form \( \frac{u_{j,\ell}}{c_j^\ell} \) in its irreducible partial fraction decomposition with respect to \( y \). According to Equation (6.9) and the uniqueness of irreducible partial fraction decomposition along with the fact that \( a_0 \in \mathbb{K}(x) \setminus \{ 0 \} \), we derive that
\[
\sum_{i=1}^M \frac{b_{i,\ell}}{c_i^\ell} = \sigma_y^n \sigma_z^{k_{2,\ell}}(h_{1,\ell}) - h_{1,\ell},
\]
where \( h_{1,\ell} = \frac{1}{a_{\ell}} \sum_{j=1}^M \beta_j \frac{u_{j,\ell}}{c_j^\ell} \). Collecting all the terms with denominator \( \langle \sigma_x, \sigma_y \rangle \)-equivalent to \( c_{\ell} \) in Equation (6.9), we obtain
\[
Q \left( \sum_{i=1}^M \frac{b_{i,\ell}}{c_i^\ell} \right) = Q \left( \sigma_y^n \sigma_z^{k_{2,\ell}}(h_{1,\ell}) - h_{1,\ell} \right)
\]
(6.10)
\[
= \sigma_y^n \sigma_z^{k_{2,\ell}}(p_{\ell}) - p_{\ell}.
\]
with \( p_{\ell} = Q(h_{1,\ell}) \). Subtracting Equation (6.11) from Equation (6.9), we obtain
\[
Q \left( \sum_{j=2}^M \frac{b_{j,\ell}}{c_j^\ell} \right) = \sigma_y^n \sigma_z^{k_{2,\ell}}(p_{\ell}) - p_{\ell}
\]
(6.12)
with \( p_{\ell} = p_{\ell} - p_{1,\ell} \). Now we can repeat the arguments for the set \( \Delta \setminus \Delta_{1} \) and Equation (6.12) to get
\[
\sum_{i=1}^M \frac{b_{i,\ell}}{c_i^\ell} = \sigma_y^n \sigma_z^{k_{2,\ell}}(h_{j,\ell}) - h_{j,\ell},
\]
for all \( j = 1, \ldots, M \) and all \( \ell \). Then \( \sum_{i=1}^M \frac{b_{i,\ell}}{c_i^\ell} \) is \((\sigma_y, \sigma_z)\)-summable by Theorem 3.1 and thus \( \sum_{i=1}^M \frac{b_{i,\ell}}{c_i^\ell} \) is \((\sigma_y, \sigma_z)\)-summable for all \( \ell \). This completes the proof. \( \blacksquare \)

Combining Lemmas 6.2, 6.3 and Theorems 6.5, 6.7, we now present an algorithm for testing the existence of telescopers for simple fractions in Figure 1.

**Remark 6.8.** For testing the existence of telescopers for a general rational function \( f \in \mathbb{K}(x, y, z) \), we first apply the algorithm in [21] to compute the additive decomposition \( f = \Delta_\ell(g) + \Delta_\ell(h) + r \), where \( g, h, r \in \mathbb{K}(x, y, z) \) and \( r \) is of the form (5.2) with the \( d_s \)’s satisfying the condition (5.3). By Lemmas 5.2 and 5.3, the existence of telescope for \( f \) can be determined by applying Algorithm ExistenceTelescoperSimple to each simple fraction of \( r \).

**Example 6.9.** Let
\[
f = \frac{x^4 + 2x^2y^2 + y^4 + x^3 + 3xy^2 + y^3 - xy^2 + x^2 - xy}{(x + y)(x^2 + y^2)(2x + 1)(x^2 + y^2)(x + y + z)^2}.
\]
Algorithm **ExistenceTelescopersimple**

**Input**: \( f = b/c^{d^\prime} \) as in (6.1).

**Output**: true if \( f \) has a telescopers; false otherwise.

1. Using partial fraction decomposition, decompose \( f \) into the form (6.6);
2. If \( d \) does not satisfy the condition (6.2), return true if \( f \) is summable (checked by the algorithm in [21]) and false otherwise; Else
   (a) if \( d \) does not satisfy the condition (6.3), return true if \( B_2 = 0 \) and \( b_i, \ell = 0 \) for all \( i, \ell \) and false otherwise; Else
      i. return true if \( r_\ell := \sum_{i=1}^{\ell} b_i, \ell \) is summable for all \( \ell \), and false otherwise.

**Figure 1**: Testing the existence of telescopers for simple fractions.

First decompose \( f \) as

\[
\frac{1}{x+y} + \frac{y+1}{x^2+y^2+2y+1} - \frac{y}{x^2+y^2}, \quad \frac{1}{(x+y+z)^2}. \]

Letting \( d = x+y+z \), we have \( \sigma_y d = \sigma_x d \) and \( \sigma_x d = \sigma_y d \). As in the proof of Lemma 6.3, we get that \( \Sigma = S_x - 1 \) is a telescopers for \( \left( \frac{1}{x+y} \right) \left( \frac{y+1}{x+y+z} \right)^2 \). Theorem 3.1 then guarantees

\[
\frac{y+1}{x^2+y^2+2y+1} - \frac{y}{x^2+y^2}, \quad \frac{1}{(x+y+z)^2}. \]

is \((\sigma_y, \sigma_x)\)-summable, so \( \Sigma = S_x - 1 \) is a telescopers for \( f \).

7. **Conclusion**

In this paper, we solve the existence problem of telescopers for rational functions in three discrete variables. We give a procedure which reduces the problem to a special shift equivalence testing problem and the summability problem of bivariate rational functions. Those problems have been recently solved. In terms of future research, the first direction is to solve the existence problem of telescopers for multivariate rational functions or a more general class of functions, for example, hypergeometric terms. This would include both efficient algorithms and implementations. A crucial step is to solve the summability problem for these functions. This is also a challenging problem in symbolic summation as noted in [9].

**Acknowledgement**. The authors would like to thank the anonymous referees for their constructive and helpful comments, which have significantly improved the presentation of this paper.

8. **References**


