A BABY STEP-GIANT STEP ROADMAP ALGORITHM FOR GENERAL ALGEBRAIC SETS

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ABSTRACT. Let R be a real closed field and D $\subset$ R an ordered domain. We give an algorithm that takes as input a polynomial $Q \subset \mathrm{D}[X_1, \ldots, X_k]$, and computes a description of a roadmap of the set of zeros, $\mathrm{Zer}(Q, \mathrm{R}^k)$, of Q in R^k. The complexity of the algorithm, measured by the number of arithmetic operations in the domain D, is bounded by $d^{O(k\sqrt{k})}$, where $d = \deg(Q) \geq 2$. As a consequence, there exist algorithms for computing the number of semi-algebraically connected components of a real algebraic set, $\mathrm{Zer}(Q, \mathrm{R}^k)$, whose complexity is also bounded by $d^{O(k\sqrt{k})}$, where $d = \deg(Q) \geq 2$. The best previously known algorithm for constructing a roadmap of a real algebraic subset of R^k defined by a polynomial of degree d had complexity $d^{O(k^2)}$.

1. Introduction

The problem of designing efficient algorithms for deciding whether two points belong to the same semi-algebraically connected component of a semi-algebraic set, as well as counting the number of semi-algebraically connected components of a given semi-algebraic set $S \subset \mathrm{R}^k$ where R is a real closed field (for example the field of real numbers), is a very important problem in algorithmic semi-algebraic geometry.

The first algorithm for solving this problem [**?**] was based on the technique of cylindrical algebraic decomposition [**?**, **?**], and consequently had doubly exponential complexity.

Algorithms with singly exponential complexity were given later in a series of papers [**?**, **?**, **?**, **?**, **?**].

They are all based on a geometric idea introduced by Canny, the construction of an one-dimensional semi-algebraic subset of the given semi-algebraic set S, called a *roadmap of S*, which has the property that it is non-empty and semi-algebraically connected inside every semi-algebraically connected component of S.

Key words and phrases. Roadmaps, Real algebraic variety, Baby step-giant step.

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In the papers mentioned above, the construction of a roadmap of a semi-algebraic set S depends on recursive calls to itself on several (in fact, singly exponentially many) $(k-1)$ dimensional slices of S, each obtained by fixing the first coordinate. For constructing the roadmap of a real algebraic variety defined by a polynomial $Q \subset \mathrm{R}[X_1, \ldots, X_k]$ with $\deg(Q) \leq d$, this technique gave an algorithm with complexity $d^{O(k^2)}$. The exponent in the complexity, $O(k^2)$, is due to the fact that the depth of the recursion in these algorithms could be as large as k. This exponent is not satisfactory since the total number of semi-algebraically connected components is $(O(d))^k$ and so there is room for trying to improve it. However, this has turned out to be a rather difficult problem with no progress till very recently.

A new construction for computing roadmaps, with an improved recursive scheme of baby step - giant step type, has been proposed, and applied successfully in the case of smooth real algebraic hypersurfaces in [**?**]. In this new recursive scheme, the dimension drops by $\sqrt{k}$ in each recursive call. As a result, the depth of the recursive calls in this new algorithm is at most $\sqrt{k}$, and consequently the algorithm has a complexity of $d^{O(k\sqrt{k})}$. The proof of correctness of the algorithm in [**?**] depends on certain results from commutative algebra and complex algebraic geometry, in order to prove smoothness of polar varieties corresponding to generic projections of a non-singular hypersurface. Choosing generic coordinates in the algorithm is necessary since the non-singularity of polar varieties does not hold for all projections, but only for a Zariski-dense set of projections. This is an important restriction, since there is no known method for making such a choice of generic coordinates deterministically within this improved complexity bound. As a result, the authors obtain a randomized (rather than a deterministic) algorithm for computing roadmaps: there might be cases where the algorithm terminates and gives a wrong result.

In contrast to these techniques which depend on complex algebraic geometry, the algorithm for constructing roadmaps described in [**?**] depend mostly on arguments which are semi-algebraic in nature. The greater flexibility of semi-algebraic geometry (as opposed to complex geometry) makes it possible to avoid genericity requirements for coordinates. More precisely, we apply the technique used in [**?**] to make an infinitesimal deformation of the given variety so that the original coordinates are good. Since the infinitesimal deformation uses only one infinitesimal, it does not affect the asymptotic complexity class of the algorithm.

The goal of this paper is to obtain a *deterministic algorithm* for computing the roadmap of a *general algebraic set*, combining a baby step - giant step recursive scheme similar to that used in [**?**] and extending techniques coming from [**?**].

We start by recalling the precise definition of what is meant by a roadmap.

Definition 1.1. Let $S \subset \mathrm{R}^k$ be a semi-algebraic set. A *roadmap* for S is a semi-algebraic set $\mathrm{RM}(S)$ of dimension at most one contained in S which satisfies the following roadmap conditions:

(1) RM_1 For every semi-algebraically connected component C of S, $C \cap \mathrm{RM}(S)$ is semi-algebraically connected.
(2) RM_2 For every $x \in \mathrm{R}$ and for every semi-algebraically connected component D of S_x, $D \cap \mathrm{RM}(S) \neq \emptyset$, where we denote by S_x the set $S \cap \pi_1^{-1}(x)$ for $x \in \mathrm{R}$, and $\pi_1 : \mathrm{R}^k \to \mathrm{R}$ the projection map onto the first coordinate.

Let $\mathcal{M} \subset \mathrm{R}^k$ be a finite set of points. A roadmap for $(S, \mathcal{M})$ is a semi-algebraic set $\mathrm{RM}(S, \mathcal{M})$ such that $\mathrm{RM}(S, \mathcal{M})$ is a roadmap of S and $\mathcal{M} \subset \mathrm{RM}(S, \mathcal{M})$.

The main result of the paper is the following theorem. The notion of real univariate representations used in the following statements is explained in Section 4.

Theorem 1.2. *Let $Z \subset \mathrm{R}^k$ be an algebraic set defined as the set of zeros of a polynomial of degree at most $d \geq 2$ in k variables with coefficients in an ordered domain* D *contained in a real closed field* R*.*

a) *There exists an algorithm for constructing a roadmap for Z using $d^{O(k\sqrt{k})}$ arithmetic operations in* D*.*
b) *Moreover, there exists an algorithm that given a finite set of points $\mathcal{M}_0 \subset Z$, with cardinality δ, and described by real univariate representations of degree at most $d^{O(k)}$, constructs a roadmap for $(Z, \mathcal{M}_0)$ using $\delta^{O(1)} d^{O(k\sqrt{k})}$ arithmetic operations in* D*.*

The following corollary is an immediate consequence of b).

Corollary 1.3. *Let $Z \subset \mathrm{R}^k$ be an algebraic set defined as the set of zeros of a polynomial of degree at most $d \geq 2$ in k variables with coefficients in an ordered domain* D *contained in a real closed field* R*.*

a) *There exists an algorithm for counting the number of semi-algebraically connected components of Z which uses $d^{O(k\sqrt{k})}$ arithmetic operations in* D*.*
b) *There exists an algorithm for deciding whether two given points, described by real univariate representations of degree at most $d^{O(k)}$ belong to the same semi-algebraically connected component of Z which uses $d^{O(k\sqrt{k})}$ arithmetic operations in* D*.*

Remark 1.4. We can always suppose without loss of generality that the zero set of a family of polynomials of degree at most d is defined by one single polynomial of degree at most $2d$ by replacing the input polynomials by their sum of squares.

Remark 1.5. Even if the input is a polynomial with coefficients in the field of real numbers, the deformation techniques by infinitesimal elements we use make it necessary to perform computations on polynomials with coefficients in some non-archimedean real closed field. This is the reason why general real closed fields provide a natural framework for our work.

2. Outline

We outline below the classical construction of a roadmap $\mathrm{RM}(\mathrm{Zer}(Q, \mathrm{R}^k))$ for a bounded algebraic set $\mathrm{Zer}(Q, \mathrm{R}^k)$, defined as the zero set of a polynomial Q inside R^k. The geometric ideas yielding this construction are due to Canny. The description below is similar to the one in [**?**, Chapter 15, Section 15.2].

A key ingredient of the algorithm is the construction of a particular finite set of points intersecting every semi-algebraically connected component of $\mathrm{Zer}(Q, \mathrm{R}^k)$. In the case of a bounded and non-singular real algebraic set in R^k (in the generic case), these points are nothing but the set of critical points of the projection to the X_1-coordinate on $\mathrm{Zer}(Q, \mathrm{R}^k)$. In more general situations, the points we consider are called X_1-pseudo-critical points, since they are obtained as limits of the critical points of the projection to the X_1-coordinate of a bounded nonsingular algebraic hypersurface defined by a particular infinitesimal deformation of the polynomial Q. Their projections on the X_1-axis are called pseudo-critical values.

We first construct the "silhouette" which is the set of X_2-pseudo-critical points on $\mathrm{Zer}(Q, \mathrm{R}^k)$ along the X_1-axis by following continuously, as x varies on the X_1-axis, the X_2-pseudo-critical points on $\mathrm{Zer}(Q, \mathrm{R}^k)_x$. This results in curves and their endpoints on $\mathrm{Zer}(Q, \mathrm{R}^k)$. The curves are continuous semi-algebraic curves parametrized by open intervals on the X_1-axis and their endpoints are points of $\mathrm{Zer}(Q, \mathrm{R}^k)$ above the corresponding endpoints of the open intervals. Since these curves and their endpoints include for every $x \in \mathrm{R}$ the X_2-pseudo-critical points of $\mathrm{Zer}(Q, \mathrm{R}^k)_x$, they meet every semi-algebraically connected component of $\mathrm{Zer}(Q, \mathrm{R}^k)_x$. Thus, the set of curves and their endpoints, already satisfy RM_2. However, it is clear that this set might not be semi-algebraically connected in a semi-algebraically connected component and so RM_1 might not be satisfied.

In order to ensure property RM_1 we need to add more curves to the roadmap. For this purpose, we define the set of *distinguished values* $\mathcal{D}$ as the union of the X_1-pseudo-critical values, and the first coordinates of the endpoints of the curves described in the previous paragraph. A *distinguished hyperplane* is an hyperplane defined by $X_1 = v$, where v is a distinguished value. The input points, the endpoints of the curves, and the intersections of the curves with the distinguished hyperplanes define the set of *distinguished points*, $\mathcal{M}$.

Let the distinguished values be $v_1 < \ldots < v_N$. Note that amongst these are the X_1-pseudo-critical values. Above each interval (v_i, v_{i+1}) we have constructed a collection of curves $\mathcal{C}_i$ meeting every semi-algebraically connected component of $\mathrm{Zer}(Q, \mathrm{R}^k)_v$ for every $v \in (v_i, v_{i+1})$. Above each distinguished value v_i we have a set of distinguished points $\mathcal{N}_i$. Each curve in $\mathcal{C}_i$ has an endpoint in $\mathcal{N}_i$ and another in $\mathcal{N}_{i+1}$. Moreover, the union of the $\mathcal{N}_i$ contains $\mathcal{N}$. We denote by $\mathcal{C}$ the union of the $\mathcal{C}_i$.

The following key connectivity result is proved in [**?**, Lemma 15.9].

Proposition 2.1. *Let* $\mathcal{R} = \mathcal{C} \cup \mathrm{Zer}(Q, \mathrm{R}^k)_{\mathcal{D}}$. *If* P *is a semi-algebraically connected component of* $\mathrm{Zer}(Q, \mathrm{R}^k)$, *then* $\mathcal{R} \cap P$ *is semi-algebraically connected.*

Thus, in order to construct a roadmap of $\mathrm{Zer}(Q, \mathrm{R}^k)$ it suffices to repeat the same construction in each distinguished hyperplane H_i defined by $X_1 = v_i$ with input $Q(v_i, X_2, \ldots, X_k)$ and the distinguished points in $\mathcal{M}_{v_i}$ by making recursive calls to the algorithm. The following proposition is proved in [**?**, Proposition 15.7].

Proposition 2.2. *The semi-algebraic set* $\mathrm{RM}(\mathrm{Zer}(Q, \mathrm{R}^k), \mathcal{M})$ *obtained by this construction is a roadmap for* $\mathrm{Zer}(Q, \mathrm{R}^k)$ *containing* $\mathcal{M}$.

To summarize, classical roadmap algorithms based on Canny's construction proceed by first considering the "silhouette", consisting of curves in the X_1-direction, and then making recursive calls to the same algorithm at certain hyperplane sections of $\mathrm{Zer}(Q, \mathrm{R}^k)$, so that the dimension of the ambient space drops by 1 at each recursive call.

The main difference between classical roadmap algorithms and the algorithms described in [**?**] and in the current paper is that instead of considering curves in the X_1-direction and making recursive calls to the same algorithm at certain hyperplane sections of $\mathrm{Zer}(Q, \mathrm{R}^k)$ corresponding to special values of X_1, so that the dimension of the ambient space drops by 1, we consider a p-dimensional subset W of $\mathrm{Zer}(Q, \mathrm{R}^k)$ where $1 \le p \le k$, and make recursive calls at certain $(k-p)$-dimensional fibers of $\mathrm{Zer}(Q, \mathrm{R}^k)$, so that the dimension of the ambient space drops by p.

The main topological result, generalizing Proposition 2.1, is that the semi-algebraic set which is the union of W and these fibers is semi-algebraically connected. This is proved in Section 3, in a special case. Thus, in order to produce a roadmap of $\mathrm{Zer}(Q, \mathrm{R}^k)$ it suffices to compute a roadmap of W passing through an appropriate set of points, and the roadmap of the corresponding fibers in a $(k-p)$-dimensional ambient space, using recursive calls.

The fact that in the new algorithm we are fixing a whole block of p variables at a time necessitates introducing a new kind of algebraic representation which we call "real block representation". This notion is defined in Section 4, where we also explain how to represent curves.

In Section 5, the roadmap of W is computed by an algorithm directly adapted from [**?**, Algorithm 15.3] which makes use of the fact that W is low dimensional, in a special case. The general case, requiring the use of a deformation technique and a limit process, is described in Section 6.

Finally, we obtain in Section 7 a baby step - giant step roadmap algorithm for a general algebraic set. We prove its correctness, as well as the improved complexity bound.

The algorithm for computing efficiently limits of curve segments is quite technical. Since, this technicality can obscure the ideas behind the main algorithm, for the sake of readability we have postponed the details behind taking limits of curves to a separate section (Section 8).

Throughout the paper, we use as a basic reference [**?**]. We cite [**?**] instead when the precise statements needed in the paper are not included in [**?**].

3. Connectivity results

In this section we prove a topological result about connectivity which will be used in proving the correctness of our algorithm later. The statement of the result, as well as the main ideas of the proof, is influenced by [**?**, Theorem 14]. It is a direct generalization of Proposition 2.1 to the case of projection onto more than one variable.

We denote by R a real closed field.

Notation 3.1. For $1 \le q \le p < k$, we denote by $\pi_{[q,p]} : \mathrm{R}^k \to \mathrm{R}^{p-q+1}$ the projection

$$(x_1, \ldots, x_k) \mapsto (x_q, \ldots, x_p).$$

In case $p = q$ we will denote by π_p the projection $\pi_{[p,p]}$. For $1 \le q \le p < k$, we denote by $\pi_{]q,p]} : \mathrm{R}^k = \mathrm{R}^k \to \mathrm{R}^{p-q}$ the projection

$$(x_1, \ldots, x_k) \mapsto (x_{q+1}, \ldots, x_p).$$

For any semi-algebraic subset $S \subset \mathrm{R}^k$, and $T \subset \mathrm{R}^p$, we denote by S_T the semi-algebraic set $\pi_{[1,p]}^{-1}(T) \cap S$, and S_y rather than $S_{\{y\}}$, for $y \in \mathrm{R}^p$. We also denote $S_{<a}$ and $S_{\le a}$ rather than $S_{(-\infty,a)}$ and $S_{(-\infty,a]}$, for $a \in \mathrm{R}$.

We denote as before by $\mathrm{Zer}(Q, \mathrm{R}^k)$ the algebraic set of zeros of a polynomial $Q \in \mathrm{R}[X_1, \ldots, X_k]$ inside R^k. Note that this does not imply that the dimension of $\mathrm{Zer}(Q, \mathrm{R}^k)$ is $k - 1$. In fact, over any real closed field, algebraic sets defined by one equation coincide with general algebraic sets since replacing several equations by their sum of squares does not modify the zero set. A Q–**singular point** is a point x such that

$$Q(x) = \frac{\partial Q}{\partial X_1}(x) = \ldots = \frac{\partial Q}{\partial X_k}(x) = 0.$$

Note that this is an algebraic property related to the equation Q rather than a geometric property of the underlying set $\mathrm{Zer}(Q, \mathrm{R}^k)$: two equations can define the same algebraic set but have a different set of singular points.

Similarly a Q-**critical point** of π_1 is a point x such that

$$Q(x) = \frac{\partial Q}{\partial X_2}(x) = \cdots = \frac{\partial Q}{\partial X_k}(x) = 0.$$

To simplify notations, when there will be no ambiguity on Q, we will simply refer to singular/critical points.

In this paper, we will be using constantly the notion of semi-algebraically connected components of a semi-algebraic set [**?**, Section 5.2]. Note that, in particular, a semi-algebraically connected component is always non-empty by definition [**?**, Theorem 5.21].

Property 3.2. We now consider a tuple

$$(V, p, W, (\mathcal{M}_i)_{1\leq i\leq 2}, (\mathcal{D}_i)_{1\leq i\leq 2})$$

with the following properties

(1) $V \subset \mathrm{R}^k$ is the union of certain bounded semi-algebraically connected components of an algebraic set $\mathrm{Zer}(Q, \mathrm{R}^k) \subset \mathrm{R}^k$, such that the Q-singular points of V, as well as the Q-critical points of the map π_1 on V form the finite set $\mathcal{M}_1 \subset V$, and $\mathcal{D}_1 = \pi_1(\mathcal{M}_1)$;
(2) $W \subset V$ is a closed semi-algebraic set of dimension p, $1 \leq p < k$, such that for each $y \in \mathrm{R}^p$, W_y is a finite set of points having non-empty intersection with every semi-algebraically connected component of V_y;
(3) $\mathcal{M}_2 \subset V$ is a finite subset such that the intersection of $\mathcal{M}_2$ with every semi-algebraically connected component of W_a is non-empty, for $a \in \mathcal{D}_2 = \pi_1(\mathcal{M}_2)$. Moreover for every interval $[a, b]$ and $c \in [a, b]$ with $\{c\} \supset \mathcal{D}_2 \cap [a, b]$, if D is a semi-algebraically connected component of $W_{[a,b]}$, then D_c is a semi-algebraically connected component of W_c.

A tuple

$$(V, p, W, (\mathcal{M}_i)_{1\leq i\leq 2}, (\mathcal{D}_i)_{1\leq i\leq 2})$$

satisfies Property 3.2 if it satisfies the above properties (1) to (3).

We state now the main result of this section. It generalizes Proposition 2.1 as well as [**?**, Theorem 14], in the special case of Property 3.2.

Proposition 3.3. *Let*

$$(V, p, W, (\mathcal{M}_i)_{1\leq i\leq 2}, (\mathcal{D}_i)_{1\leq i\leq 2})$$

satisfy Property 3.2,

$$\mathcal{N}_1 = \pi_{[1,p]}(\mathcal{M}_1), \mathcal{N}_2 = \pi_{[1,p]}(\mathcal{M}_2),$$

and

$$\mathcal{S} = W \cup V_{\mathcal{N}_1 \cup \mathcal{N}_2}.$$

For every semi-algebraically connected component C of V, $C \cap \mathcal{S}$ is non-empty and semi-algebraically connected.

Remark 3.4. In order to understand the situation, the following example of a tuple satisfying Property 3.2 can be useful

(1) the torus $V \subset \mathrm{R}^3$ defined as the set of zeros of the equation

$$Q = 36(X_1^2 + \left(\frac{12X_2 + 5X_3}{13}\right)^2 - (X_1^2 + X_2^2 + X_3^2 + 8)^2,$$

([**?**], page 40, figure 2.5), the four critical points $\mathcal{M}_1 \subset V$, of the map π_1 restricted to V and $\mathcal{D}_1 = \pi_1(\mathcal{M}_1)$;

(2) the silhouette $W \subset V$ defined by

$$Q = \frac{\partial Q}{\partial X_3} = 0;$$

(3) the six critical values $\mathcal{D}_2 \subset \mathrm{R}$ of the map π_1 restricted to the silhouette W, and the intersection $\mathcal{M}_2$ of the corresponding six fibers with the silhouette W.

The tuple

$$(V, 1, W, (\mathcal{M}_i)_{1 \le i \le 2}, (\mathcal{D}_i)_{1 \le i \le 2})$$

satisfies Property 3.2.

Finally, $\mathcal{S}$ is the union of the silhouette and the intersection of the torus with the six curves which are the fibers of V at the distinguished values $\mathcal{D}_2$.

The end of this section is devoted to prove Proposition 3.3. We need preliminaries about non-archimedean extensions of the base real closed field R.

Remark 3.5. A typical non-archimedean extension of R is the field $\mathrm{R}\langle\varepsilon\rangle$ of algebraic Puiseux series with coefficients in R, which coincide with the germs of semi-algebraic continuous functions (see [**?**, Chapter 2, Section 6 and Chapter 3, Section 3]). An element $x \in \mathrm{R}\langle\varepsilon\rangle$ is bounded over R if $|x| \le r$ for some $0 \le r \in \mathrm{R}$. The subring $\mathrm{R}\langle\varepsilon\rangle_b$ of elements of $\mathrm{R}\langle\varepsilon\rangle$ bounded over R consists of the Puiseux series with non-negative exponents. We denote by $\lim_\varepsilon$ the ring homomorphism from $\mathrm{R}\langle\varepsilon\rangle_b$ to R which maps $\sum_{i\in\mathbb{N}} a_i \varepsilon^{i/q}$ to a_0. So, the mapping $\lim_\varepsilon$ simply replaces ε by 0 in a bounded Puiseux series. Given $S \subset \mathrm{R}\langle\varepsilon\rangle^k$, we denote by $\lim_\varepsilon(S) \subset \mathrm{R}^k$ the image by $\lim_\varepsilon$ of the elements of S whose coordinates are bounded over R.

More generally, let R' be a real closed field extension of R. If $S \subset \mathrm{R}^k$ is a semi-algebraic set, defined by a boolean formula Φ with coefficients in R, we denote by $\mathrm{Ext}(S, \mathrm{R}')$ the extension of S to R', i.e. the semi-algebraic subset of R'^k defined by Φ. The first property of $\mathrm{Ext}(S, \mathrm{R}')$ is that it is well defined, i.e. independent on the formula Φ describing S [**?**, Proposition 2.87]. Many properties of S can be transferred to $\mathrm{Ext}(S, \mathrm{R}')$: for example S is non-empty if and only if $\mathrm{Ext}(S, \mathrm{R}')$ is non-empty, S is semi-algebraically connected if and only if $\mathrm{Ext}(S, \mathrm{R}')$ is semi-algebraically connected [**?**, Proposition 5.24].

Moreover, if Property 3.2 (2) holds for V, W, i.e. for every $y \in \mathrm{R}^p$, W_y is a finite set of points having non-empty intersection with every semi-algebraically connected component of V_y, then Property 3.2 (2) holds for $\mathrm{Ext}(V, \mathrm{R}'), \mathrm{Ext}(W, \mathrm{R}')$, i.e for each $y' \in \mathrm{R}'^p$, $\mathrm{Ext}(W, \mathrm{R}')_{y'}$ is a finite set of points having non-empty intersection with every semi-algebraically connected component of $\mathrm{Ext}(V,\mathrm{R}')_{y'}$. Indeed, by Hardt's semi-algebraic triviality theorem [**?**, Theorem 5.45], one can find a finite partition of R^p in semi-algebraic sets T_i, $i=1,\ldots,r$, a finite partition of V_{T_i} into semi-algebraic sets $S_{i,j}$ and an integer $n_i>0$ such that $S_{i,j}$ is semi-algebraically homeomorphic to $T_i \times (S_{i,j})_{y_i}$ for some $y_i \in T_i$, and for all $y \in T_i$, the semi-algebraically connected components of V_y are $(S_{i,j})_y$ and W_y has n_i points. By Tarski-Seidenberg's transfer principle [**?**, Theorem 2.80], $\mathrm{Ext}(S_{i,j},\mathrm{R}')$ is semi-algebraically homeomorphic to $\mathrm{Ext}(T_i,\mathrm{R}')\times \mathrm{Ext}(S_{i,j},\mathrm{R}')_{y_i}$, and for all $y' \in \mathrm{R}'^p$, there exists i such that $y' \in \mathrm{Ext}(T_i,\mathrm{R}')$, the sets $\mathrm{Ext}(S_{i,j},\mathrm{R}')_{y'}$ are the semi-algebraically connected components of $\mathrm{Ext}(V,\mathrm{R}')_{y'}$ and the intersection of $\mathrm{Ext}(W,\mathrm{R}')_{y'}$ and $\mathrm{Ext}(S_{i,j},\mathrm{R}')_{y'}$ has exactly n_i points.

We now prove a few preliminary results about V defined as the union of certain bounded semi-algebraically connected components of an algebraic set $\mathrm{Zer}(Q,\mathrm{R}^k)\subset \mathrm{R}^k$, supposing that the set $\mathcal{M}_1$ of points which are singular points or critical points of π_1 on $\mathrm{Zer}(Q,\mathrm{R}^k)$ inside V is finite.

In this paper a *semi-algebraic path* is a semi-algebraic continuous function γ from a closed interval $[a,b]\subset \mathrm{R}$ to R^k. Note that a semi-algebraic set is semi-algebraically connected if and only it is semi-algebraically path connected [**?**, Theorem 5.23].

Lemma 3.6. *Suppose that $b \notin \mathcal{D}_1 = \pi_1(\mathcal{M}_1)$. Let C be a semi-algebraically connected component of $V_{\le b}$. If $a<b$ and $(a,b]\cap \mathcal{D}_1$ is empty, then $C_{\le a}$ is semi-algebraically connected.*

Proof. Let x and y be two points of $C_{\le a}$ and $\gamma:[0,1]\to C$ be a semi-algebraic path connecting x to y inside C. We want to prove that there is a semi-algebraic path connecting x to y inside $C_{\le a}$.

If $\mathrm{Im}(\gamma)\subset C_{\le a}$ there is nothing to prove.

If $\mathrm{Im}(\gamma)\not\subset C_{\le a}$,

$$\exists c \in \mathrm{R}\ \forall a<d<c\ \mathrm{Im}(\gamma)\cap \mathrm{Zer}(Q)_d \neq \emptyset.$$

Let ε be a positive infinitesimal. Then

$$\mathrm{Ext}(\gamma([0,1]),\mathrm{R}\langle\varepsilon\rangle)\cap \mathrm{Zer}(Q,\mathrm{R}\langle\varepsilon\rangle^k)_{a+\varepsilon}\neq\emptyset$$

using [**?**, Proposition 3.17] . Since

$$\{u\in[0,1]\subset \mathrm{R}\langle\varepsilon\rangle \mid \mathrm{Ext}(\gamma,\mathrm{R}\langle\varepsilon\rangle)(u)\in \mathrm{Zer}(Q,\mathrm{R}\langle\varepsilon\rangle^k)_{<a+\varepsilon}\}$$

and

$$\{u\in[0,1]\subset \mathrm{R}\langle\varepsilon\rangle \mid \mathrm{Ext}(\gamma,\mathrm{R}\langle\varepsilon\rangle)(u)\in \mathrm{Zer}(Q,\mathrm{R}\langle\varepsilon\rangle^k)_{[a+\varepsilon,b]}\}$$

are semi-algebraic subsets of $[0,1]\subset \mathrm{R}\langle\varepsilon\rangle$ there exists by [**?**, Corollary 2.79] a finite partition $\mathfrak{P}$ of $[0,1]\subset \mathrm{R}\langle\varepsilon\rangle$ such that for each open interval (u,v) of $\mathfrak{P}$, $\mathrm{Ext}(\gamma,\mathrm{R}\langle\varepsilon\rangle)(u,v)$ is either contained in $\mathrm{Zer}(Q,R\langle\varepsilon\rangle^k)_{<a+\varepsilon}$, or in $\mathrm{Zer}(Q,\mathrm{R}\langle\varepsilon\rangle^k)_{[a+\varepsilon,b]}$, with $\gamma(u)$ and $\gamma(v)$ in $C_{a+\varepsilon}$.

If $\mathrm{Ext}(\gamma,\mathrm{R}\langle\varepsilon\rangle)(u,v)$ is contained in $\mathrm{Zer}(Q,\mathrm{R}\langle\varepsilon\rangle^k)_{[a+\varepsilon,b]}$, we can replace γ by a semi-algebraic path $\gamma'_{[a,b]}$ connecting $\gamma(u)$ to $\gamma(v)$ inside $C_{a+\varepsilon}$. Note that there is no Q-critical point of π_1 in $\mathrm{Ext}(V, \mathrm{R}\langle\varepsilon\rangle)_{[a+\varepsilon,b]}$ and $\mathrm{Ext}(V, \mathrm{R}\langle\varepsilon\rangle)_{[a+\varepsilon,b]}$ contains no Q-singular point by [?, Proposition 3.17] while $\mathrm{Ext}(V, \mathrm{R}\langle\varepsilon\rangle) \subset \mathrm{Zer}(Q, \mathrm{R}\langle\varepsilon\rangle^k)$ by [?, Proposition 2.87] .

By [?, Proposition 15.1 b] if D is a semi-algebraically connected component of $\mathrm{Ext}(V, \mathrm{R}\langle\varepsilon\rangle)_{[a+\varepsilon,b]}$, $D_{a+\varepsilon}$ is a semi-algebraically connected component of $\mathrm{Ext}(V, \mathrm{R}\langle\varepsilon\rangle)_{a+\varepsilon}$.

Construct a semi-algebraic path γ' from x to x' inside $C_{\leq a+\varepsilon}$, obtained by concatenating pieces of γ inside $\mathrm{Zer}(Q, \mathrm{R}\langle\varepsilon\rangle^k)_{<a+\varepsilon}$ and the paths $\gamma'_{(u,v)}$ connecting $\gamma(u)$ to $\gamma(v)$ for (u, v) such that $\mathrm{Ext}(\gamma, \mathrm{R}\langle\varepsilon\rangle)(u, v) \subset \mathrm{Ext}(V, \mathrm{R}\langle\varepsilon\rangle)_{[a+\varepsilon,b]}$. Note that such a semi-algebraically connected path γ' is closed and bounded. Applying [?, Proposition 12.43], $\lim_\varepsilon(\gamma'([0, 1]))$ is semi-algebraically connected, contains x and x' and is contained in $\lim_\varepsilon(C_{\leq a+\varepsilon}) = C_{\leq a}$. This is enough to prove the lemma. □

We continue to suppose that $\mathcal{M}_1$ is finite.

Lemma 3.7. *Let C be a semi-algebraically connected component of $V_{\leq b}$. such that $C \cap V_b$ is not empty.*

1. *If $\dim(C) = 0$, C is a point contained in $\mathcal{M}_1$.*
2. *If $\dim(C) \neq 0$, $C_{<b}$ is non-empty. Let $B_1, \ldots, B_r$ be the semi-algebraically connected components of $C_{<b}$. Then,*
   - (a) *for each $i, 1 \leq i \leq r$, $\overline{B_i} \cap \mathcal{M}_1 \neq \emptyset$;*
   - (b) *if there exist $i, j, 1 \leq i < j \leq r$ such that $\overline{B_i} \cap \overline{B_j} \neq \emptyset$, then $\overline{B_i} \cap \overline{B_j} \subset \mathcal{M}_1$;*
   - (c) *$\cup_{i=1}^r \overline{B_i} = C$, and hence $\cup_{i=1}^r \overline{B_i}$ is semi-algebraically connected.*

Proof. Part 1 follows immediately from [?, Proposition 7.3]. Let us prove Part 2: since $\mathcal{M}_1$ is finite, there is a non-singular point $x \in C$ which is non-critical for π_1 on V. Let T_xV denote the tangent space to V at x. So T_xV is not orthogonal to the X_1 axis, and the semi-algebraic implicit function theorem [?, Theorem 3.25] implies that $C_{<b}$ is non-empty.

Part 2) a) and 2 b) are immediate consequences of Proposition 7.3 in [?].

We prove 2) c). Clearly, $\cup_{i=1}^r \overline{B_i} \subset C$. Suppose that $x \in C \setminus \cup_{i=1}^r \overline{B_i}$. For $r > 0$ and small enough, $\mathcal{B}_k(x, r) \cap C_{<b} = \emptyset$ (where $\mathcal{B}_k(x, r)$ is the k-dimensional open ball of center x and radius r). Note that $\pi_1(x) = b$, since otherwise x belongs to $C_{<b}$, and thus to one of the B_i's.

Applying [?, Proposition 7.3], we deduce from the fact that $\mathcal{B}_k(x, r) \cap C_{<b} = \mathcal{B}_k(x, r)_{<b} \cap C = \emptyset$ that x is either a Q-singular point, or a Q-critical point of π_1 on V. In other words $x \in \mathcal{M}_1$. But since by assumption $\mathcal{M}_1$ is finite, this implies that $C \setminus \cup_{i=1}^r \overline{B_i}$ is a finite set. Since C is semi-algebraically connected and of positive dimension, $C \setminus \cup_{i=1}^r \overline{B_i}$ must be empty. □

Notation 3.8. If $S \subset \mathrm{R}^k$ is semi-algebraic set and $x \in S$, then we denote by $\mathcal{C}(S, x)$ the semi-algebraically connected component of S containing x.

We are now ready to prove Proposition 3.3.

Proof of Proposition 3.3. For a in R, we say that property $\mathbf{P}(a)$ holds if: for any semi-algebraically connected component C of $V_{\leq a}$, $C \cap \mathcal{S}$ is non-empty and semi-algebraically connected.

We prove that for all a in R, $\mathbf{P}(a)$ holds; taking $a \geq \max_{x \in V} \pi_1(x)$ suffices to prove the proposition since V is bounded.

Let $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 = \pi_1(\mathcal{M}_1 \cup \mathcal{M}_2)$ (see Property 3.2).

The proof uses two intermediate results:

Step 1: For every $a \in \mathcal{D}$, $\mathbf{P}(a)$ implies $\mathbf{P}(b)$ for all $b \in$ R with $(a, b] \cap \mathcal{D} = \emptyset$.

Step 2: For every $b \in \mathcal{D}$, if $\mathbf{P}(a)$ holds for all $a < b$, then $\mathbf{P}(b)$ holds.

Since for $a < \min_{x \in V} \pi_1(x)$, property $\mathbf{P}(a)$ holds vacuously, and the combination of these two results gives by an easy induction $\mathbf{P}(a)$ for all a in R, thereby proving the proposition.

We now prove the two steps.

Step 1. We suppose that $a \in \mathcal{D}$, $\mathbf{P}(a)$ holds, take $b \in$ R, $a < b$ with $(a, b] \cap \mathcal{D} = \emptyset$ and prove that $\mathbf{P}(b)$ holds. Let C be a semi-algebraically connected component of $V_{\leq b}$. We have to prove that $C \cap \mathcal{S}$ is semi-algebraically connected.

Since $(a, b] \cap \mathcal{D} = \emptyset$, it follows that $(\mathcal{M}_1)_{(a,b]} = \emptyset$, and $C_{\leq a}$ is a semi-algebraically connected component of $V_{\leq a}$ using Lemma 3.6. So, using property $\mathbf{P}(a)$, we see that $C_{\leq a} \cap \mathcal{S}$ is non-empty and semi-algebraically connected.

If $C_{\leq a} \cap \mathcal{S} = C \cap \mathcal{S}$, there is nothing to prove. Otherwise, let $x \in C \cap \mathcal{S}$ such that $x \notin C_{\leq a}$. We prove that x can be semi-algebraically connected to a point in $C_{\leq a} \cap \mathcal{S}$ by a semi-algebraic path in $C \cap \mathcal{S}$, which is enough to prove that $C \cap \mathcal{S}$ is semi-algebraically connected.

Since $\pi_1(x) \in (a, b]$ and $(a, b] \cap \mathcal{D} = \emptyset$, $\pi_1(x) \notin \mathcal{D}$ and $x \notin V_{\mathcal{N}_1 \cup \mathcal{N}_2}$. So, from $x \in \mathcal{S}$, we get $x \in W$. We note that $\mathcal{C}(W_{[a,b]}, x) \subset C$. By Property 3.2 (3) applied to $\mathcal{C}(W_{[a,b]}, x)$ we have that $a \in \pi_1(\mathcal{C}(W_{[a,b]}, x))$ and $\mathcal{C}(W_{[a,b]}, x)_a$ is non-empty. Hence there exists a semi-algebraic path connecting x to a point in $\mathcal{C}(W_{[a,b]}, x)_a$ inside $\mathcal{C}(W_{[a,b]}, x)$. Since $\mathcal{C}(W_{[a,b]}, x) \subset W \subset \mathcal{S}$ and $\mathcal{C}(W_{[a,b]}, x) \subset C$, if follows that $\mathcal{C}(W_{[a,b]}, x) \subset C \cap \mathcal{S}$ and we are done.

Step 2. We suppose that $b \in \mathcal{D}$, and $\mathbf{P}(a)$ holds for all $a < b$, and prove that $\mathbf{P}(b)$ holds.

Let C be a semi-algebraically connected component of $V_{\leq b}$. If $C_b = \emptyset$ there is nothing to prove. Suppose that C_b is non-empty; we have to prove that $C \cap \mathcal{S}$ is semi-algebraically connected.

If $\dim(C) = 0$, C is a point, belonging to $\mathcal{M}_1 \subset \mathcal{S}$ by Lemma 3.7. So $C \cap \mathcal{S}$ is semi-algebraically connected.

Hence, we can assume that $\dim(C) > 0$, so that $C_{<b}$ is non-empty by Lemma 3.7.

Our aim is to prove that $C \cap \mathcal{S}$ is semi-algebraically connected. We do this in two steps. We prove the following statements:

(a) If B is a semi-algebraically connected component of $C_{<b}$, then $\overline{B} \cap \mathcal{S}$ is non-empty and semi-algebraically connected, and

(b) and, using (a) $C \cap \mathcal{S}$ is semi-algebraically connected.

Proof of (a) We prove that if B is a semi-algebraically connected component of $V_{<b}$, then $\overline{B} \cap \mathcal{S}$ is non-empty and semi-algebraically connected.

Since $\overline{B}$ contains a point of $\mathcal{M}_1$ it follows that $\overline{B} \cap \mathcal{S}$ is not empty.

Note that if $\overline{B} \cap \mathcal{S} = B \cap \mathcal{S}$, then there exists a with

$$\max(\{\pi_1(x) \mid x \in B \cap \mathcal{S}\}) < a < b,$$

with $B \cap \mathcal{S} = (B \cap \mathcal{S})_{\leq a}$ and $B_{\leq a}$ semi-algebraically connected using Lemma 3.6. So $B \cap \mathcal{S}$ is semi-algebraically connected since $\mathbf{P}(a)$ holds.

We now suppose that $(\overline{B} \setminus B) \cap \mathcal{S}$ is non-empty. Taking $x \in (\overline{B} \setminus B) \cap \mathcal{S}$, we are going to show that x can be connected to a point z in $B \cap \mathcal{S}$ by a semi-algebraic path γ inside $\overline{B} \cap \mathcal{S}$. Notice that $\pi_1(x) = b$.

We first prove that we can assume without loss of generality that $x \in W$. Otherwise, since $x \in \mathcal{S}$ and $\mathcal{S} = W \cup V_{\mathcal{N}_1 \cup \mathcal{N}_2}$, we must have that $x \in V_y$ with $y = \pi_{[1,p]}(x)$, and $V_y \subset \mathcal{S}$. Let $A = \mathcal{C}(V_y \cap \overline{B}, x)$. We now prove that $A \cap W_y \neq \emptyset$. Using the curve section lemma choose a semi-algebraic path $\gamma : [0, \varepsilon] \to \mathrm{Ext}(\overline{B}, \mathrm{R}\langle\varepsilon\rangle)$ such that $\gamma(0) = x$, $\lim_\varepsilon \gamma(\varepsilon) = x$ and $\gamma((0, \varepsilon]) \subset \mathrm{Ext}(B, \mathrm{R}\langle\varepsilon\rangle)$. Let $y_\varepsilon = \pi_{[1,p]}(\gamma(\varepsilon))$ and

$$A_\varepsilon = \mathcal{C}(\mathrm{Ext}(B, \mathrm{R}\langle\varepsilon\rangle)_{y_\varepsilon}, \gamma(\varepsilon)).$$

Note that $x \in \lim_\varepsilon A_\varepsilon \subset A$.

By Remark 3.5, $\mathrm{Ext}(B, \mathrm{R}\langle\varepsilon\rangle)$ is a semi-algebraically connected component of $\mathrm{Ext}(V_{<a}, \mathrm{R}\langle\varepsilon\rangle)$ which implies that A_ε is a semi-algebraically connected component of $\mathrm{Ext}(V, \mathrm{R}\langle\varepsilon\rangle)_{y_\varepsilon}$. By Property 3.2 (2) and Remark 3.5, $\mathrm{Ext}(W, \mathrm{R}\langle\varepsilon\rangle)_{y_\varepsilon} \cap A_\varepsilon \neq \emptyset$. Then, since $\mathrm{Ext}(W, \mathrm{R}\langle\varepsilon\rangle)_{y_\varepsilon} \cap A_\varepsilon$ is bounded over R, $\lim_\varepsilon(\mathrm{Ext}(W, \mathrm{R}\langle\varepsilon\rangle)_{y_\varepsilon} \cap A_\varepsilon)$ is a non-empty subset of $W_y \cap A$.

Now connect x to a point in $x' \in W_y$ by a semi-algebraic path whose image is contained in $A \subset \overline{B}_y \subset (\overline{B} \setminus B) \cap \mathcal{S}$. Thus, replacing x by x' if necessary we can assume that $x \in W$ as announced.

There are four cases, namely

(1) $x \in \mathcal{M}_1 \cup \mathcal{M}_2$;
(2) $x \notin \mathcal{M}_1 \cup \mathcal{M}_2$ and $\mathcal{C}(W_b, x) \not\subset \overline{B}$;
(3) $x \notin \mathcal{M}_1 \cup \mathcal{M}_2$, $\mathcal{C}(W_b, x) \subset \overline{B}$ and $b \in \mathcal{D}_2$;
(4) $x \notin \mathcal{M}_1 \cup \mathcal{M}_2$, $\mathcal{C}(W_b, x) \subset \overline{B}$ and $b \notin \mathcal{D}_2$;

that we consider now.

(1) $x \in \mathcal{M}_1 \cup \mathcal{M}_2$:
Define $y = \pi_{[1,p]}(x) \in \mathrm{R}^p$, and note that $V_y \subset \mathcal{S}$. Since $x \in \overline{B}$, and B is bounded, $y \in \pi_{[1,p]}(\overline{B}) = \overline{\pi_{[1,p]}(B)}$. Now let $\varepsilon > 0$ be an infinitesimal. By applying the curve selection lemma to the set B and $x \in \overline{B}$, and then projecting to R^p using $\pi_{[1,p]}$ we obtain that there exists $y_\varepsilon \in \mathrm{R}\langle\varepsilon\rangle^p$ infinitesimally close to y with $\pi_1(y_\varepsilon) < \pi_1(y)$, and $x \in \lim_\varepsilon \mathrm{Ext}(V, \mathrm{R}\langle\varepsilon\rangle)_{y_\varepsilon}$. Let $x_\varepsilon \in \mathrm{Ext}(V, \mathrm{R}\langle\varepsilon\rangle)_{y_\varepsilon}$ be such that $\pi_{[1,k]}(\lim x_\varepsilon) = x$. Moreover, by Property 3.2 (2) and Remark 3.5 we have that $\mathrm{Ext}(W, \mathrm{R}\langle\varepsilon\rangle)_{y_\varepsilon}$ is non-empty and

meets every semi-algebraically component of $\mathrm{Ext}(V, \mathrm{R}\langle\varepsilon\rangle)_{y_\varepsilon}$. Let $x'_\varepsilon \in \mathrm{Ext}(W, \mathrm{R}\langle\varepsilon\rangle)_{y_\varepsilon} \cap \mathcal{C}(\mathrm{Ext}(B, \mathrm{R}\langle\varepsilon\rangle)_{y_\varepsilon}, x_\varepsilon)$, and $x' = \pi_{[1,k]}(\lim_\varepsilon x'_\varepsilon)$. Since $\pi_{[1,k]}(\lim x_\varepsilon) = x$ and $\lim_\varepsilon\ \mathcal{C}(\mathrm{Ext}(B, \mathrm{R}\langle\varepsilon\rangle)_{y_\varepsilon}, x_\varepsilon)$ is semi-algebraic-

ally connected,

$$\lim_\varepsilon \mathcal{C}(\mathrm{Ext}(B, \mathrm{R}\langle\varepsilon\rangle)_{y_\varepsilon}, x_\varepsilon) \subset \mathcal{C}(\overline{B}_y, x).$$

Now choose a semi-algebraic path γ_1 connecting x to x' inside $\mathcal{C}(\overline{B}_y, x)$ (and hence inside $\mathcal{S}$ since $\mathcal{C}(\overline{B}_y, x) \subset V_y \subset \mathcal{S}$), and a semi-algebraic path $\gamma_2(\varepsilon)$ joining x' to x_ε inside $\mathrm{Ext}(W, \mathrm{R}\langle\varepsilon\rangle)$. The concatenation of $\gamma_1, \gamma_2(\varepsilon)$ gives a semi-algebraic path γ having the required property, after replacing ε in $\gamma_2(\varepsilon)$ by a small enough positive element of R.

(2) $x \notin \mathcal{M}_1 \cup \mathcal{M}_2$ and $\mathcal{C}(W_b, x) \not\subset \overline{B}$:
There exists $x' \in \mathcal{C}(W_b, x)$, $x' \notin \overline{B}$ and a semi-algebraic path $\gamma : [0,1] \to \mathcal{C}(W_b, x)$, with $\gamma(0) = x, \gamma(1) = x'$. Since $x' \notin \overline{B}$, it follows from Lemma 3.7 2) that for $t_1 = \max\{0 \le t < 1 \mid \gamma(t) \in \overline{B}\}$, $\gamma(t_1) \in \mathcal{M}_1$. We can now connect x' to a point in $B \cap \mathcal{S}$ by a semi-algebraic path inside $\overline{B} \cap \mathcal{S}$ using (1).

(3) $x \notin \mathcal{M}_1 \cup \mathcal{M}_2$, $\mathcal{C}(W_b, x) \subset \overline{B}$ and $b \in \mathcal{D}_2$:
Since $b \in \mathcal{D}_2$ by Property 3.2 (2) there exists $x' \in \mathcal{C}(W_b, x) \cap \mathcal{M}_2$. Thus, there exists a semi-algebraic path connecting x to $x' \in \mathcal{M}_2$ with image contained in $\overline{B} \cap W \subset \overline{B} \cap \mathcal{S}$. We can now connect x' to a point in $B \cap \mathcal{S}$ by a semi-algebraic path inside $\overline{B} \cap \mathcal{S}$ using (1).

(4) $x \notin \mathcal{M}_1 \cup \mathcal{M}_2$, $\mathcal{C}(W_b, x) \subset \overline{B}$ and $b \notin \mathcal{D}_2$:
Since $b \notin \mathcal{D}_2$, for all $a < b$ such that $[a,b] \cap \mathcal{D}_2 = \emptyset$, $\mathcal{C}(W_{[a,b]}, x)_b = \mathcal{C}(W_b, x)$ and $\mathcal{C}(W_{[a,b]}, x)_a \neq \emptyset$ by Property 3.2 (3). Let $x' \in \mathcal{C}(W_{[a,b]}, x)_a$. We can choose a semi-algebraic path $\gamma : [0,1] \to \mathcal{C}(W_{[a,b]}, x)$ with $\gamma(0) = x, \gamma(1) = x'$. Let $t_1 = \max\{0 \le t < 1 \mid \gamma(t) \in W_b\}$. Then, either $\gamma(t_1) \in \mathcal{M}_1$ and we can connect $\gamma(t_1)$ to a point in $B \cap \mathcal{S}$ by a semi-algebraic path inside $\overline{B} \cap \mathcal{S}$ using (1). Otherwise, by Lemma 3.7 (2 b), for all small enough $r > 0$, $\mathcal{B}_k(\gamma(t_1), r) \cap C_{<b}$ is non-empty and contained in B. Then, there exists $t_2 \in (t_1, 1]$ such that $\gamma(t_2) \in B \cap W \subset B \cap \mathcal{S}$, and the semi-algebraic path $\gamma|_{[0,t_2]}$ gives us the required path in this case.

Taking x and x' in $\overline{B} \cap \mathcal{S}$, they can be connected to points z and z' in $B \cap \mathcal{S}$ by semi-algebraic path γ and γ' inside $\overline{B} \cap \mathcal{S}$ such that, without loss of generality, $\pi_1(z) = \pi_1(z')) = a$. Using $\mathbf{P}(a)$, we conclude that $\mathbf{P}(b)$ holds.

Proof of (b) We have to prove that $C \cap \mathcal{S}$ is semi-algebraically connected.

Let x and x' be in $C \cap \mathcal{S}$. We prove that it is possible to connect them by a semi-algebraic path inside $C \cap \mathcal{S}$.

Since we suppose that $\dim(C) > 0$, $C_{<b}$ is non-empty by Lemma 3.7 (2). Using Lemma 3.7 (2.c), let B_i (resp. B_j) be a semi-algebraically connected component of $C_{<b}$ such that $x \in \overline{B_i}$ (resp. $x' \in \overline{B_j}$).

If $i = j$, x and x' both lie in $\overline{B}_i \cap \mathcal{S}$ which is semi-algebraically connected by (a). Hence, they can be connected by a semi-algebraically connected path in $\overline{B}_i \cap \mathcal{S} \subset C \cap \mathcal{S}$.

So let us suppose that $i \neq j$. Note that:

- by Lemma 3.7 (2.a), $\overline{B_i} \cap \mathcal{M}_1$ and $\overline{B_j} \cap \mathcal{M}_1$ are not empty,
- by (a) $\overline{B_i} \cap \mathcal{S}$ and $\overline{B_j} \cap \mathcal{S}$ are semi-algebraically connected,
- by definition of $\mathcal{S}$, $\mathcal{M}_1 \subset \mathcal{S}$.

Then, one can connect x (resp. x') to a point in $\overline{B}_i \cap \mathcal{M}_1$ (resp. $\overline{B}_j \cap \mathcal{M}_1$). This shows that one can suppose without loss of generality that $x \in \overline{B}_i \cap \mathcal{M}_1$ and $x' \in \overline{B}_j \cap \mathcal{M}_1$.

Let $\gamma : [0, 1] \to C$ be a semi-algebraic path that connects x to x', and let $G = \gamma^{-1}(C \cap \mathcal{M}_1)$ and $H = [0, 1] \setminus G$.

Since $\mathcal{M}_1$ is finite, we can assume without loss of generality that G is a finite set of points, and H is a union of a finite number of open intervals.

Since $\gamma(G) \subset \mathcal{M}_1 \subset \mathcal{S}$, it suffices to prove that if t and t' are the end points of an interval in H, then $\gamma(t)$ and $\gamma(t')$ are connected by a semi-algebraic path inside $C \cap \mathcal{S}$.

Notice that $\gamma((t, t')) \cap \mathcal{M}_1 = \emptyset$, so that $\gamma(t)$ and $\gamma(t')$ belong to the same $\overline{B}_\ell$ by Lemma 3.7 2 b). Recall now that $\gamma(t)$ and $\gamma(t')$ both lie in $\overline{B}_\ell \cap \mathcal{S}$ and that $\overline{B}_\ell \cap \mathcal{S}$ is semi-algebraically connected by (a). Consequently, $\gamma(t)$ and $\gamma(t')$ can be connected by a semi-algebraic path in $\overline{B}_\ell \cap \mathcal{S} \subset C \cap \mathcal{S}$. □

We are going to need the following corollary.

Corollary 3.9. *Let*

$$(V, p, W, (\mathcal{M}_i)_{1 \le i \le 2}, (\mathcal{D}_i)_{1 \le i \le 2})$$

satisfy Property 3.2,

$$\mathcal{N}_1 = \pi_{[1,p]}(\mathcal{M}_1), \mathcal{N}_2 = \pi_{[1,p]}(\mathcal{M}_2),$$

and $\mathcal{N} \subset \mathrm{R}^p$ a finite set containing $\mathcal{N}_1 \cup \mathcal{N}_2$. For every semi-algebraically connected component C of V,

$$C \cap (W \cup V_{\mathcal{N}})$$

is non-empty and semi-algebraically connected.

Proof. Follows immediately from Proposition 3.3 and Property 3.2 b). □

4. Block representations and curve segments

We denote by D an ordered domain contained in a real closed field R and by **C** the algebraically closed field R$[i]$. All the polynomials in the input and output of our algorithms have coefficients in D and the complexity of our algorithms is measured by the number of arithmetic operations (addition, multiplication, sign determination) in D.

In this section, we first define certain representations of points, as well as semi-algebraic curves, that are going to be used in the inputs and outputs of our algorithms. Several of these representations share the common property that a certain initial number of coordinates are fixed by a triangular system of equations, along with certain Thom encodings and the remaining coordinates are defined by rational functions to be evaluated at a fixed real root of another polynomial (see Definitions 4.1 and 4.8 below). The structure of these representations reflect the recursive structure of our main algorithms described in Section 7.

After defining these representations we recall the input, output and complexity of a key algorithm, Algorithm 1(Curve Segments), which is described in full detail in [**?**]. Algorithm 1 accepts as input a polynomial defining a bounded real algebraic variety (with some coordinates fixed by a triangular system as mentioned above), and outputs a semi-algebraic partition of the first (non-fixed) coordinate, as well as descriptions of semi-algebraic curve segments (as well as points) parametrized by this coordinate satisfying certain properties – which are key to the construction of the main roadmap algorithm. Indeed, the curve segments appearing in the main roadmap algorithm would be limits of the curve segments output by the various calls to Algorithm 1.

We begin with a few definitions.

Definition 4.1. A *Thom encoding* f, σ *representing an element* $\alpha \in \mathrm{R}$ consists of

(1) a polynomial $f \in \mathrm{D}[T]$ such that α is a root of f in R,
(2) a sign condition σ on the set $\mathrm{Der}(f)$ of derivatives of f, such that σ is the sign condition satisfied by $\mathrm{Der}(f)$ at α.

Distinct roots of f in R correspond to distinct Thom encodings [**?**, Proposition 2.28].

A *real univariate representation* g, τ, G *representing* $x \in \mathrm{R}^k$ consists of

(1) a Thom encoding g, τ representing $\beta \in \mathrm{R}$,
(2) $G = (g_0, g_1, \ldots, g_k) \in \mathrm{D}[T]^{k+1}$ where g and g_0 are co-prime and such that
$$x = \left(\frac{g_1(\beta)}{g_0(\beta)}, \ldots, \frac{g_k(\beta)}{g_0(\beta)}\right) \in \mathrm{R}^k.$$

4.1. **Block representations.** In our algorithms, we make recursive calls, where we fix blocks of several coordinates. This makes necessary the following rather technical definitions.

Definition 4.2. A *triangular Thom encoding* $\mathcal{F} = (f_{[1]}, \ldots, f_{[m]}), \sigma$ *representing* $t = (t_1, \ldots, t_m)$ *in* R^m consists of

(1) a triangular system $\mathcal{F} = (f_{[1]}, \ldots, f_{[m]})$, i.e. $f_{[i]} \in \mathrm{D}[T_1, \ldots, T_i]$ for $i = 1, \ldots, m$, such that the zero set of $\mathcal{F}$ in $\mathbf{C}^r$ is finite;
(2) a list, $\sigma = (\sigma_1, \ldots, \sigma_m)$, where for $i = 1, \ldots, m$, σ_i is the Thom encoding of the root t_i of $f_{[i]}(t_1, \ldots, t_{i-1}, T_i)$.

A triangular Thom encoding is *quasi-monic* if the leading coefficient of $f_{[i]} \in \mathrm{D}[T_1, \ldots, T_i]$ with respect to T_i is a strictly positive element in D.

Let P, Q be two polynomials in $\mathrm{D}[T]$ with D a domain, the *pseudo-remainder* of P by Q with respect to T is defined as

$$\mathrm{PsRem}_T(P,Q) = \mathrm{Rem}_T(b_q^{p-q-1}P, Q)$$

where $q = \deg_T(Q)$ and the leading coefficient of Q with respect to T is b_q. Note that, with

$$P = CQ + \mathrm{PsRem}_T(P,Q)$$

both C and $\mathrm{PsRem}_T(P,Q)$ have coefficients in D.

The *pseudo-reduction modulo* $\mathcal{F}$ associated to $f \in \mathrm{D}[T_1,\ldots,T_m]$ is defined as

$$\mathrm{PsRed}(f,\mathcal{F}) = \mathrm{PsRem}_{T_1}(\ldots(\mathrm{PsRem}_{T_m}(f, g_{[m]}),\ldots), g_{[1]}). \tag{4.1}$$

The pseudo-reduction involves only coefficients in the domain D since the triangular Thom encoding is quasi-monic and pseudo-division is used. Note that at the zeros of $\mathcal{F}$ the signs of f and $\mathrm{PsRed}(f,\mathcal{F})$ coincide.

Remark 4.3. If $f \in \mathrm{D}[T_1,\ldots,T_m]$ is a polynomial of degree D, and d is a bound on the degree of the f_i with respect to each T_i, the complexity of computing $\mathrm{PsRed}(f,\mathcal{F})$ is $(Dd)^{O(m)}$ (see Section 8 Proposition 8.4)).

Definition 4.4. A *real block representation* $\mathcal{F},\sigma,L,F$ *representing* $y \in \mathrm{R}^\ell$ consists of

(1) a triangular Thom encoding $\mathcal{F} = (f_{[1]},\ldots,f_{[m]}),\sigma$ representing a root $t = (t_1,\ldots,t_m)$ of $\mathcal{F}$ in R^m;

(2) a list of natural numbers $L = (\ell_1,\ldots,\ell_m)$ such that
$$\ell = \ell_1 + \cdots + \ell_m;$$

(3) a list of polynomials $F = (F_{[1]},\ldots,F_{[m]})$, where
$$F_{[i]} = (f_{[i]0},\ldots,f_{[i]\ell_i}), f_{[i]j} \in \mathrm{D}[T_1,\ldots,T_i], 0 \le j \le \ell_i,$$
with $f_{[i]}(t_1,\ldots,t_{i-1},T_i), f_{[i]0}(t_1,\ldots,t_{i-1},T_i)$ coprime (as polynomials in T_i), such that
$$y = (y_{[1]},\ldots,y_{[m]}) \in \mathrm{R}^\ell,$$
with
$$y_{[i]} = \left(\frac{f_{[i]1}(t_1,\ldots,t_i)}{f_{[i]0}(t_1,\ldots,t_i)},\ldots,\frac{f_{[i]\ell_i}(t_1,\ldots,t_i)}{f_{[i]0}(t_1,\ldots,t_i)}\right), 1 \le i \le m.$$

In case $\ell_1 = \cdots = \ell_m = p$, then we will write

$$L = [p^m]. \tag{4.2}$$

Notation 4.5 (Substituting a real block representation in a polynomial)**.** Let $\mathcal{F},\sigma,L,F$ be a real block representation representing $y \in \mathrm{R}^\ell$, and $t \in \mathrm{R}^m$ represented by $\mathcal{F},\sigma$.

Let

$$f_{[i]}(T_1,\ldots,T_i) = \left(\frac{f_{[i]1}(T_1,\ldots,T_i)}{f_{[i]0}(T_1,\ldots,T_i)},\ldots,\frac{f_{[i]\ell_i}(T_1,\ldots,T_i)}{f_{[i]0}(T_1,\ldots,T_i)}\right).$$

Given $Q \in \mathrm{D}[X_1,\ldots,X_k]$ with $\ell \le k$, we define $T=(T_1,\ldots,T_m)$, $Q_F \in \mathrm{D}[T,X_{\ell+1},\ldots,X_k]$ by

$$Q_F := f_0(T)Q\left(f_{[1]}(T_1),\ldots,f_{[m]}(T_1,\ldots,T_m),X_{\ell+1},\ldots,X_k\right), \tag{4.3}$$

where

$$f_0(T)=\prod_{i=1}^{m} f_{[i]0}(T_1,\ldots,T_i)^{e_i},$$

and e_i is the smallest even number $\ge \deg_{X_{[i]}}(Q)$, where $X_{[i]}$ is the block of variables $X_{\ell_1+\cdots+\ell_{i-1}+1},\ldots,X_{\ell_1+\cdots+\ell_i}$.

Note that

$$Q_F(t,X_{\ell+1},\ldots,X_k)=f_0(t)Q(y,X_{\ell+1},\ldots,X_k),$$

with $f_0(t)>0$.

Notation 4.6 (Substituting a real block representation in a parametrized univariate representation)**.** Let $\mathcal{F},\sigma,L,F$ be as above and let

$$g,G=(g_0,g_{\ell+1},\ldots,g_k),$$

be a parametrized univariate representation with $g,g_i \in \mathrm{D}[X_1,\ldots,X_\ell,U]$, where $X_1,\ldots,X_\ell$ are the parameters.

We denote by G_F the tuple $(g_{0,F},\ldots,g_{k,F})$, where each $g_{i,F} \in \mathrm{D}[T,U]$ and is defined by

$$g_{i,F} := f_0(T)g_i\left(f_{[1]}(T_1),\ldots,f_{[m]}(T_1,\ldots,T_m),X_{\ell+1},\ldots,X_k\right), \tag{4.4}$$

where

$$f_0(T)=\prod_{i=1}^{m} f_{[i]0}(T_1,\ldots,T_i)^{e_i},$$

and e_i is the smallest number $\ge \max_j \deg_{X_{[i]}}(g_j)$.

Definition 4.7. Let $t \in \mathrm{R}^m$ be represented by a triangular Thom encoding $\mathcal{F},\sigma$.

A *Thom encoding* g,τ *representing* β *over* t consists of (using the same notation as above)

1. a polynomial $g \in \mathrm{D}[T_1,\ldots,T_m,T]$ such that $g(t,\beta)=0$,
2. a sign condition τ on $\mathrm{Der}_T(g)$ such that τ is the sign condition satisfied by the set $\mathrm{Der}_T(g(t,T))$ at β.

A *real univariate representation representing* $x \in \mathrm{R}^k$ *over* t, consists of

1. a Thom encoding g,τ representing β over t,
2. $G=(g_0,g_1,\ldots,g_k) \in \mathrm{D}[T_1,\ldots,T_m,U]^{k+1}$ such that $g(t,U),g_0(t,U)$ are coprime, and such that

$$x=\left(\frac{g_1(t,\beta)}{g_0(t,\beta)},\ldots,\frac{g_k(t,\beta)}{g_0(t,\beta)}\right) \in \mathrm{R}^k.$$

A real univariate representation over t is *quasi-monic* if the leading monomial of g with respect to U is in D.

A *triangular Thom encoding representing* $z = (z_1, \ldots, z_r)$ *over* t *with variables* $Y_1, \ldots, Y_r$ consists of

(1) a triangular system $\mathcal{H} = (h_1, \ldots, h_r)$, with
$$h_i \in \mathrm{D}[T_1, \ldots, T_m, Y_1, \ldots, Y_i]$$
for $i = 1, \ldots, r$, such that the zero set of $\mathcal{H}(y, Z_1, \ldots, Z_r)$ in $\mathbf{C}^r$ is finite;

(2) a list, $\rho = (\rho_1, \ldots, \rho_r)$, where for $i = 1, \ldots, r$, ρ_i is the Thom encoding of the root z_i of $h_i(t, z_1, \ldots, z_{i-1}, Y_i)$.

4.2. **Curve segments.**

Definition 4.8. Let $t \in \mathrm{R}^m$ be represented by a triangular Thom encoding $\mathcal{F}, \sigma$. A *curve segment with parameter* X_j *over* t *on* (α_1, α_2) *in* R^k,
$$f_1, \sigma_1, f_2, \sigma_2, g, \tau, G$$
is given by

(1) $\alpha_1, \alpha_2 \in \mathrm{R}$ represented by Thom encodings f_1, σ_1 and f_2, σ_2 over t;

(2) a parametrized univariate representation with parameter X_j, i.e.
$$g, G = (g_0, g_1, \ldots, g_k),$$
with $g_j = X_j g_0$ and $g, g_0, \ldots, g_k$ in $\mathrm{D}[T_1, \ldots, T_m, X_j, U]$;

(3) a sign condition τ on $\mathrm{Der}_U(g)$ such that for every $x_j \in (\alpha_1, \alpha_2)$ there exists a real root $u(x_j)$ of $g(t, x_j, U)$ with Thom encoding τ, and $g_0(t, x_j, u(x_j)) \neq 0$.

The *curve represented by* $f_1, \sigma_1, f_2, \sigma_2, g, \tau, G$ is the image of the smooth injective semi-algebraic function γ which maps a point x_j of (α_1, α_2) to the point of R^k defined by
$$\gamma(x_j) = \left(\frac{g_1(t, x_j, u(x_j))}{g_0(t, x_j, u(x_j))}, \ldots, \frac{g_k(t, x_j, u(x_j))}{g_0(t, x_j, u(x_j))} \right).$$

Let $Q \in \mathrm{R}[X_1, \ldots, X_k]$. For $0 \leq \ell < k$ and $y \in \mathrm{R}^\ell$, we denote
$$Q(y, -) \stackrel{\text{def}}{=} Q(y_1, \ldots, y_\ell, X_{\ell+1}, \ldots, X_k). \tag{4.5}$$

Remark 4.9. Abusing notation slightly, we will occasionally identify
$$\mathrm{Zer}(Q(y, -), \mathrm{R}^{k-\ell}) \subset \mathrm{R}^{k-\ell}$$
with
$$\{y\} \times \mathrm{Zer}(Q(y, -), \mathrm{R}^{k-\ell}) = \mathrm{Zer}(Q, \mathrm{R}^k)_y \subset \mathrm{R}^k,$$
and more generally, for a semi-algebraic set $A \subset \mathrm{R}^k$, $A_y \subset \mathrm{R}^k$ with $\{x \in \mathrm{R}^{k-\ell} \mid (y, x) \in A_y\}$.

We now recall the input, output and complexity of [**?**, Algorithm 15.2 (Curve segments)].

Algorithm 1. **[Curve Segments]**

Input. (1) a point $t \in \mathrm{R}^m$ represented by a triangular Thom encoding $\mathcal{F}, \sigma$;
(2) a polynomial $P \in \mathrm{D}[T_1 \ldots, T_m, X_1, \ldots, X_k]$ for which $\mathrm{Zer}(P(t,-), \mathrm{R}^k)$ is bounded;
(3) a finite set of points contained in $\mathrm{Zer}(P(t,-), \mathrm{R}^k)$ represented by real univariate representations $\mathcal{U}$ over t.

Moreover, all the polynomials describing the input are with coefficients in D.

Output. (1) An ordered list of points $c_1 < \ldots < c_N$ of R with c_i, $i = 1, \ldots, N$ represented by a Thom encoding g_i, τ_i over t. The c_i's are called *distinguished values*.
(2) For every $i = 1, \ldots, N$, a finite set of real univariate representations $\mathcal{D}_i$ with parameter X_1 over t representing a finite number of points, called *distinguished points*.
(3) For every $i = 1, \ldots, N-1$ a finite set of curve segments $\mathcal{C}_i$ defined on (c_i, c_{i+1}) with parameter X_1, over t. The represented curves are called *distinguished curves*.
(4) For every $i = 1, \ldots, N-1$ a list of pairs of elements of $\mathcal{C}_i$ and $\mathcal{D}_i$ (resp. $\mathcal{D}_{i+1}$) describing the adjacency relations between distinguished curves and distinguished points.

The distinguished curves and points are contained in $\mathrm{Zer}(P(t,-), \mathrm{R}^k)$. The sets of distinguished values, distinguished curves, and distinguished points satisfy the following properties.

CS_1. If $c \in \mathrm{R}$ is a distinguished value, the set of distinguished points output intersect every semi-algebraically connected component of $\mathrm{Zer}(P(t,c,-), \mathrm{R}^{k-1})$.
If $c \in \mathrm{R}$ is not distinguished, the set of distinguished curves output intersect every semi-algebraically connected component of $\mathrm{Zer}(P(t,c,-), \mathrm{R}^k)$.

CS_2. For each distinguished curve output over an interval with endpoint a given distinguished value, there exists a distinguished point over this distinguished value which belongs to the closure of the curve.

Complexity. If $d = \deg_X(P) \geq 2$, $\deg_T(P) = D$, and the degree of the polynomials in $\mathcal{F}$ and the number of elements of $\mathcal{U}$ are bounded by D, the number of arithmetic operations in D is bounded by $D^{O(m)} d^{O(mk)}$. Moreover, the degree in T_i of the polynomials appearing in the output is bounded by $Dd^{O(k)}$.

5. Low dimensional roadmap in a special case

In this section we describe an algorithm for computing the roadmap of a variety described by equations having a special structure. Although, this algorithm is very similar to [**?**, Algorithm 15.3 (Bounded Algebraic

Roadmap)], the complexity analysis differs because of the special structure assumed for the input.

Let $Q \in \mathrm{R}[X_1, \ldots, X_k]$ and suppose that $V = \mathrm{Zer}(Q, \mathrm{R}^k)$ is bounded.

For $0 \leq \ell < k$, $0 \leq p \leq k - \ell$, and $y \in \mathrm{R}^\ell$, we denote

$$\mathrm{Cr}_p(Q(y,-)) \stackrel{\text{def}}{=} \left(Q(y,-), \frac{\partial Q(y,-)}{\partial X_{\ell+p+2}}, \ldots, \frac{\partial Q(y,-)}{\partial X_k}\right). \tag{5.1}$$

We assume that Q satisfies the following property.

Property 5.1. For every $\ell, 0 \leq \ell < k$, $0 \leq p \leq k - \ell$, and $y \in \mathrm{R}^\ell$, the algebraic set

$$W_y^p = \mathrm{Zer}(\mathrm{Cr}_p(Q(y,-)), \mathrm{R}^{k-\ell})$$

is of dimension p.

Remark 5.2. Note that for every $y \in \mathrm{R}^\ell$, $z \in \mathrm{R}^r$, $(W_y^r)_z = W^0_{(y,z)}$ has a finite number of points and intersects every semi-algebraically connected component of $V_{(y,z)}$ by [**?**, Proposition 7.4].

Now suppose that Q satisfies Property 5.1. For every $p, 1 \leq p \leq k$, and $y \in \mathrm{R}^\ell$, with $\ell + p < k$ we are going to define $(\mathcal{M}_{y,i})_{1 \leq i \leq 2}, (\mathcal{D}_{y,i})_{1 \leq i \leq 2}$ so that the tuple

$$\left(V_y, p, W_y^p, (\mathcal{M}_{y,i})_{1 \leq i \leq 2}, (\mathcal{D}_{y,i})_{1 \leq i \leq 2}\right)$$

satisfies Property 3.2 using a slight abuse of notation (cf. Remark 4.9).

Definition 5.3. Let

1. (1) $\mathcal{M}_{y,1} = W_y^0 \subset V_y$, and $\mathcal{D}_{y,1} = \pi_{\ell+1}(W_y^0)$;
2. (2) $\mathcal{D}_{y,2} \subset \mathrm{R}$ the set pseudo-critical values (see [**?**, Definition 12.41]) of $\pi_{\ell+1}$ on $\pi_{[\ell+1,k]}(W_y^p)$ and $\mathcal{M}_{y,2}$ a set of points such that for every $c \in \mathcal{D}_2$, $\mathcal{M}_2$ intersects every semi-algebraically connected component D of $(W_y^p)_c$.

Note that by Property 5.1, W_y^p is of dimension p (in particular, W_y^0 is finite), and satisfies Property 3.2 2): for each $z \in \mathrm{R}^p$, $(W_y^p)_z = W^0_{(y,z)}$ is a finite set of points having non-empty intersection with every semi-algebraically connected component of $V_{(y,z)}$ by Remark 5.2. Moreover, $\mathrm{Zer}(Q(y,-), \mathrm{R}^{k-\ell})$ is clearly bounded (since $\mathrm{Zer}(Q, \mathrm{R}^k)$ is bounded), and the finite set $\mathcal{M}_{y,1} = W_y^0$ is the union of the $Q(y,-)$-singular points of $\mathrm{Zer}(Q(y,-), \mathrm{R}^{k-\ell})$ and the $Q(y,-)$-critical points of the map $\pi_{\ell+1}$ on $V_y = \mathrm{Zer}(Q(y,-), \mathrm{R}^{k-\ell})$. Thus, $\mathcal{M}_{y,1}$ and $\mathcal{D}_{y,1}$ satisfies Property 3.2 1).

Note also that $\mathcal{M}_{y,2}$ satisfies Property 3.2 3). Indeed, the intersection of $\mathcal{M}_{y,2}$ with every semi-algebraically connected component of $(W_y^p)_c$ is non-empty, by [**?**, Proposition 12.42]. Moreover for every interval $[a,b]$ and $c \in [a,b]$ such that $[a,b]$ contains no point of $\mathcal{D}_{y,2}$, except maybe c, and for every semi-algebraically connected component D of $W_{\{y\} \times [a,b]}$, $D_{(y,c)}$ is a semi-algebraically connected component of $(W_y^p)_c$, by [**?**, Proposition 15.4].

So we have proved

Proposition 5.4. *If Q satisfies Property 5.1, using the notation above,*

$$\big(V_y, p, W_y^p, (\mathcal{M}_{y,i})_{1\le i\le 2}, (\mathcal{D}_{y,i})_{1\le i\le 2}\big)$$

satisfies Property 3.2.

We are going to describe below, in the special case where Q satisfies Property 5.1, an algorithm directly adapted from [**?**, Algorithm 15.3] for computing a roadmap of certain subvarieties of $\mathrm{Zer}(Q, \mathrm{R}^k)$ of dimension at most p : this is Algorithm 2 (Roadmap for Lower Dimensional Special Algebraic Sets)

Remark 5.5. In all our algorithms, the roadmaps output are represented by a finite number of real univariate representations and curve segments over a point defined by a triangular Thom encoding (see Definitions 4.4, 4.7, and 4.8 above).

ALGORITHM 2. **[Roadmap for Lower Dimensional Special Algebraic Sets]**

INPUT.
1. (1) a polynomial $Q \in \mathrm{D}[X_1, \ldots, X_k]$ satisfying Property 5.1, and for which $V = \mathrm{Zer}(Q, \mathrm{R}^k) \subset \mathcal{B}_k(0, 1/c)$ (where $c \in \mathrm{R}$);
2. (2) numbers $p, m, r \ge 0$ satisfying $mp \le k, 0 \le r < p$;
3. (3) $y \in \mathrm{R}^{mp}$ represented by a real block representation $\mathcal{F}, \sigma, [p^m], F$ (see (4.2)) with $t \in \mathrm{R}^m$ represented by a quasi-monic triangular system $\mathcal{F}, \sigma$;
4. (4) $z \in \mathrm{R}^r$ represented by a triangular Thom encoding $\mathcal{H}, \rho$ over t, with variables $X_{mp+1}, \ldots, X_{mp+r}$;
5. (5) a finite set of points $\mathcal{M}_0$ contained in $W_{(y,z)}^{p-r}$ represented by real univariate representations $\mathcal{U}_0$, over (t, z) (using the notation of Property 5.1).

OUTPUT. a roadmap $\mathrm{RM}(W_{(y,z)}^{p-r}, \mathcal{M}_0)$ for $(W_{(y,z)}^{p-r}, \mathcal{M}_0)$ represented as a union of of curve segment representations and real univariate representations over points defined by triangular Thom encodings. The adjacencies between the images of the associated curves and points are also output.

COMPLEXITY. $D^{O(m+p)} d^{O((m+p)k}$ where $d = \deg(Q) \ge 2$, and D is a bound on the degree of $\mathcal{H}, \mathcal{F}, F$ and the number and degrees of the elements in $\mathcal{U}_0$.

PROCEDURE.

Step 0. Define

$$P := \sum_{A \in \mathrm{Cr}_p(Q_F)} A^2 \in \mathrm{D}[T_1, \ldots, T_m, X_{mp+1}, \ldots, X_k]$$

using Notation 4.5 and (5.1), and initialize $i := 1$.

Step 1. If $r - 1 + i < p$, call Algorithm 1 (Curve Segments) with input

$$(\mathcal{F}, \mathcal{H}), (\sigma, \rho), P, \mathcal{U}_0.$$

Pseudo-reduce modulo $\mathcal{F}$ using (4.1) and place the output in the description of $\mathrm{RM}(W_{(y,z)}^{p-r}, \mathcal{M}_0)$.

Step 2. Set $i := i + 1$. Using the notation in the output of Algorithm 1, for every $j = 1, \dots, N$, define

$$\begin{aligned} z &:= (z, c_j), \\ \mathcal{H} &:= (\mathcal{H}, g_j(T_1, \dots, T_m, X_1, \dots, X_{r+i})), \\ \rho &:= (\rho, \tau_j), \\ \mathcal{U}_0 &:= \mathcal{D}_j, \end{aligned}$$

and call Step 1 of Algorithm 2 (Roadmap for Lower Dimensional Special Algebraic Sets) recursively, with input

$$(\mathcal{F}, \mathcal{H}), P, (\sigma, \rho), \mathcal{U}_0.$$

Proof of correctness. Notice that

$$W^{p-r}_{(y,z)} = \mathrm{Zer}(P(t, z, -)), \mathrm{R}^{k-(mp+r)}).$$

The correctness of the algorithm then follows from the correctness of Algorithm 1 (Curve Segments) and Proposition 2.2. The only additional fact that needs to be checked is that when the recursion ends with $r = p$, the algebraic variety $\mathrm{Zer}(P((t, z, z'), -), \mathrm{R}^{k-p(m+1)})$ is zero-dimensional, where $z' = (c_{j_1}, \dots, c_{j_{p-r}}) \in \mathrm{R}^{p-r}$ and the various $c_{j_i} \in \mathrm{R}$ are associated to the Thom encodings computed in Step 1 of the algorithm. This is the case because $\mathrm{Zer}(P((t, z, z'), -), \mathrm{R}^{k-p(m+1)}) = W^0_{(y,z,z')}$, and $W^0_{(y,z,z')}$ is zero-dimensional by Property 5.1. □

Complexity analysis. The depth i of the recursion is bounded by $p - r$, and the total number of recursive calls at depth i is bounded by $d^{O(ik)}$. Thus, there are at most $d^{O((p-r)k)}$ calls to Algorithm 1 (Curve Segments).

In each of the calls to Algorithm 1 (Curve Segments), the number of arithmetic operations in D is bounded by $D^{O(m+p)} d^{O((m+p)k)}$ using the complexity analysis of Algorithm 1 (Curve Segments). Moreover the number of arithmetic operations needed for each pseudo-reduction is $(Dd^k)^{O(m)}$ since the degree in T_i of the output of Algorithm 1 (Curve Segments) is $Dd^{O(k)}$ using Remark 4.3.

Thus, the total number of arithmetic operations in D for Algorithm 2 is bounded by $D^{O(m+p)} d^{O((m+p)k)}$. □

6. Low dimensional roadmap in general

In this section, we first explain how to perform an infinitesimal deformation of any given polynomial $Q \in \mathrm{R}[X_1, \dots, X_k]$ such that the deformed polynomial satisfies Property 5.1.

We then sketch how to compute the limit of a curve, and finally how to compute the limits of roadmaps of certain algebraic sets which are the critical locus of dimension p of certain projection maps restricted to the algebraic hypersurfaces obtained after performing an infinitesimal deformation.

6.1. **Deformation.** We consider a bounded algebraic set defined by a non-negative polynomial Q. Our aim is to define a deformation of Q defining a polynomial satisfying Property 5.1.

Suppose that the polynomial $Q \in \mathrm{R}[X_1, \ldots, X_k]$, and the tuple $(d_1, \ldots, d_k)$ satisfy the following conditions:

(1) $Q(x) \geq 0$ for every $x \in \mathrm{R}^k$,
(2) $\mathrm{Zer}(Q, \mathrm{R}^k)$ is bounded,
(3) $d_1 \geq d_2 \cdots \geq d_k$,
(4) $\deg(Q) \leq d_1$, $\mathrm{tDeg}_{X_i}(Q) \leq d_i$, for $i = 2, \ldots, k$.

Let $\bar{d}_i$ be an even number $> d_i, i = 1, \ldots, k$, and $\bar{d} = (\bar{d}_1, \ldots, \bar{d}_k)$. Let

$$G_k(\bar{d}) = X_1^{\bar{d}_1} + \cdots + X_k^{\bar{d}_k} + X_2^2 + \cdots + X_k^2 + X_{k+1}^2 + 2k,$$

and note that $\forall\ x\ G_k(\bar{d})(x) > 0$.

We denote

Notation 6.1.

$$\mathrm{Def}(Q, \varepsilon) = -\varepsilon G_k(\bar{d}) + Q, \tag{6.1}$$

$$V_{y,\varepsilon} = \mathrm{Zer}(\mathrm{Def}(Q, \varepsilon)(y, -), \mathrm{R}\langle\varepsilon\rangle^{k+1}) \tag{6.2}$$

$$\mathrm{Cr}_\ell(Q, \varepsilon) = \left(\mathrm{Def}(Q, \varepsilon), \frac{\partial \mathrm{Def}(Q, \varepsilon)}{\partial X_{p+\ell+1}}, \ldots, \frac{\partial \mathrm{Def}(Q, \varepsilon)}{\partial X_k}.\right) \tag{6.3}$$

$$W^p_{y,\varepsilon} = \mathrm{Zer}(\mathrm{Cr}_p(Q, \varepsilon)(y, -), \mathrm{R}\langle\varepsilon\rangle^{k+1}) \tag{6.4}$$

Proposition 6.2. *For every $\ell, 0 \leq \ell \leq k$, and every $y \in \mathrm{R}^\ell$,*

a) $\mathrm{Def}(Q, \varepsilon)(y, -)$ *satisfies Property 5.1;*
b) $\lim_\varepsilon$ *induces a 1-1 correspondence between the bounded semi-algebraically connected components of*

$$V_{y,\varepsilon} = \mathrm{Zer}(\mathrm{Def}(Q, \varepsilon)(y, -), \mathrm{R}\langle\varepsilon\rangle^{k+1})$$

and the semi-algebraically connected components of

$$Z_y = \mathrm{Zer}(Q(y, -), \mathrm{R}^k)$$

Proof. a) follows from [**?**, Proposition 12.44] and b) from [**?**, Lemma 15.6]. □

We are going to describe in Section 6.3 an algorithm for computing the limit, under the $\lim_\varepsilon$ map, of a roadmap of the critical locus of dimension p, $W^p_{y,\varepsilon}$, of $V_{y,\varepsilon} = \mathrm{Zer}(\mathrm{Def}(Q, \varepsilon)(y, -), \mathrm{R}\langle\varepsilon\rangle^{k+1-\ell})$. In order to achieve this we first need to compute limits of curves, which is the purpose of Section 6.2.

6.2. **Limits of points and curve segments.** The general problem of computing the image of a semi-algebraic set $S \subset \mathrm{R}\langle\varepsilon\rangle^k$ which is bounded over R under the $\lim_\varepsilon$ map reduces to the problem of computing the closure of a one-parameter family of semi-algebraic sets, which can be done using quantifier elimination algorithms (see, for example, [**?**, pg. 556]). However, the complexity of this general algorithm, $d^{k^{O(1)}}$, is not good enough for our purposes in this paper. Fortunately, we need efficient algorithms for computing limits only in two very special situations, where we can do better than in the general case.

These two special cases are the following :

(1) when the set is a point represented by a real univariate representation,
(2) when the set is a curve represented by curve segments.

We give now the input output and complexity of Algorithm 3 (Limit of a Bounded Point) and Algorithm 4 (Limit of a Curve). A full description of these algorithms, their correctness and complexity analysis appear in Section 8.

Algorithm 3. **[Limit of a Bounded Point]**

Input. (1) a quasi-monic triangular Thom encoding $\mathcal{F}, \sigma$, with coefficients in D, representing a point $t \in \mathrm{R}^m$;
(2) a real univariate representation $g_\varepsilon, \tau_\varepsilon, G_\varepsilon$ over t with coefficients in $\mathrm{D}[\varepsilon]$, representing a point $z_\varepsilon \in \mathrm{R}\langle\varepsilon\rangle^p$ bounded over R.

Output. (1) a quasi-monic triangular Thom encoding $\mathcal{F}', \sigma'$, representing the point $t \in \mathrm{R}^m$;
(2) a quasi-monic real univariate representation (h, H) representing

$$z = \lim_\varepsilon z_\varepsilon \in \mathrm{R}^p.$$

Complexity. If D_1 (resp. D_2) is a bound on the degrees of the polynomials in $\mathcal{F}, g_\varepsilon$ and G_ε with respect to $T_1, \ldots, T_m$ (resp. ε, U), then D_1 (resp. D_2) is a bound on the degrees of the polynomials appearing in the output, and the number of arithmetic operations in D is bounded by $D_1^{O(m)} D_2^{O(1)}$.

Remark 6.3. Note that there is a possibly new representation of t in the output of Algorithm 3 (Limit of a Bounded Point). The reason for this peculiarity is explained in Section 8 (cf Proposition 8.4).

Algorithm 4. **[Limit of a Curve]**

Input. (1) a quasi-monic triangular Thom encoding $\mathcal{F}, \sigma$ with coefficients in D representing $t \in \mathrm{R}^m$;
(2) a triangular Thom encoding $\mathcal{H}_\varepsilon, \rho_\varepsilon$ over t with coefficients in $\mathrm{D}[\varepsilon]$ representing $z_\varepsilon \in \mathrm{R}\langle\varepsilon\rangle^r$ over t;
(3) a curve segment with parameter X_{r+1} and coefficients in $\mathrm{D}[\varepsilon]$ over (t, z_ε), representing a curve bounded over R.

Output. (1) a real univariate representation p_z, ρ_z, P_z of $z = \lim_\varepsilon(z_\varepsilon)$, with u the root of p_z with Thom encoding ρ_z;
(2) a finite set $\{d_1, \ldots, d_{N-1}\}$ where each d_i is a real univariate representation over (t, u, c_i), and c_i is given by a Thom encoding over t fixing $X_{m(i)}$;

(3) a finite set $\{w_1, \dots, w_N\}$, of curve segments over (t, u) with w_i parametrized by $X_{\ell(i)}$.
Moreover, the union of the curves represented by $\mathcal{W}$, and the points represented by $\mathcal{D}$ define a partition of $S = \lim_\varepsilon(S_\varepsilon)$. All the coefficients of the polynomials in the output belong to D.

Complexity If the polynomials occurring in the input have degrees bounded by D, then the complexity of the algorithm is bounded by $k^{O(1)} D^{O(m+r)}$.

6.3. **Low dimensional roadmap algorithm.** We are going to describe an algorithm computing the limit of a roadmap of the critical locus of dimension p, $W^p_{y,\varepsilon}$, of the deformation $V_{y,\varepsilon} = \mathrm{Zer}(\mathrm{Def}(Q,\varepsilon)(y,-), \mathrm{R}\langle\varepsilon\rangle^{k+1-\ell})$ of $Z_y = \mathrm{Zer}(Q(y,-), \mathrm{R}^{k-\ell})$. The algorithm proceeds by first calling Algorithm 2 (Roadmap for Lower Dimensional Special Algebraic Sets) in order to compute a roadmap for $W^p_{y,\varepsilon}$, and then computes the image of the resulting roadmap under the $\lim_\varepsilon$ map. Note that this limit is not necessarily a roadmap of V_y, since a semi-algebraically connected component of V_y might contain the image under $\lim_\varepsilon$ of more than one semi-algebraically connected components of $W^p_{y,\varepsilon}$.

Algorithm 5. **[Limit of Roadmaps of Special Low Dimensional Varieties]**

Input. (1) a natural number $p \le k$;
(2) a polynomial $Q \in \mathrm{D}[X_1, \dots, X_k]$ for which $Z = \mathrm{Zer}(Q, \mathrm{R}^k) \subset \mathcal{B}_k(0, 1/c)$ (with $c \in \mathrm{R}$);
(3) $y \in \mathrm{R}^{mp}$ represented by the real block representation

$$\mathcal{F}, \sigma, [p^m], F,$$

(see (4.2)) with coefficients in D, such that $t \in \mathrm{R}^m$ is represented by a quasi-monic triangular Thom encoding $\mathcal{F}, \sigma$;
(4) a finite set of points $\mathcal{N}_{y,\varepsilon} \in \{y\} \times \mathrm{R}\langle\varepsilon\rangle^p$ represented by quasi-monic real univariate representations $\mathcal{V}_\varepsilon$, over t.

Output. Real univariate representations and curve segment representing the set of points

$$\mathcal{R} = (\pi_{[1,k]} \circ \lim_\varepsilon)(\mathrm{RM}(W^p_{y,\varepsilon}, W_{\mathcal{N}_{y,\varepsilon}}))$$

where $W^p_{y,\varepsilon}$ is the zero set of $\mathrm{Cr}_p(Q(y,-),\varepsilon)$,

$$W_{\mathcal{N}_{y,\varepsilon}} = (W^p_{y,\varepsilon})_{\mathcal{N}_{y,\varepsilon}} = \bigcup_{z_\varepsilon \in \mathcal{N}_{y,\varepsilon}} W^0_{y,z_\varepsilon,\varepsilon}, \qquad (6.5)$$

and $\mathrm{RM}(W^p_{y,\varepsilon}, W_{\mathcal{N}_{y,\varepsilon}})$ is a roadmap for $(W^p_{y,\varepsilon}, W_{\mathcal{N}_{y,\varepsilon}})$

Complexity. $D^{O(m+p)} d^{O((m+p)k}$ where $d = \deg(Q) \ge 2$ and D is a bound on the degree of $\mathcal{F}, F$ and the number and degrees (including that in ε) of the elements in $\mathcal{N}_{y,\varepsilon}$.

Procedure.

Step 1. Let $T = (T_1, \ldots, T_m)$, and, using (4.5) and (6.3),

$$P = \sum_{A \in \mathrm{Cr}_p(Q_F, \varepsilon)} A^2 \in \mathrm{D}[\varepsilon, T, X_{mp+1}, \ldots, X_k].$$

Call [?, Algorithm 12.18 (Parametrized Bounded Algebraic Sampling)] with input P and parameters $\varepsilon, T, X_{mp+1}, \ldots, X_{(m+1)p}$ and output a set of parametrized univariate representations with variable U.

Pseudo-reduce them modulo $\mathcal{F}$ using (4.1) and place the result in $\mathcal{U}'_\varepsilon$.

For every $(h_\varepsilon, H_\varepsilon) \in \mathcal{U}_\varepsilon$, and every $z_\varepsilon \in \mathcal{N}_{y,\varepsilon}$ represented by a real univariate representation $(g_\varepsilon, \tau, G_\varepsilon) \in \mathcal{V}_\varepsilon$, use [?, Algorithm 12.20 (Triangular Thom Encoding)] with input the triangular system $(\mathcal{F}, g_\varepsilon, h_\varepsilon)$ to compute the Thom encodings of the real roots of $h_\varepsilon(y, z_\varepsilon, U)$. Let $\mathcal{U}'_{y,z_\varepsilon}$ be the set of real univariate representations over y, z_ε so obtained. Define

$$\mathcal{U}'_{y,\varepsilon} = \bigcup_{z_\varepsilon \in \mathcal{N}_{y,\varepsilon}} \mathcal{U}'_{y,z_\varepsilon}.$$

The set of points represented by $\mathcal{U}'_{y,\varepsilon}$ is $W_{\mathcal{N}_{y,\varepsilon}}$ (see (6.5)).

Step 2. Call Algorithm 2 (Roadmap for Lower Dimensional Special Algebraic Sets) with input $\mathrm{Def}(Q, \varepsilon)$ (see (6.1)) and p, the real block representation $\mathcal{F}, \sigma, [p^m], F$, $r := 0$ and $\mathcal{U}'_{y,\varepsilon}$. The output of Algorithm 2 (Roadmap for Lower Dimensional Special Algebraic Sets) consists of a set of real univariate representations and curve segments over triangular Thom encodings. Each such curve segment, $\gamma = (f_1, \sigma_1, f_2, \sigma_2, g, \tau, G)$, is defined over some (t, z_γ) with $r_\gamma < p$ and $z_\gamma \in \mathrm{R}\langle\varepsilon\rangle^{r_\gamma}$, represented by a triangular system $\mathcal{F}, \mathcal{H}_\gamma$.

Step 3. For each such curve segment γ over (t, z_γ), output in the previous step over call Algorithm 4 (Limit of a Curve) with input the triangular system $\mathcal{F}, \mathcal{H}_\gamma$ and γ. Finally, project on R^k by forgetting the last coordinate.

Remark 6.4. The role played by the set of points $W_{\mathcal{N}_{y,\varepsilon}}$ which are included in the roadmap of $W^p_{y,\varepsilon}$, whose limit is computed by Algorithm 5 (Limit of Roadmaps of Special Low Dimensional Varieties), will become clear in the proof of correctness of Algorithm 6 (Baby-giant Roadmap for Bounded Algebraic Sets) (see (7.2)).

Proof of correctness. First note that it follows from Proposition 6.2 that $\mathrm{Def}(Q_F, \varepsilon)$ satisfies Property 5.1, and hence $(W^p_{y,\varepsilon})_{\mathcal{N}_\varepsilon}$ is a finite set of points. The correctness of the algorithm now follows from the correctness of Algorithm 2 (Roadmap for Lower Dimensional Special Algebraic Sets) and Algorithm 4 (Limit of Curve). □

Complexity analysis. The number of arithmetic operations performed in D[ε] in Step 1 is bounded by $D^{O(m+p)}d^{O((m+p)k)}$ arithmetic operations in D[ε] according to the complexity analysis of [**?**, Algorithm 12.18 (Parametrized Bounded Algebraic Sampling)] and [**?**, Algorithm 12.20 (Triangular Thom Encoding)]. Since the degree in ε in the output of [**?**, Algorithm 12.18 (Parametrized Bounded Algebraic Sampling)] is $d^{O(k)}$ and does not change during the pseudo-reduction, the number of arithemtic operations in D in Step 1 (and hence the complexity) is bounded by $D^{O(m+p)}d^{O((m+p)k)}$.

The number of arithmetic operations performed in D[ε] in Step 2 is bounded by $D^{O(m+p)}d^{O((m+p)k)}$ according to the complexity analysis of Algorithm 2 (Roadmap for Lower Dimensional Special Algebraic Sets). Moreover the degree in ε is bounded by $O(d)^k$ by [**?**, Algorithm 15.10 (Parametrized Curve Segments)], since the computations of Algorithm (Curve segments) with coefficients in D[ε] is contained in the computations of [**?**, Algorithm 15.10 (Parametrized Curve Segments)] in D with parameter ε. So the the number of arithmetic operations in D in Step 2 (and hence the complexity) is bounded by $D^{O(m+p)}d^{O((m+p)k)}$.

The complexity of Step 3 is also bounded by $D^{O(m+p)}d^{O((m+p)k}$ according to the complexity analysis of Algorithm 4 (Limit of Curve).

Thus the total complexity of the algorithm is $D^{O(m+p)}d^{O((m+p)k)}$. □

7. Main result

We now describe our main result Algorithm 7 (Baby-giant Roadmap for General Algebraic Sets). It is based on Algorithm 6 (Baby-giant Roadmap for Bounded Algebraic Sets), computing a baby step - giant step roadmap algorithm for a bounded algebraic set. The algorithm for computing roadmaps of general (i.e. not necessarily bounded) algebraic sets, Algorithm 7 (Baby-giant Roadmap for General Algebraic Sets) is then obtained from Algorithm 6 (Baby-giant Roadmap for Bounded Algebraic Sets) following a method similar to the one in [**?**] to go from the bounded case to the general case.

Algorithm 6 (Baby-giant Roadmap for Bounded Algebraic Sets) proceeds roughly as follows. We denote by y the fixed coordinates. If the number of non-fixed coordinates is too small (i.e. less than the number p which is prescribed in the input), then we compute the roadmap using [**?**, Algorithm 15.3 (Bounded Algebraic Roadmap)]. Otherwise, we compute representations of points in $\mathcal{N}_y \subset \mathrm{R}^p$ defining the fibers at which we make recursive calls to the same algorithm; these are the giant steps.

For the baby steps, the algorithm uses Algorithm 5 (Limit of Roadmaps of Special Low Dimensional Varieties) to compute the limit (under the $\lim_\varepsilon$ map) of the roadmap of the critical set $W^p_{y,\varepsilon}$ going through a well chosen finite set of points.

We are now ready to proceed to the description of Algorithm 6 (Baby-giant Roadmap for Bounded Algebraic Sets)

Let as in Notation 6.1

$$V_{y,\varepsilon} = \mathrm{Zer}(\mathrm{Def}(Q,\varepsilon)(y,-), \mathrm{R}\langle\varepsilon\rangle^{k+1})$$
$$W^p_{y,\varepsilon} = \mathrm{Zer}(\mathrm{Cr}_p(Q,\varepsilon)(y,-), \mathrm{R}\langle\varepsilon\rangle^{k+1})$$

and define

$$(\mathcal{M}_{y,\varepsilon,i})_{1\le i\le 2}, (\mathcal{D}_{y,\varepsilon,i})_{1\le i\le 2})$$

from $V_{y,\varepsilon}, W^p_{y,\varepsilon}$ as in Definition 5.3.

It follows from Proposition 6.2 and Proposition 5.4 that

$$\big(V_{y,\varepsilon}, p, W^p_{y,\varepsilon}, (\mathcal{M}_{y,\varepsilon,i})_{1\le i\le 2}, (\mathcal{D}_{y,\varepsilon,i})_{1\le i\le 2}\big)$$

satisfies Property 3.2.

Algorithm 6. **[Baby-giant Roadmap for Bounded Algebraic Sets]**

Input. (1) a polynomial $Q \in \mathrm{D}[X_1,\dots,X_k]$ which $Z = \mathrm{Zer}(Q,\mathrm{R}^k) \subset \mathcal{B}_k(0,1/c)$ (where $c \in \mathrm{R}$);
(2) $y \in \mathrm{R}^{mp}$ represented by a real block representation

$$\mathcal{F}, \sigma, [p^m], F,$$

(see (4.2)) such that $t \in \mathrm{R}^m$ is represented by a quasi-monic triangular Thom encoding $\mathcal{F}, \sigma$;
(3) a finite set of points $\mathcal{M}_{y,0}$ in $Z_y = \mathrm{Zer}(Q(y,-),\mathrm{R}^{k-mp})$ represented by quasi-monic real univariate representations $\mathcal{U}_0$ over t.
All the coefficients of the input polynomials are in D.

Output. a roadmap, $\mathrm{BGRM}(Z_y, \mathcal{M}_{y,0})$, for $(Z_y, \mathcal{M}_{y,0})$.

Complexity. $d^{O(k^2/p+pk)}$ operations in D where $d = \deg(Q) \ge 2$ and the degrees of the polynomials in $\mathcal{F}, F$, as well as the degrees of the polynomials and the number of elements in $\mathcal{U}_0$ are all bounded by $d^{O(k)}$.

Procedure.

Step 1. If $(m+1)p \ge k$ call [**?**, Algorithm 15.3 (Bounded Algebraic Roadmap)] with input
(1) the quasi-monic triangular Thom encoding $\mathcal{F}, \sigma$ representing $t \in \mathrm{R}^m$,
(2) the polynomial Q_F, using Notation 4.5,
(3) the finite set of points $\mathcal{M}_{y,0}$ in $Z_y = \mathrm{Zer}(Q_F(t,-),\mathrm{R}^{k-mp})$ represented by real univariate representations $\mathcal{U}_0$ over t.

Otherwise set $i := 1$ and do the following.

Step 2. Determination of the finite set of points $\mathcal{N}_y$ used in the recursive call. Let $T = (T_1,\dots,T_m)$, and, using (4.5) and (6.3),

$$P = \sum_{A\in \mathrm{Cr}_p(Q_F,\varepsilon)} A^2 \in \mathrm{D}[\varepsilon, T, X_{mp+1},\dots,X_k].$$

Step 2 a). Call [**?**, Algorithm 12.18 (Parametrized Bounded Algebraic Sampling)] with input P and parameters ε, T, and output a set of parametrized univariate representations with variable U. Pseudo-reduce them modulo $\mathcal{F}$ using (4.1) and place the result in $\mathcal{U}_{\varepsilon,1}$.

For every $(h_\varepsilon, H_\varepsilon) \in \mathcal{U}_{\varepsilon,1}$, use [**?**, Algorithm 12.20 (Triangular Thom Encoding)] with input the triangular system $(\mathcal{F}, h_\varepsilon)$ to compute the Thom encodings of the real roots of $h_\varepsilon(y, U)$. Let $\mathcal{U}_{y,\varepsilon,1}$ be the set of real univariate representations over y so obtained. Let $\mathcal{M}_{y,\varepsilon,1} \subset\subset V_{y,\varepsilon}$ be the set of points represented by $\mathcal{U}_{y,\varepsilon,1}$.

Projecting $\mathcal{U}_{y,\varepsilon,1}$, by forgetting its last components, obtain a set of quasi-monic real univariate representations $\mathcal{V}_{y,\varepsilon,1}$ representing

$$\mathcal{N}_{y,\varepsilon,1} = \pi_{[mp+1,(m+1)p]}(\mathcal{M}_{y,\varepsilon,1})$$

over t. Then apply Algorithm 3 (Limit of a Bounded Point) with $\mathcal{V}_{\mathbf{y},eps,1}$ as input to obtain a set of quasi-monic real univariate representations $\mathcal{V}_{y,1}$ representing

$$\mathcal{N}_{y,1} = \lim_{\varepsilon}(\mathcal{N}_{y,\varepsilon,1})$$

over t.

Step 2 b). Perform Algorithm 1 (Curve Segment) with input P and the triangular Thom encoding $\mathcal{F}, \sigma$ and retain the set of univariate representations, $\mathcal{U}_{y,\varepsilon,2}$, representing $\mathcal{M}_{y,\varepsilon,2} \subset V_{y,\varepsilon}$, which are the distinguished points in the output.

Projecting $\mathcal{U}_{y,\varepsilon,2}$, by forgetting its last components, obtain a set of real univariate representations $\mathcal{V}_{y,\varepsilon,2}$ representing

$$\mathcal{N}_{y,\varepsilon,2} = \pi_{[mp+1,(m+1)p]}(\mathcal{M}_{y,\varepsilon,2}).$$

Then apply Algorithm 3 (Limit of a Bounded Point) with $\mathcal{V}_{y,\varepsilon,2}$ as input to obtain a set of quasi-monic real univariate representations $\mathcal{V}_{y,2}$ representing

$$\mathcal{N}_{y,2} = \lim_{\varepsilon}(\mathcal{N}_{y,\varepsilon,2}).$$

Step 2 c). Projecting $\mathcal{U}_0$, by forgetting its last components, obtain a set of quasi-monic real univariate representations $\mathcal{V}_{y,0}$ representing

$$\mathcal{N}_{y,0} = \pi_{[mp+1,(m+1)p]}(\mathcal{M}_{y,0})$$

over t.

Let

$$\begin{aligned}\mathcal{N}_y &= \mathcal{N}_{y,0} \cup \mathcal{N}_{y,1} \cup \mathcal{N}_{y,2},\\ \mathcal{V}_y &= \mathcal{V}_{y,0} \cup \mathcal{V}_{y,1} \cup \mathcal{V}_{y,2},\\ \mathcal{N}_{y,\varepsilon} &= \mathcal{N}_y \cup \mathcal{N}_{y,\varepsilon,1} \cup \mathcal{N}_{y,\varepsilon,2},\end{aligned}$$

and

$$\mathcal{V}_{y,\varepsilon} = \mathcal{V}_y \cup \mathcal{V}_{y,\varepsilon,1} \cup \mathcal{V}_{y,\varepsilon,2}.$$

Step 3. Call Algorithm 5 (Limit of Roadmaps of Special Low Dimensional Varieties) with input p, Q, the real block representation $\mathcal{F}, \sigma, [p^m], F$, and $\mathcal{V}_{y,\varepsilon}$ and note that it contains

$$W_{\mathcal{N}_{y,\varepsilon}} = (W^p_{y,\varepsilon})_{\mathcal{N}_{y,\varepsilon}} = \bigcup_{z_\varepsilon \in \mathcal{N}_{y,\varepsilon}} W^0_{y,z,\varepsilon}.$$

Recursive call For every element $u = ((\mathcal{F},h),(\sigma,\tau),(F,H)) \in \mathcal{V}_y$, representing $(y,z) \in \mathrm{R}^{(m+1)p}$, compute a set, $\mathcal{U}_{y,z}$, of quasi-monic univariate representations over $((\mathcal{F},h),(\sigma,\tau),(F,H))$, representing

$$\mathcal{M}_{(y,z)} = (\pi_{[1,k]} \circ \lim_{\varepsilon}) W^0_{y,z,\varepsilon}). \tag{7.1}$$

using Algorithm 3 (Limit of a Bounded Point).

Call Algorithm 6 (Baby-giant Roadmap for Bounded Algebraic Sets) recursively with input

$$Q, \mathcal{F} := (\mathcal{F},h), \sigma := (\sigma,\tau), L := [p^{m+1}], F := (F,H), i := i+1$$

and $\mathcal{U}_{(y,z),0} := \mathcal{U}_{(y,z)} \cup (\mathcal{U}_{y,0})_z$, where $(\mathcal{U}_0)_z$ is a set of quasi-monic real univariate representations representing $(\mathcal{M}_{y,0})_z$.

Output the set of curve segments computed in the last two steps.

Remark 7.1. Algorithm 6 would have been much simpler if we could make recursive calls to Algorithm 6 at the fibers over the points in $\mathcal{N}_{y,\varepsilon}$, and thus obtain a roadmap first of $V_{y,\varepsilon}$, and finally take the image of the resulting roadmap under the $\lim_\varepsilon$ map. In this case the proof of correctness of the algorithm would be an immediate consequence of the main connectivity result, Corollary 5.4, and the fact that the image under $\lim_\varepsilon$ of a bounded, semi-algebraically connected semi-algebraic set is also semi-algebraically connected.

However we are unable to compute limits of semi-algebraic curves given by curve segments over a real block representations depending on ε with number as well sizes of the blocks bounded by $O(\sqrt{k})$ with complexity $d^{O(k\sqrt{k})}$, because we would obtain a degree $d^{O(k^2)}$ in ε.

We overcome this difficulty by making recursive calls to Algorithm 6, not at the fibers over the points in $\mathcal{N}_{y,\varepsilon}$, but at the fibers over $\mathcal{N}_y = \lim_\varepsilon(\mathcal{N}_{\varepsilon,y})$, so that the algebraic sets specified in the input to the various recursive calls are then $Z_{(y,z)}$ for $z \in \mathcal{N}_y$. In this approach, the only limits of curve segments that are computed are those of the roadmap of $W^p_{y,\varepsilon}$, and we can compute the limits of these curve segments without spoiling the complexity, as they are not defined over real block representations depending on ε. However, since the recursive calls are made with fibers of Z_y (instead of $V_{y,\varepsilon}$), Corollary 5.4 is not directly applicable, and we need to be more careful about choosing the set of points in the input to the recursive calls. It also makes the proof of correctness more complicated.

Proof of correctness.

Base case.

If $\lceil (k-mp)/p \rceil = 1$ then the correctness of the algorithm is a consequence of the correctness of [**?**, Algorithm 15.3 (Bounded Algebraic Roadmap)].

General case.

Suppose that $\lceil (k-mp)/p \rceil > 1$.

Denote by $\mathrm{BGRM}(Z_y, \mathcal{M}_0)$ the union of the curve segments output by Algorithm 6 (Baby-giant Roadmap for Bounded Algebraic Sets).

We have that

$$\mathrm{BGRM}(Z_y, \mathcal{M}_{y,0}) = \mathcal{R}_y \cup \bigcup_{(y,z)\in\mathcal{N}_y} \mathrm{BGRM}(Z_{(y,z)}, \mathcal{M}_{(y,z)} \cup (\mathcal{M}_{y,0})_z),$$

with

$$\mathcal{R}_y = (\pi_{[1,k]} \circ \lim_{\varepsilon})(\mathrm{RM}(W^p_{y,\varepsilon}, W_{\mathcal{N}_{y,\varepsilon}})),$$

denoting by $W^p_{y,\varepsilon}$ the zero set of $\mathrm{Cr}_p(Q_F, \varepsilon)$ and

$$W_{\mathcal{N}_{y,\varepsilon}} = (W^p_{y,\varepsilon})_{\mathcal{N}_{y,\varepsilon}} = \bigcup_{z_\varepsilon \in \mathcal{N}_{y,\varepsilon}} W^0_{y,z,\varepsilon},$$

(see (6.3) and (6.5)).

Proof of $\mathcal{M}_{y,0} \subset \mathrm{BGRM}(Z_y, \mathcal{M}_{y,0})$.

The proof is by induction on $\lceil (k-mp)/p \rceil$.

We suppose by induction hypothesis that for every $(y,z) \in \mathcal{N}_y$ that

$$\mathcal{M}_{(y,z)} \cup (\mathcal{M}_{y,0})_z) \subset \mathrm{BGRM}(Z_{(y,z)}, \mathcal{M}_{(y,z)} \cup (\mathcal{M}_{y,0})_z).$$

Since, $\mathcal{M}_{y,0} \subset \bigcup_{(y,z)\in\mathcal{N}_y} (\mathcal{M}_{y,0})_z$, and by induction hypothesis we have that $(\mathcal{M}_{y,0})_z) \subset \mathrm{BGRM}(Z_{(y,z)}, \mathcal{M}_{(y,z)} \cup (\mathcal{M}_{y,0})_z)$, it is clear that $\mathrm{BGRM}(Z_y, \mathcal{M}_{y,0})$ contains $\mathcal{M}_{y,0}$.

Proof of RM_1.

The property RM_1 of $\mathrm{BGRM}(Z_y, \mathcal{M}_{y,0})$ is also proved by induction on $\lceil (k-mp)/p \rceil$.

Let C be a semi-algebraically connected component of Z_y, and $D = \mathrm{BGRM}(Z_y, \mathcal{M}_{(y,z)} \cup (\mathcal{M}_{y,0})_z) \cap C$. We want to prove that D is semi-algebraically connected.

Suppose that $x, x' \in D$, we are going to prove that there exists a semi-algebraic path $\gamma : [0,1] \to D$ with $\gamma(0) = x, \gamma(1) = x'$.

Without loss of generality we can suppose that $x \in \mathcal{R}_y$.

Since

$$\mathrm{BGRM}(Z_y, \mathcal{M}_{(y,z)} \cup (\mathcal{M}_{y,0})_z) = \mathcal{R}_y \cup \bigcup_{(y,z)\in\mathcal{N}_y} \mathrm{BGRM}(Z_{(y,z)}, \mathcal{M}_{(y,z)} \cup (\mathcal{M}_{y,0})_z),$$

we have that x (resp. x') either belongs to

$$\mathcal{R}_y$$

or to some

$$\mathrm{BGRM}(Z_{(y,z)}, \mathcal{M}_{(y,z)} \cup (\mathcal{M}_{y,0})_z)$$

with $(y,z) \in \mathcal{N}_y$.

If $x \in \mathrm{BGRM}(Z_{(y,z)}, \mathcal{M}_{(y,z)} \cup (\mathcal{M}_{y,0})_z)$ we show that x can be connected to a point in $\mathcal{M}_{(y,z)}$ inside

$$\mathrm{BGRM}(Z_{(y,z)}, \mathcal{M}_{(y,z)} \cup (\mathcal{M}_{y,0})_z)$$

by a semi-algebraic path. It follows from Proposition 5.4 that $\mathrm{Def}(Q,\varepsilon)_F$ satisfies Property 3.2 (2), and hence we have that $W^0_{(y,z),\varepsilon}$ meets every semi-algebraically connected component of $V_{(y,z),\varepsilon}$. By [**?**, Lemma 15.6] each semi-algebraically connected component of $Z_{(y,z)}$ is image under $\pi_{[1,k]} \circ \lim_\varepsilon$ of a unique semi-algebraically connected component of $V_{(y,z),\varepsilon}$. It follows that each semi-algebraically connected component of $Z_{(y,z)}$ meets

$$\lim_\varepsilon (W^0_{(y,z),\varepsilon}) \subset \mathcal{M}_{(y,z)},$$

since

$$W^0_{(y,z),\varepsilon} = (W^p_{y,\varepsilon})_z$$

Finally, applying the induction hypothesis to $\mathrm{BGRM}(Z_{(y,z)}, \mathcal{M}_{(y,z)} \cup (\mathcal{M}_{y,0})_z)$ we have that the intersection of $\mathrm{BGRM}(Z_{(y,z)}, \mathcal{M}_{(y,z)} \cup (\mathcal{M}_{y,0})_z)$ with each semi-algebraically connected component of $Z_{(y,z)}$ is non-empty and semi-algebraically connected, and meets $\mathcal{M}_{(y,z)}$. Thus, there exists a semi-algebraic path with image in $\mathrm{BGRM}(Z_{(y,z)}, \mathcal{M}_{(y,z)} \cup (\mathcal{M}_{y,0})_z)$ joining x to a point in $\mathcal{M}_{(y,z)}$.

Since $\mathcal{M}_{(y,z)} \subset \mathcal{R}_y$ we can assume that x (and similarly x') is contained in $\mathcal{R}_y$.

Connectivity when x and x' are contained in $\mathcal{R}_y$.

Since

$$\mathcal{R}_y = (\pi_{[1,k]} \circ \lim_\varepsilon)(\mathrm{RM}(W^p_{y,\varepsilon}, W_{\mathcal{N}_{y,\varepsilon}})),$$

there exists $x_\varepsilon \in \mathrm{RM}(W^p_{y,\varepsilon}, W_{\mathcal{N}_{y,\varepsilon}})$ (resp. $x'_\varepsilon \in \mathrm{RM}(W^p_{y,\varepsilon}, W_{\mathcal{N}_{y,\varepsilon}})$) such that $\lim_\varepsilon(x_\varepsilon) = x$ (resp. $\lim_\varepsilon(x'_\varepsilon) = x'$).

Let

$$\mathcal{S}_\varepsilon = W^p_{y,\varepsilon} \cup (V_{y,\varepsilon})_{\mathcal{N}_{y,\varepsilon}},$$

and C_ε the unique semi-algebraically connected component of $V_{y,\varepsilon}$ such that $(\pi_{[1,k]} \circ \lim_\varepsilon)(C_\varepsilon) = C$.

By Corollary 3.9, since $\mathcal{N}_{y,\varepsilon,1} \cup \mathcal{N}_{y,\varepsilon,2} \subset \mathcal{N}_{y,\varepsilon}$, $\mathcal{S}_\varepsilon \cap C_\varepsilon$ is semi-algebraically connected. So there exists a semi-algebraic path $\gamma_\varepsilon : [0,1] \to \mathcal{S}_\varepsilon \cap C_\varepsilon$, with $\gamma_\varepsilon(0) = x_\varepsilon, \gamma_\varepsilon(1) = x'_\varepsilon$. Moreover, there exists a partition of $(0,1) \subset \mathrm{R}\langle\varepsilon\rangle$ into a finite number of open intervals and points, such that for every open interval I in the partition one of the following holds :

Case 1:

$$\gamma_\varepsilon(I) \subset W^p_{y,\varepsilon}.$$

Case 2: there exists $z_\varepsilon \in \mathcal{N}_{y,\varepsilon}$ such that

$$\gamma_\varepsilon(I) \subset V_{(y,z_\varepsilon),\varepsilon}.$$

Since $W^p_{y,\varepsilon} \subset V_{y,\varepsilon}$, for each point $a \in (0,1)$ defining the partition

$$\gamma_\varepsilon(a) \in W_{\mathcal{N}_{y,\varepsilon}} \subset \mathrm{RM}(W^p_{y,\varepsilon}, W_{\mathcal{N}_{y,\varepsilon}}). \tag{7.2}$$

Hence, by definition of $\mathcal{M}_{(y,z)}$ (see (7.1))

$$\pi_{[1,k]} \circ \lim_\varepsilon)(\gamma_\varepsilon(a)) \in \mathcal{M}_{(y,z)}, \tag{7.3}$$

where $\lim_\varepsilon(z_\varepsilon) = z$.

In Case 1, we can replace $\gamma_\varepsilon(I)$ by a semi-algebraic path having the same endpoints and whose image is contained in

$$\mathrm{RM}(W^p_{y,\varepsilon}, W_{\mathcal{N}_{y,\varepsilon}})$$

using RM_1 for

$$\mathrm{RM}(W^p_{y,\varepsilon}, W_{\mathcal{N}_{y,\varepsilon}})$$

as well as (7.2) Taking the image under $\pi_{[1,k]} \circ \lim_\varepsilon$ of this new path we obtain a semi-algebraic path

$$\gamma : \lim_\varepsilon(I) \to \mathcal{R}_y.$$

In Case 2, $(\pi_{[1,k]} \circ \lim_\varepsilon)(\gamma_\varepsilon(a)), (\pi_{[1,k]} \circ \lim_\varepsilon)(\gamma_\varepsilon(b))$ both belong to

$$\mathrm{BGRM}(Z_{(y,z)}, \mathcal{M}_{(y,z)} \cup (\mathcal{M}_{y,0})_z)$$

using (7.3). Using the induction hypothesis for $\mathrm{BGRM}(Z_{(y,z)}, \mathcal{M}_{(y,z)} \cup (\mathcal{M}_{y,0})_z)$ we have that there exists a semi-algebraic path

$$\gamma : [\lim_\varepsilon(a), \lim_\varepsilon(b)] \to \mathrm{BGRM}(Z_{(y,z)}, \mathcal{M}_{(y,z)} \cup (\mathcal{M}_{y,0})_z).$$

Finally, we have constructed a semi-algebraic path $\gamma : [0,1] \to D$ with $\gamma(0) = x, \gamma(1) = x'$.

This proves that $\mathrm{BGRM}(Z_{(y,z)}, \mathcal{M}_{(y,z)} \cup (\mathcal{M}_{y,0})_z) \cap C$ is non-empty and semi-algebraically connected proving RM_1.

Proof of RM_2.

Let $c \in \mathrm{R}$ such that $Z_{(y,c)}$ is not empty, and let C be a semi-algebraically connected component of $Z_{(y,c)}$. We prove that

$$\mathrm{BGRM}(Z_{(y,z)}, \mathcal{M}_{(y,z)} \cup (\mathcal{M}_{y,0})_z) \cap C$$

is not empty. It follows from [**?**, Lemma 15.6] that there exists a semi-algebraically connected component, C_ε, of $V_{(y,c),\varepsilon}$ such that

$$C = (\pi_{[1,k]} \circ \lim_\varepsilon)(C_\varepsilon).$$

Since C_ε is non-empty, let $x_\varepsilon \in C_\varepsilon$ and let $z_\varepsilon = \pi_{[mp+1,(m+1)p]}(x_\varepsilon)$. It follows from Proposition 5.4 that $(W^p_{y,\varepsilon})_{z_\varepsilon} = W^0_{(y,z_\varepsilon),\varepsilon}$ meets every semi-algebraically connected component of $V_{(y,z_\varepsilon),\varepsilon}$. Since C_ε contains a semi-algebraically connected component of $V_{(y,z_\varepsilon),\varepsilon}$, we have that

$$W^0_{(y,z_\varepsilon),\varepsilon} \cap C_\varepsilon \neq \emptyset,$$

and thus C_ε contains a semi-algebraically connected component of $(W^p_{y,\varepsilon})_c$ (since $W^p_{y,\varepsilon} \subset V_{y,\varepsilon}$). Now, since the roadmap

$$\mathrm{RM}(W^p_{y,\varepsilon}, W_{\mathcal{N}_{y,\varepsilon}})$$

satisfies RM_2, $\mathrm{RM}(W^p_{y,\varepsilon}, W_{\mathcal{N}_{y,\varepsilon}})$ has a non-empty intersection with every semi-algebraically connected component of $(W^p_{y,\varepsilon})_c$, and in particular with the one contained in C_ε. Taking the image under $\pi_{[1,k]} \circ \lim_\varepsilon$ map we get that $\mathcal{R}_y = (\pi_{[1,k]} \circ \lim_\varepsilon)(\mathrm{RM}(W^p_{y,\varepsilon}, W_{\mathcal{N}_{y,\varepsilon}}))$ has a non-empty intersection with

$(\pi_{[1,k]} \circ \lim_\varepsilon)(C_\varepsilon) = C$. Since, $\mathrm{BGRM}(Z_{(y,z)}, \mathcal{M}_{(y,z)} \cup (\mathcal{M}_{y,0})_z)_c$ contains $\mathcal{R}_y$, this finishes the proof. □

Complexity analysis. We first bound the number of arithmetic operations in Step 1. Since we assume that the degrees of the polynomials in $\mathcal{F}, F$ are bounded by $d^{O(k)}$, it follows from the complexity analysis of [**?**, Algorithm 15.3 (Bounded Algebraic Roadmap)], and [**?**, Algorithm 15.2 (Curve Segments)], that the number of arithmetic operations in this step is bounded by

$$d^{O(km)} d^{O((k-mp)^2)} = d^{O(km+p^2)}$$

since $k - mp \le p$.

The number of arithmetic operations in $\mathrm{D}[\varepsilon]$ in in Step 2 is bounded by $d^{O(mk)}$ and the degree and number of univariate representations produced is bounded by $O(d)^{k-mp}$. Moreover the degree in ε is bounded by $O(d)^k$ by [**?**, Algorithm 15.10 (Parametrized Curve Segments)], since the computations of Algorithm (Curve segments) with coefficients in $\mathrm{D}[\varepsilon]$ is contained in the computations of [**?**, Algorithm 15.10 (Parametrized Curve Segments)] in D with parameter ε. So the number of arithmetic operations in D in in Step 2 is bounded by $d^{O(mk)}$.

The complexity of computing $\mathcal{R}_y$ in Step 3 is bounded by $d^{O((m+p)k)}$ given that the number of arithmetic operations of Algorithm 5 (Limit of Roadmaps of Special Low Dimensional Varieties) is $d^{O((m+p)k)}$.

The total number of recursive calls at depth i is $d^{O(ki)}$, and for each such call the number of arithmetic operations in D in Steps 1, 2 and 3 is bounded by $d^{O((m+i+p)k+p^2}$, where $0 \le i \le \lfloor k/p \rfloor - m$. Since the depth of the recursion is at most $\lfloor k/p \rfloor - m$, we obtain that the total number of arithmetic operations in the domain D is bounded by

$$d^{O(k^2/p)} d^{O((k/p+p)k+p^2)} = d^{O(k^2/p+pk)}.$$

□

We now describe Algorithm 7 (Baby-giant Roadmap for General Algebraic Sets) for computing a roadmap of a general (i.e. not necessarily bounded algebraic set). This algorithm is essentially the same algorithm as [**?**, Algorithm 15.5 (Algebraic Roadmap)], except that we call Algorithm 6 (Baby-giant Roadmap for Bounded Algebraic Sets) after reducing to the bounded case instead of of [**?**, Algorithm 15.3 (Bounded Algebraic Roadmap)]. We first need a notation. Let $P \in \mathrm{R}[X]$ be given by

$$P = a_p X^p + \cdots + a_q X^q, p > q, a_q a_p \neq 0.$$

Notation 7.2. We denote

$$c(P) = \left(\sum_{q \le i \le p} \left| \frac{a_i}{a_q} \right| \right)^{-1}.$$

Algorithm 7. **[Baby-giant Roadmap for General Algebraic Sets]**

Input. (1) a polynomial $Q \in \mathrm{D}[X_1, \dots, X_k]$;
(2) a finite set of points $\mathcal{M}_{y,0}$ in $\mathrm{Zer}(Q, \mathrm{R}^k)$, represented by real univariate representations $\mathcal{U}_0$.

Output. a roadmap, $\mathrm{BGRM}(\mathrm{Zer}(Q, \mathrm{R}^k), \mathcal{M}_0)$, for $(\mathrm{Zer}(Q, \mathrm{R}^k), \mathcal{M}_0)$.

Complexity. $d^{O(k^2/p+pk)}$ operations in D.

Procedure.

Step 1. Introduce new variables X_{k+1} and ε and replace Q by the polynomial

$$Q_\varepsilon = Q^2 + (\varepsilon^2(X_1^2 + \dots + X_{k+1}^2) - 1)^2.$$

Replace $\mathcal{M}_0 \subset \mathrm{R}^k$ by the set of real univariate representations representing the elements of $\mathrm{Zer}(\varepsilon^2(X_1^2+\dots+X_{k+1}^2)-1, \mathrm{R}\langle\varepsilon\rangle^{k+1})$ above the points represented by $\mathcal{M}_0$ using [**?**, Algorithm 12.18 (Parametrized Bounded Algebraic Sampling)].

Step 2. Call Algorithm 6 (Baby-giant Roadmap for Bounded Algebraic Sets) with input Q_ε, $\mathcal{M}_0$, $m = 0$, performing arithmetic operations in the domain $\mathrm{D}[\varepsilon]$. The algorithm outputs a roadmap

$$\mathrm{BGRM}(\mathrm{Zer}(Q_\varepsilon, \mathrm{R}\langle\varepsilon\rangle^{k+1}), \mathcal{M}_0)$$

composed of points and curves whose description involves ε.

Step 3. Denote by $\mathcal{L}$ the set of polynomials in $\mathrm{D}[\varepsilon]$ whose signs have been determined in the preceding computation and take

$$a = \min_{P \in \mathcal{L}} c(P)$$

(Notation 7.2). Replace ε by a in the polynomial Q_ε to get a polynomial Q_a. Replace ε by a in the output roadmap to obtain a roadmap $\mathrm{BGRM}(\mathrm{Zer}(Q_a, \mathrm{R}^{k+1}), \mathcal{M}_{y,0})$. When projected to R^k, this roadmap gives a roadmap

$$\mathrm{BGRM}(\mathrm{Zer}(Q, \mathrm{R}^k), \mathcal{M}_{y,0}) \cap \mathcal{B}_k(0, 1/a).$$

Step 4. In order to extend the roadmap outside the ball $B(0, 1/a)$ collect all the points $(y_1, \dots, y_k, y_{k+1}) \in \mathrm{R}\langle\varepsilon\rangle^{k+1}$ in the roadmap

$$\mathrm{BGRM}(\mathrm{Zer}(Q_\varepsilon, \mathrm{R}\langle\varepsilon\rangle^{k+1}), \mathcal{M}_0)$$

which satisfies $\varepsilon(y_1^2 + \ldots + y_k^2) = 1$. Each such point is described by a real univariate representation involving ε. Add to the roadmap the curve segment obtained by first forgetting the last coordinate and then treating ε as a parameter which varies vary over $(0, a]$ to get a roadmap $\mathrm{BGRM}(\mathrm{Zer}(Q, \mathrm{R}^k), \mathcal{M}_0)$.

Proof of correctness. The proof of correctness follows from the proof of correctness of Algorithm 6 (Baby-giant Roadmap for Bounded Algebraic Sets). □

Complexity analysis. The complexity is dominated by the complexity of Step 2. □

Proofs of Theorem 1.2 and Corollary 1.3. Follows directly from the correctness and complexity analysis of Algorithm 7 (Baby-giant Roadmap for General Algebraic Sets), after substituting $m = 0$ and $p = \sqrt{k}$. □

8. Appendix: computing the limit of bounded points and curve segments

8.1. **Limit of a bounded point.** Before computing the limit of a bounded point we need to explain how to perform some useful computations modulo a quasi-monic triangular Thom encoding $\mathcal{F}, \sigma$ representing a point $t \in \mathrm{R}^m$.

We associate to $t \in \mathrm{R}^m$ specified by a triangular Thom encoding $\mathcal{F}, \sigma$,

$$\mathcal{F} = (f_{[1]}, \ldots, g_{[m]}), f_{[i]} \in \mathrm{D}[T_1, \ldots, T_i],$$

the ordered domain $\mathrm{D}[t]$ contained in R and generated by t.

We now aim at describing the pseudo-inversion of a non-zero element in the domain $\mathrm{D}[t]$ specified by $\mathcal{F}, \sigma$.

Definition 8.1. A pseudo-inverse of $f \in \mathrm{D}[t]$ is an element $g \in \mathrm{D}[t]$ such that $fg \in \mathrm{D}$ is strictly positive.

This notion is delicate as the computation of the pseudo-inverse sometimes requires us to update the triangular Thom encoding specifying t, in the spirit of dynamical methods in algebra (see for example [**?**]). We start with a motivating example.

Example 8.2. We consider t, specified as the root of

$$f(T) = T^4 - T^2 - 2$$

giving signs $(+, +, +, +)$ to the set $\mathrm{Der}(f)$ of derivatives of f.

Consider $T^2 + 1$. It is easy to see, using for example [**?**, Algorithm 10.13 (Sign Determination Algorithm)] applied to f and the list $\mathrm{Der}(f), T^2 + 1$, that the sign of $T^2 + 1$ at t is positive. In order to compute its pseudo-inverse, we perform [**?**, Algorithm 8.22 (Extended Signed Subresultant)] of f and $T^2 + 1$. If $f(T)$ and $T^2 + 1$ were coprime, we would obtain the pseudo-inverse of $T^2 + 1$ modulo $f(T)$ since the last subresultant would be a non zero constant in D. But $f(T)$ and $T^2 + 1$ are not coprime and their gcd is $T^2 + 1$. So we divide $f(T)$ by $T^2 + 1$, obtain a new polynomial $g(T) = T^2 - 2$ and check that the root t of $f(T)$ giving signs $(+, +, +, +)$ to the set $\mathrm{Der}(f)$ coincides with $\sqrt{2}$ which is the root of $T^2 - 2$ making the derivative $g'(T) = 2T$ positive, using again -for example- [**?**, Algorithm 10.13 (Sign Determination Algorithm)]. It is now possible to pseudo-reduce $T^2 + 1$ modulo $g(T)$, which gives 3.

In other words, during the process of computing the pseudo-inverse of $T^2 + 1$ we discovered the factor $g(T)$ of $f(T)$ having t as a root and coprime with $T^2 + 1$. Using this new description of t we have been able to compute a pseudo-inverse of $T^2 + 1$.

We can now describe the computation of the pseudo-inverse in general.

Description 8.3. Given $t = (t_1, \ldots, t_m) \in \mathrm{R}^m$ specified by the quasi-monic triangular Thom encoding $\mathcal{F} = (f_{[1]}, \ldots, f_{[m]}), \sigma = (\sigma_1, \ldots, \sigma_m)$, we describe how to compute a pseudo-inverse of a non-zero elements of $\mathrm{D}[t]$.

We proceed by induction on the number m of variables of $\mathcal{F}$.

If $m = 0$ there is nothing to do since D is an ordered domain.

If $m \neq 0$, let $t' = (t_1, \ldots, t_{m-1})$ specified by $\mathcal{F}' = (f_{[1]}, \ldots, f_{[m-1]}), \sigma = (\sigma_1, \ldots, \sigma_{m-1})$.

We consider f as a polynomial in T_m whose coefficients, which are elements of

$$\{h \in \mathrm{D}[T_1, \ldots, T_{m-1}] \mid \deg_{T_i}(h) < \deg_{T_i}(f_{[i]}), i = 1, \ldots, m-1\}$$

represent elements of $\mathrm{D}[t']$.

We first decide the sign of f at t, which is done by [**?**, Algorithm 12.19 (Triangular Sign Determination Algorithm)].

If $f(t) \neq 0$, we try to pseudo-invert f modulo $\mathcal{F}$. We perform [**?**, Algorithm 8.22 (Extended Signed Subresultant)] for f and $f_{[m]}$, with respect to the variable T_m and compute a $\gcd(f, f_{[m]}) \in \mathrm{D}[t']$ (the last non zero subresultant polynomial) as well as the cofactors $u, v \in \mathrm{D}[t']$ with $uf + vf_{[m]} = \gcd(f, f_{[m]})$.

(1) If $\gcd(f, f_{[m]})$ is of degree 0 in T_m, u is a quasi-inverse of f.
(2) If $\gcd(f, f_{[m]})$ is of degree > 0 in T_m, we have discovered a factor of $f_{[m]}$. We define h as the quasi-monic polynomial proportional to $f_{[m]}/\gcd(f, f_{[m]})$ obtained by [**?**, Algorithm 8.22 (Extended Signed Subresultant)] (see [**?**, Algorithm 10.1 (Gcd and Gcd-free part)]). We perform [**?**, Algorithm 12.19 (Triangular Sign Determination)] applied to $f_{[m]}$ and $\mathrm{Der}(f_{[m]}), \mathrm{Der}(h)$ to identify the Thom encoding τ of $t_{[m]}$ as a root of h. We replace $f_{[m]}$ by h and $\sigma_{[m]}$ by τ in $\mathcal{F}$. Now f and the new $f_{[m]}$, considered as polynomials in $T_{[m]}$ are coprime and we can invert f module $f_{[m]}$.

Proposition 8.4. *Let* D *be an ordered domain contained in a real closed field* R*, and* $t = (t_1, \ldots, t_m) \in \mathrm{R}^m$ *specified by a triangular Thom encoding* $\mathcal{F}, \sigma$*,*

$$\mathcal{F} = (f_{[1]}, \ldots, f_{[m]}), f_{[i]} \in \mathrm{D}[T_1, \ldots, T_i].$$

Let d be a bound of the degree of $f_{[i]}$ with respect to each T_j, $i = 1, \ldots, m, j \leq i$.

a) If $f \in \mathrm{D}[T_1, \ldots, T_m]$ is a polynomial of degree D, the complexity of computing the pseudo-reduction $\mathrm{PsRed}(f, \mathcal{F})$*, is $(O((D-d)d))^m$ arithmetic operations in* D*.*

b) The complexity of the computation of the pseudo-inverse of an element of $\mathrm{D}[t]$ *is $d^{O(m)}$ arithmetic operations in* D*.*

Proof. The proof of a) is made by induction on the number of variables m of $\mathcal{F}$. If $m = 0$, there is nothing to do. If $m \neq 0$, the pseudo-division process by $g_{[m]}$ takes $O((D-d)d)$ arithmetic operations in $\mathrm{D}[T_1, \ldots, T_m]$

using the complexity analysis of [**?**, Algorithm 8.3 (Euclidean Division)] and produces d polynomials of degree $D+(D-d+1)d$ in $\mathrm{D}[T_1,\dots,T_m]$, so the complexity is $(O((D-d)d))^m$. The end of the reduction of f coincides with the reduction of these d polynomials modulo

$$\mathcal{F}' = (f_{[1]},\dots,f_{[m-1]}), f_{[i]} \in \mathrm{D}[T_1,\dots,T_i],$$

and costs $d(O((D-d)d))^{(m-1)}$ by the induction hypothesis.

The proof of b) proceeds also by induction on the number of variables m of $\mathcal{F}$.

If $m=1$, the computation of a gcd takes $(d+1)^c$ operations in the domain D, for some universal constant $c>0$, using the complexity analysis of [**?**, Algorithm 8.22 (Extended Signed Subresultant)] and [**?**, Algorithm 10.13 (Sign Determination)].

If $m>1$, let $t=(t',u)$, and we suppose by induction hypothesis that the complexity of arithmetic operations in $\mathrm{D}[t']$ is $(d+1)^{c(m-1)}$ arithmetic operations in the ordered domain D. The statement is clear since the arithmetic operations in the domain $\mathrm{D}[t]$ are using $(d+1)^c$ operations in the domain $\mathrm{D}[t']$ using the complexity analysis of [**?**, Algorithm 8.22 (Extended Signed Subresultant)] and [**?**, Algorithm 10.13 (Sign Determination)]. □

We can now give the description of Algorithm 3 (Limit of a Bounded Point).

PROCEDURE. Remove from $g_\varepsilon(T,U)$ the monomials vanishing at t, using [**?**, Algorithm 12.19 (Triangular Sign Determination)]. Supposing without loss of generality that all the coefficients of $g_\varepsilon(t,U)$ are not multiple of ε, denote by $g(T_1,\dots,T_m,U)$ the polynomial obtained by substituting 0 to ε in $g_\varepsilon(T_1,\dots,T_m,U)$. Similarly denote by $G(T_1,\dots,T_m,U)$ the polynomials obtained by substituting 0 to ε in $G_\varepsilon(T_1,\dots,T_m,U)$.

Compute the set Σ of Thom encodings of roots of $g(t,U)$ using [**?**, Algorithm 12.19 (Triangular Sign Determination)]. Denoting by μ_σ the multiplicity of the root of $g(t,U)$ with Thom encoding σ, define G_σ as the $\mu_\sigma-1$-th derivative of G with respect to U.

Identify the Thom encoding σ and G_σ representing z using [**?**, Algorithm 12.19 (Triangular Sign Determination)], by checking whether a ball of infinitesimal radius δ $(1 \gg \delta \gg \varepsilon > 0)$ around the point x represented by the real univariate representation g,σ,G_σ contains z_ε.

Pseudo-invert the leading coefficient of the univariate representation, denote by $\mathcal{F}',\sigma'$ the new triangular Thom encoding describing t and pseudo-reduce by $\mathcal{F}'$.

Complexity analysis: Follows from the complexity of [**?**, Algorithm 12.19 (Triangular Sign Determination)]. □

8.2. **Limit of a curve.** Computing the limit of a curve is not immediate when some part of the curve has a vertical limit, as seen in the following example.

Example 8.5. Consider the semi-algebraic curve $\gamma : [0, \varepsilon] \to \mathrm{R}\langle\varepsilon\rangle^3$, parametrized by the X_1 coordinate defined by

$$\gamma(x_1) = (x_1, \gamma_2(x_1), \gamma_3(x_1)), x_1 \in [0, \varepsilon]$$

where $(\gamma_2(x_1), \gamma_3(x_1)))$ is the solution of the triangular system,

$$\begin{aligned} X_2 - x_1/\varepsilon &= 0, \\ X_2^2 + X_3^2 - 1 &= 0, \end{aligned}$$

with Thom encoding $(0, +), (0, +, +)$.

Notice that the image of γ is contained in the cylinder of unit radius with axis the X_1-axis and is bounded over R. The image of γ under the $\lim_\varepsilon$ map is contained in a circle in the plane $X_1 = 0$, and can no longer be described as a curve parametrized by the X_1-coordinate.

However, it is possible to reparametrize γ by the X_2-coordinate. By doing so we obtain another semi-algebraic curve $\varphi : [0, 1] \to \mathrm{R}\langle\varepsilon\rangle^3$ (having the same image as γ) defined by

$$\varphi(x_2) = (\varphi_1(x_2), x_2, \varphi_3(x_2)), x_2 \in [0, 1]$$

where $(\varphi_1(x_2), \varphi_3(x_2))$ is the real solution of the triangular system

$$\begin{aligned} X_1 - \varepsilon x_2 &= 0, \\ X_3^2 + x_2^2 - 1 &= 0, \end{aligned}$$

with Thom encoding $(0, -), (0, +, +)$. Notice that the image under $\lim_\varepsilon$ of the curve which is the graph of φ can be easily described as the curve represented by the following triangular system parametrized by $x_2 \in [0, 1]$

$$\begin{aligned} X_1 &= 0, \\ X_3^2 + x_2^2 - 1 &= 0, \end{aligned}$$

and Thom encoding $(0, -1), (0, +, +)$.

This is the reason why some kind of reparametrization is necessary before computing the limit.

8.2.1. *Reparametrization of curve segments.* We define the notion of well-parametrized curve, and prove that the limit of a well-parametrized curve is easy to describe.

Definition 8.6. A differentiable semi-algebraic curve

$$\gamma = (\gamma_1, \ldots, \gamma_k) : (a, b) \to \mathrm{R}^k$$

parametrized by X_1 (i.e. $\gamma_1(x_1) = x_1$) is *well-parametrized* if for every $x_1 \in (a, b)$,

$$\sum_{i=1}^{k} \left(\frac{\partial \gamma_i}{\partial X_1}\right)^2 \leq k.$$

Let $t \in \mathrm{R}^m$ be represented by a triangular Thom encoding $\mathcal{F}, \sigma$, and

$$f_1, \sigma_1, f_2, \sigma_2, g, \tau, G$$

be a curve segment with parameter X_j over t on (α_1, α_2) where α_1 and α_2 are the elements of R represented by the Thom encodings f_1, σ_1 and f_2, σ_2.

The curve segment

$$f_1, \sigma_1, f_2, \sigma_2, g, \tau, G$$

is *well-parametrized* if the semi-algebraic curve $\gamma : (\alpha_1, \alpha_2) \to \mathrm{R}\langle\varepsilon\rangle^k$ defined by

$$\gamma(x_j) = \left(\frac{g_1(t, x_j, u(x_j))}{g_0(t, x_j, u(x_j))}, \ldots, \frac{g_k(t, x_j, u(x_j))}{g_0(t, x_j, u(x_j))}\right)$$

is well-parametrized, where $u : (\alpha_1, \alpha_2) \to \mathrm{R}^{k-(\ell+r)}$ maps each $x_j \in (\alpha_1, \alpha_2)$ to the root of $g(t, x_j, U)$ with Thom encoding τ. This means that

$$\sum_{i=1}^{k} \left(\left(\frac{g_i(t, x_j, u(x_j))}{g_0(t, x_j, u(x_j))}\right)'\right)^2 \leq k,$$

where the derivative is taken with respect to x_j.

Example 8.5 is not a well-parametrized curve segment.

If a curve segment defined over $\mathrm{R}\langle\varepsilon\rangle$ is well-parametrized, and represents a curve bounded over R, then the image of the curve under the $\lim_\varepsilon$ map can be easily described. The following proposition explains why this is true.

Proposition 8.7. *Let* $(a_\varepsilon, b_\varepsilon) \subset \mathrm{R}\langle\varepsilon\rangle$, $r < j \leq k$, $z_\varepsilon = (z_{\varepsilon,1}, \ldots, z_{\varepsilon,r}) \in \mathrm{R}\langle\varepsilon\rangle^r$, *and*

$$\gamma_\varepsilon = (z_\varepsilon, \gamma_{\varepsilon,r+1}, \ldots, \gamma_{\varepsilon,k}) : (a_\varepsilon, b_\varepsilon) \to \{z_\varepsilon\} \times \mathrm{R}\langle\varepsilon\rangle^{k-r}$$

a semi-algebraic differentiable curve parametrized by X_j *and bounded over* R*. If* γ_ε *is well-parametrized,*

1. *for each* $x \in (\lim_\varepsilon a_\varepsilon, \lim_\varepsilon b_\varepsilon)$ *and any* $x_\varepsilon \in (a_\varepsilon, b_\varepsilon)$ *with* $\lim_\varepsilon x_\varepsilon = x$, $\gamma(x) := \lim_\varepsilon \gamma_\varepsilon(x) = \lim_\varepsilon \gamma_\varepsilon(x_\varepsilon)$,
2. $\lim_\varepsilon \gamma_\varepsilon((a_\varepsilon, b_\varepsilon)) = \gamma([\lim_\varepsilon a_\varepsilon, \lim_\varepsilon b_\varepsilon])$.

In other words, the graph of the semi-algebraic function $\gamma(x) := \lim_\varepsilon \gamma_\varepsilon(x)$ *is the image under* $\lim_\varepsilon$ *of the graph of* γ_ε.

Proof. It follows from the definition of being well-parametrized that $||\gamma'_\varepsilon(x)|| \leq \sqrt{k}$ for all $x \in (a_\varepsilon, b_\varepsilon)$. By the semi-algebraic mean value theorem [**?**, Exercice 3.4] we have that for each $x \in (\lim_\varepsilon a_\varepsilon, \lim_\varepsilon b_\varepsilon)$ and any $x_\varepsilon \in (a_\varepsilon, b_\varepsilon)$ with $\lim_\varepsilon x_\varepsilon = x$,

$$||\gamma_\varepsilon(x) - \gamma_\varepsilon(x_\varepsilon)|| = ||\gamma'_\varepsilon(w_\varepsilon)|||x - x_\varepsilon|,$$

for some $w \in (x, x_\varepsilon)$ (assuming without loss of generality that $x < x_\varepsilon$). Taking the image under $\lim_\varepsilon$ and noticing that $||\gamma'_\varepsilon(w_\varepsilon)||$ is bounded over R by the previous observation, we obtain that $\lim_\varepsilon \gamma_\varepsilon(x) = \lim_\varepsilon \gamma_\varepsilon(x_\varepsilon)$, proving (1). This implies that the function $\gamma : (\lim_\varepsilon a_\varepsilon, \lim_\varepsilon b_\varepsilon) \to \mathrm{R}^k$ defined by $\gamma(x) = \lim_\varepsilon \gamma_\varepsilon(x)$ is a continuous, bounded (since γ_ε is bounded over R) semi-algebraic function, and hence can be extended to a continuous, bounded semi-algebraic function on the closed interval $[\lim_\varepsilon a_\varepsilon, \lim_\varepsilon b_\varepsilon]$, and (2) follows. □

A semi-algebraic curve is in general not well-parametrized. However, subdividing if necessary the curve into several pieces, it is possible to choose for each such piece a parametrizing coordinate which makes the piece well-parametrized. This is what we do in Algorithm 8 (Reparametrization of a Curve).

ALGORITHM 8. **[Reparametrization of a Curve]**

INPUT. (1) $t \in \mathrm{R}^m$ represented by a triangular Thom encoding $\mathcal{F}, \sigma$,
(2) a bounded curve S represented by a curve segment,

$$f_1, \sigma_1, f_2, \sigma_2, g, \tau, G$$

with parameter X_1 in R^k over t on (a, b).
All the polynomials in the input have coefficients in D.

OUTPUT. (1) A finite set $\mathcal{V} = \{v_1, \ldots, v_{N-1}\}$, of real univariate representation over (t, c_i), where each c_i is a Thom encoding over t fixing $X_{m(i)}$.
(2) A finite set $\mathcal{W} = \{w_1, \ldots, w_N\}$, of curve segments with w_i parametrized by $X_{\ell(i)}$.
Moreover, the union of the curves represented by $\mathcal{W}$, and the points represented by $\mathcal{V}$ define a partition of S.

COMPLEXITY If the polynomials occurring in the input have degrees bounded by D then the complexity of the algorithm is bounded by $k^{O(1)} D^{O(m)}$.

PROCEDURE.

Step 1. Let $g_1(X_1, T) = X_1 g_0(X_1, T)$, and for each $i, 1 \leq i \leq k$, let

$$F_i := \left(\frac{\partial g}{\partial T}\right) \left(\frac{\partial g_i}{\partial X_1} g_0 - g_i \frac{\partial g_0}{\partial X_1}\right) - \left(\frac{\partial g_i}{\partial T} g_0 - g_i \frac{\partial g_0}{\partial T}\right) \left(\frac{\partial g}{\partial X_1}\right)$$

(which is proportional to the projection on the i-th coordinate of the tangent vector to the input curve by the chain rule) and

$$G_i := k F_i^2 - \sum_{j=1}^{k} F_j^2.$$

Step 2. Computing $\mathrm{RElim}_T(G_i, g)$, $1 \leq i \leq k$, using [?, Algorithm 11.19 (Restricted Elimination)], obtain a family $\mathcal{L}$ of polynomials in $\mathrm{D}[T_1, \ldots, T_m, X_1]$. Subdivide (a, b) in a finite union of points and intervals over which the signs of the polynomials in $\mathcal{L}$ are fixed using [?, Algorithm 12.23 (Triangular Sampling Points)] and get $a = c_1 < \ldots < c_L = b$, where each c_j is represented by a Thom encoding (C_j, σ_j) over $t \in \mathrm{R}^m$, such that for each $j, 1 \le j \le L$, there exists an $\ell(j), 1 \le \ell(j) \le k$, such that for all $x_1 \in (c_{j-1}, c_j)$, $G_{\ell(j)}(t, x_1, u(x_1)) \ge 0$, denoting by $u(x_1)$ the root of $g(t, x_1, U)$ with Thom encoding τ. For each j fix an $\ell(j)$ satisfying this property.

Step 3. For each $j, 1 \le j \le L$, reparametrize the segment of the input curve over the interval (c_{j-1}, c_j) using the coordinate $X_{\ell(j)}$. Suppose without loss of generality from here on that $\ell(j) = 2$.

Step 3 a). Set

$$H := g^2 + (X_2 \cdot g_0(T, X_1, U) - g_2(T, X_1, U))^2 \in \mathrm{D}[T, X_1, X_2, U].$$

Note that $\mathrm{Zer}(H(t, -), \mathrm{R}^3)$ is a curve bounded over R (by assumption on the input). Call Algorithm 1 (Curve Segments) with input the polynomial H, and the triangular system $\mathcal{F}, \sigma$, noticing that X_2 is now the parameter.

Step 3 b). For each element

$$(h(T, X_2, V), \sigma_h, H(T, X_2, V)) \in \mathcal{D}_i,$$

where

$$H(T, X_2, V) = (h_0(T, X_2, V), h_1(T, X_2, V), h_2(T, X_2, V)),$$

and $\mathcal{D}_i$ is a set of distinguished points in the output of the call to Algorithm 1 (Curve Segments) in the previous step, use [**?**, Algorithm 12.19 (Triangular Sign Determination)] to check if the point (x_1, x_2, u) represented by $(h, \sigma_h, (h_0, h_1, X_2 h_0, h_2))$ over t, coincides with

$$(x_1, \frac{g_2(t, x_1, u(x_1))}{g_0(t, x_1, u(x_1))}, u(x_1)).$$

Retain only the element

$$(h(T, X_2, V), \sigma_h, H(T, X_2, V)) \in \mathcal{D}_i$$

for which this is the case, and add to the set $\mathcal{V}$ the real univariate representation $u = (h, \sigma_h, G_H)$ (see Notation 4.6) representing a point $v_h \in \mathrm{R}^k$, with parameter X_2 over t.

Step 3 c). For each element

$$(f_1(T, V), \sigma_1, f_2(T, V), \sigma_2, h(T, X_2, V), \sigma_h, H(T, X_2, V)) \in \mathcal{C}_i,$$

where

$$H(T, X_2, V) = (h_0(T, X_2, V), h_1(T, X_2, V), h_2(T, X_2, V))$$

use [**?**, Algorithm 12.19 (Triangular Sign Determination)] to check if the point (x_1, x_2, u) represented by

$$h(T, X_2, V), \sigma_h, (h_0, h_1, X_2 h_0, h_2)$$

over t, coincides with

$$(x_1, \frac{g_2(t, x_1, u(x_1))}{g_0(t, x_1, u(x_1))}, u(x_1))$$

for $x_2 = (v_1 + v_2)/2$ where v_1, v_2 are represented by (f_1, σ_1) and (f_2, σ_2) respectively. Retain only the element of $\mathcal{C}_i$ for which this is the case, and add to the set $\mathcal{W}$ the curve segment, $(f_1, \sigma_1, f_2, \sigma_2, h, \sigma_h, G_H)$, with parameter X_2 over t (see Notation 4.6).

Proof of correctness. Let $(f_1, \sigma_1, f_2, \sigma_2, g, \tau, G)$ be a curve segment parametrized by X_1 over t representing the curve $\gamma : (a, b) \to \mathrm{R}^k$.

Let (c, d) be a sub-interval of (a, b) such that for every $x_1 \in (a, b)$

$$G_\ell(t, x_1, u(x_1)) = kF_\ell^2(t, x_1, u(x_1)) - \sum_{j=1}^{k} F_j^2(t, x_1, u(x_1)) \geq 0. \tag{8.1}$$

(using the notation of Step 1 and Step 2).

Since

$$\left|\frac{\partial \gamma_\ell}{\partial X_1}\right| \geq \frac{1}{\sqrt{k}},$$

the mapping γ_ℓ from (c, d) to (c', d'); with $c' = \gamma_\ell(c), d' = \gamma_\ell(d)$ is invertible. Defining $\bar{\gamma}(x_\ell) = \gamma(\gamma_\ell^{-1}(x_\ell))$, $\bar{\gamma}((c', d')) = \gamma((c, d))$ is well-parametrized by X_ℓ.

Moreover, at each point $x_1 \in (a, b)$ such a choice of ℓ exists, since there must exist an $\ell, 1 \leq \ell \leq k$ such that $\left(\frac{\partial \gamma_\ell}{\partial X_1}\right)^2$ is at least the average value $\frac{1}{k}\sum_{i=1}^{k}\left(\frac{\partial \gamma_i}{\partial X_1}\right)^2$. In Step 2 of the algorithm we obtain a partition of the interval (a, b) into points and open intervals, such that over each sub-interval (c_{j-1}, c_j) of the partition, there exists an index $\ell = \ell(j)$ such that (8.1) is satisfied at each point$v \in (c_{j-1}, c_j)$.

Each curve segment corresponding to elements of $\mathcal{V}$ output by the algorithm is thus well-parametrized. The remaining property of the output is a consequence of the correctness of Algorithm 1 (Curve Segments), and [**?**, Algorithm 12.19 (Triangular Sign Determination)]. □

Complexity analysis.

Let D be a bound on the degrees of the polynomials in the input. The complexity of Steps 1 and 2 is bounded by $k^{O(1)} D^{O(m)}$ from the complexity of [**?**, Algorithm 11.19 (Restricted Elimination)], and [**?**, Algorithm 12.23 (Triangular Sampling Points)], noting that the number of polynomials in $\mathcal{L}$ is bounded by $k^{O(1)} D^{O(m)}$.

In Steps 3-4 the Algorithm 1 (Curve Segments) and [**?**, Algorithm 12.19 (Triangular Sign Determination)] are both called with a constant number of variables in the input. Using the complexity analysis of these algorithms, the total complexity is bounded by $k^{O(1)} D^{O(m)}$. □

8.2.2. *Limit of a curve.* We are now ready to describe Algorithm 4 (Limit of a Curve). The algorithm proceeds by reparametrizing the curve and computing the limit of the well-parametrized curve segments so obtained, as explained below.

PROCEDURE.

Step 1. Let $T = (T_1, \ldots, T_m)$, $X' = (X_1, \ldots, X_r)$. Call a slight variant of [**?**, Algorithm 12.18 (Parametrized Bounded Algebraic Sampling)], pseudo-reducing intermediate computations modulo $\mathcal{F}$ using (4.1), with input

$$\sum_{A \in \mathcal{H}_\varepsilon} A^2 \in \mathrm{D}[\varepsilon, T, X']$$

and parameters ε, T, and output the set $\mathcal{U}_\varepsilon$ of parametrized univariate representations with variable U.

For every $(h_\varepsilon, H_\varepsilon) \in \mathcal{U}_\varepsilon$, use [**?**, Algorithm 12.20 (Triangular Thom Encoding)] with input the triangular system $(\mathcal{F}, h_\varepsilon)$ to compute the Thom encodings of the real roots of $h_\varepsilon(y, U)$. If

$$\mathcal{H}_\varepsilon = (h_{[1]}, \ldots, h_{[r]})$$

with $h_{[i]} \in \mathrm{D}[T, X_1, \ldots, X_i,]$ substitute the variables X' in

$$\bigcup_{1,\ldots,r} \mathrm{Der}_{X_i}(h_{[i]})$$

using H_ε by (4.5) and define a family $\mathcal{A}$ of polynomials in ε, T, U. Using [**?**, Algorithm 12. (Triangular Sign Determination)], compute the signs of the polynomials of $\mathcal{A}$ at the roots of $h_\varepsilon(y, U)$. Comparing the Thom encodings, identify a specific $(h_\varepsilon, \tau_\varepsilon, H_\varepsilon)$ representing z_ε over t.

Then apply Algorithm 3 (Limit of a Bounded Point) with input $(h\varepsilon, \tau_\varepsilon, H_\varepsilon)$ representing z_ε over t to obtain a real univariate representation p_z, ρ_z, P_z representing z over t.

Step 2. Using Algorithm 8 (Reparametrization of a Curve) reparametrize the input curve segment.

Step 3. For every well parametrized curve segment output in Step 2, S_ε, represented by

$$(f_{\varepsilon,1}, \sigma_{\varepsilon,1}, f_{\varepsilon,2}, \sigma_{\varepsilon,2}, g_\varepsilon, \tau_\varepsilon, G_\varepsilon),$$

do the following.

First reorder the variables to ensure that the parameter of S_ε is X_{r+1}.

Then compute a description of $\lim_\varepsilon(S_\varepsilon)$. This process is going to generate a finite list of open intervals and points above which the representation of the restriction of the curve $\lim_\varepsilon(S_\varepsilon)$ by a curve segment is fixed. This is done as follows.

Step 3 a). Denote by $\alpha_{\varepsilon,1}$ the element of $\mathrm{R}\langle\varepsilon\rangle$ represented by $f_{\varepsilon,1}(\varepsilon,T,X',X_{r+1}),\sigma_{\varepsilon,1}$ over (t,z_ε).

Call a slight variant of [**?**, Algorithm 12.18 (Parametrized Bounded Algebraic Sampling)], pseudo-reducing intermediate computations modulo $\mathcal{F}$ using (4.1), with input

$$\sum_{A\in\mathcal{H}_\varepsilon} A^2 + f_{\varepsilon,1}(T,X',X_{r+1})^2 \in \mathrm{D}[\varepsilon,T,X',X_{r+1}]$$

and parameters ε,T, and output a set $\mathcal{U}'_\varepsilon$ of parametrized univariate representations with variable U.

For every $(h_\varepsilon,H_\varepsilon)\in\mathcal{U}'_\varepsilon$, use [**?**, Algorithm 12.20 (Triangular Thom Encoding)] with input the triangular system $(\mathcal{F},h_\varepsilon)$ to compute the Thom encodings of the real roots of $h_\varepsilon(y,U)$.

If

$$\mathcal{H}_\varepsilon = (h_{[1]},\dots,h_{[r]})$$

with $h_{[i]}\in\mathrm{D}[T,X_1,\dots,X_i,]$ substitute the variables X',X_{r+1} in

$$\mathrm{Der}_{X_{r+1}}(f_{\varepsilon,1}(\varepsilon,T,X',X_{r+1}),X_{r+1}\cup\bigcup_{1,\dots,r}\mathrm{Der}_{X_i}(h_{[i]})$$

using (4.5) and define a family $\mathcal{B}$ of polynomials in ε,T,U. Using [**?**, Algorithm 12. (Triangular Sign Determination)], compute the signs of the polynomials of $\mathcal{B}$ at the roots of $h_\varepsilon(y,U)$. Comparing the Thom encodings, identify a specific $(h_\varepsilon,\tau_\varepsilon,H_\varepsilon)$ representing $(z_\varepsilon,\alpha_{\varepsilon,1})$ over t.

Then apply Algorithm 3 (Limit of a Bounded Point) with input $(h_\varepsilon,\tau_\varepsilon,H_\varepsilon)$ representing $(z_\varepsilon,\alpha_{\varepsilon,1})$ over t to obtain a quasi-monic real univariate representation $p_{z,\alpha_1},\rho_{z,\alpha_1},P_{z,\alpha_1}$ representing (z,α_1) over t with $\alpha_1=\lim_\varepsilon(\alpha_{\varepsilon,1})$. Obtain a Thom encoding over t, of α_1 using [**?**, Algorithm 15.1 (Projection)].

Similarly, for $\alpha_{\varepsilon,2}$ the element of $\mathrm{R}\langle\varepsilon\rangle$ represented by $f_{\varepsilon,2}(T,X',X_{r+1}),\sigma_{\varepsilon,2}$ over (t,z_ε), compute a Thom encoding over t, of $\alpha_2=\lim_\varepsilon(\alpha_{\varepsilon,2})$.

Step 3 b). Perform a slight variant of [**?**, Algorithm 12.18 (Parametrized Bounded Algebraic Sampling)], pseudo-reducing intermediate computations modulo $\mathcal{F}$ using (4.1), with input

$$\sum_{A\in\mathcal{H}_\varepsilon} A^2 + g_\varepsilon(T,X',X_{r+1},V)^2 \in \mathrm{D}[\varepsilon,T,X',X_{r+1},U]$$

with parameters ε,T,X_{r+1} and output a set $\mathcal{V}_\varepsilon$ of parametrized univariate representations with parameter ε,T,X_{r+1} and variable V. Denote by $\mathcal{F}_\varepsilon$ the set of polynomials f_ε such that there exists F_ε such that $(f_\varepsilon,F_\varepsilon)\in\mathcal{V}_\varepsilon$. Note that $f_\varepsilon\in\mathrm{D}[\varepsilon,T,X_{r+1},V]$.

Step 3 c). Compute the family of coefficients $\mathcal{C}\subset\mathrm{D}[T,X_{r+1}]$ of the polynomials $f_\varepsilon\in\mathcal{F}_\varepsilon$ considered as elements of $\mathrm{D}[T,X_{r+1}][\varepsilon,V]$ and the list

$\mathcal{L}$ of non-empty conditions $= 0, \neq 0$ satisfied by $\mathcal{C}$ in R using [?, Algorithm 12.23 (Triangular Sample Points)]. Note that for every x_{r+1} in the realization of $\tau \in \mathcal{L}$, the orders in ε of the coefficients of the polynomials in $\mathcal{F}_\varepsilon(t, x_{r+1}) \subset \mathrm{D}[V]$ are fixed. For every $f_\varepsilon \in \mathcal{F}_\varepsilon$ we denote by $o(f_\varepsilon, \tau)$ the maximal order in ε of the coefficients of $f_\varepsilon(t, x_{r+1})$ on the realization of τ and by $\mathcal{F}_\tau \subset \mathrm{D}[T, X_{r+1}, V]$ the set of polynomials obtained by substituting 0 for ε in $\varepsilon^{-o(f_\varepsilon,\tau)} f_\varepsilon$.

Step 3 d). Define

$$\mathcal{F} = \bigcup_{\tau \in \mathcal{L}} \mathcal{F}_\tau \subset \mathrm{D}[T, X_{r+1}, V].$$

Compute

$$\mathcal{E} = \mathcal{C} \bigcup_{f \in \mathcal{F}} \mathrm{RElim}_V(f, \mathrm{Der}(f)) \subset \mathrm{D}[T, X_{r+1}].$$

using [?, Algorithm 11.19 (Restricted Elimination)], so that the Thom encodings of the real roots of $f(t, x_{r+1}, V)$ are fixed when x_{r+1} varies in an open interval defined by the roots of the polynomials $\mathcal{E}(t)$.

Step 3 e). Compute using [?, Algorithm 12.19 (Triangular Sign Determination)] the Thom encodings of the real roots of the polynomials in $\mathcal{E}(t)$, and the ordered list $c_1 < \cdots < c_{h-1}$ of the roots of the polynomials in $\mathcal{E}(t)$ in the interval (c_0, c_h), with $c_0 = \alpha_1, c_h = \alpha_2$. Denote by C_j, ρ_j a polynomial in $\mathcal{E}(t)$ and a Thom encoding representing c_j.

Step 3 f). For every j from 1 to $h-1$ and for every $f \in \mathcal{F}$, determine, using [?, Algorithm 12.19 (Triangular Sign Determination)] the Thom encoding $f(t, c_j, V), \tau_j$ of a root v_j such that $v_j = \lim_\varepsilon(v_\varepsilon)$, where v_ε is the root of $f_\varepsilon(\varepsilon, t, c_j, V)$ with Thom encoding τ_ε. The multiplicity μ_j of the root v_j is determined by τ_j.

Step 3 g). For every j from 1 to h, define $I = (c_{j-1}, c-j)$. For every $f \in \mathcal{F}$ determine, using [?, Algorithm 12.19 (Triangular Sign Determination)] the Thom encoding $f_I(t, x_{r+1}, V), \tau_I$ of a root $v_I(x_{r+1})$, of multiplicity μ_I such that for every $x_{r+1} \in I$, $v_I(x_{r+1}) = \lim_\varepsilon(v_\varepsilon)$ where v_ε is the root of $f_\varepsilon(\varepsilon, t, x_{r+1}, V)$ with Thom encoding τ_ε. The multiplicity μ_I of the root $v_I(x_{r+1})$ is determined by τ_I.

Step 3 h). Given $(f_\varepsilon, F_\varepsilon)$ in $\mathcal{U}_\varepsilon$ denote by $(g_{F_\varepsilon}, G_{F_\varepsilon})$ the $k-r+1$-tuple of polynomials obtained by substituting in $(g_\varepsilon, G_\varepsilon)$ the variables X', U by F_ε (see Notation 4.5). Denote by $\mathcal{V}'_\varepsilon \subset \mathrm{D}[\varepsilon, T, X_j, V]$ the set of $k-r+1$-tuples of polynomials $(g_{F_\varepsilon}, G_{F_\varepsilon})$.

Step 3 i). For every j from 1 to $h-1$ and every $(h_\varepsilon, H_\varepsilon) \in \mathcal{V}'_\varepsilon$, with $H_\varepsilon = (h_{\varepsilon,0}, h_{\varepsilon,r+2}, \ldots, h_{\varepsilon,k})$ determine the order in ε of

$$h_\varepsilon(\varepsilon, t, c_j, v_j), h_{\varepsilon,i}(\varepsilon, t, c_j, v_j).$$

This is done by determining the signs of the coefficient $h_\ell, h_{i,\ell}$ of ε^ℓ in $h(\varepsilon, t, c_j, v_j), h_i(\varepsilon, t, c_j, v_j)$ using [?, Algorithm 12.19 (Triangular Sign Determination)]. Retain those $(h_\varepsilon, H_\varepsilon)$ such that $o(h_{\varepsilon,0}) \geq$

$o(h_{\varepsilon,i})$ for all i from $r+2$ to k and replace ε by 0 in $(\varepsilon^{-o(h_\varepsilon)}h_\varepsilon, \varepsilon^{-o(h_{\varepsilon,0})}H_\varepsilon)$, which defines a set $\mathcal{H}_j$. Inspecting every $(h, H) \in \mathcal{H}_j$, determine, using [**?**, Algorithm 12.19 (Triangular Sign Determination)], a $k-r+1$-tuple (h_j, H_j) with the following property. Let d_j be the point represented by the real univariate representation

$$(h_j(T, X_{r+1}, V), \tau_j, H_j^{(\mu_j-1)}(T, X_{r+1}, V))$$

over t, u. The image under $\lim_\varepsilon$ of the point of S_ε with X_{r+1}-coordinate (c_j) is (z, c_j, d_j).

Step 3 j). For every j from 1 to h define $I = (c_{j-1}, c-j)$. For every $(h_\varepsilon, H_\varepsilon) \in \mathcal{V}'_\varepsilon$, with $H_\varepsilon = (h_{\varepsilon,0}, h_{\varepsilon,r+2}, \ldots, h_{\varepsilon,k})$ subdivide I so that the order in ε of $h_\varepsilon(\varepsilon, t, c_j, v_j)$ and $h_i(\varepsilon, t, x_{r+1}, v_I(x_{r+1}))$ is fixed. This is done by computing

$$\mathcal{E}_I = \bigcup_{f\in\mathcal{F},(h,H_\varepsilon)\in\mathcal{V}'_\varepsilon,0\le\ell\le\deg_\varepsilon h_i} \mathrm{RElim}_V(f, h_\ell) \subset \mathrm{D}[T, X_{r+1}],$$

and

$$\mathcal{E}_{I,i} = \bigcup_{f\in\mathcal{F},(h,H_\varepsilon)\in\mathcal{V}'_\varepsilon,0\le\ell\le\deg_\varepsilon h_i} \mathrm{RElim}_V(f, h_{i,\ell}) \subset \mathrm{D}[T, X_{r+1}],$$

using [**?**, Algorithm 11.19 (Restricted Elimination)].

Defining

$$\mathcal{E}'_I = \mathcal{E}_I \bigcup_{i\in 0,r+2,\ldots,k} \mathcal{E}_{I,i},$$

compute the Thom encodings of the roots of the polynomials in $\mathcal{E}'_I(t)$, using [**?**, Algorithm 12.19 (Triangular Sign Determination)]. On each open interval J between two successive roots, the order in ε, denoted by $o(h_\varepsilon), o(h_{\varepsilon,i})$ of the polynomials

$$h(\varepsilon, t, x_{r+1}, v_J(x_{r+1})), h_i(\varepsilon, t, x_{r+1}, v_J(x_{r+1}))$$

remains fixed. Retain those $(h_\varepsilon, H_\varepsilon)$ such that $o(h_{\varepsilon,0}) \ge o(h_{\varepsilon,i})$ for all i from $r+2$ to k and replace ε by 0 in $\varepsilon^{-o(h_{\varepsilon,0})}(h, H_\varepsilon)$, which defines a set $\mathcal{H}_J$. Inspecting every $(h, H) \in \mathcal{H}_J$, determine, using [**?**, Algorithm 12.19 (Triangular Sign Determination)], a $k-r+1$-tuple (h_J, H_J) such that the point represented by

$$(h_J(t, x_{r+1}, v_I), H_J^{(\mu_J-1)}(t, x_{r+1}, v_I))$$

is the image under $\lim_\varepsilon$ of the point of S_ε with X_{r+1}-coordinate x_{r+1}, where μ_J is the multiplicity of $u_J(x_{r+1})$ as a root of $h_J(x_{r+1}, V)$.

Let w_J be the curve represented by the curve segment representation

$$h_I(T, X_{r+1}, U), \tau_j, H_J^{(\mu_J-1)}(T, X_{r+1}, U)$$

with parameter X_{r+1} over t, u.

Step 3 k). Let $c_1 < \cdots < c_{N-1}$ denote the set of all the elements of R computed in Steps 2 d), and 2 i) above, and $c_N = c$. Re-index each v_j computed in Step 3 h), such that d_j lies above c_j. Similarly, re-index each w_I computed in Step 3 i) by some $j, 1 \le j \le N$, so that w_j lies above the interval (c_{j-1}, c_j).

Output the list of $d_1, \ldots, d_{N-1}$, and $w_1, \ldots, w_N$.

Proof of correctness. Let $\gamma_\varepsilon : (\alpha_{\varepsilon,1}, \alpha_{\varepsilon,2}) \to \mathrm{R}\langle\varepsilon\rangle^k$ be the curve represented by a well-parametrized curve segment

$$f_{\varepsilon,1}, \sigma_{\varepsilon,1}, f_{\varepsilon,2}, \sigma_{\varepsilon,2}, g_\varepsilon, \tau_\varepsilon, G_\varepsilon$$

computed in Step 2.

Let $G : (\alpha_1, \alpha_2) \to \mathrm{R}^k$ be the curve whose image equals the image of γ_ε under $\lim_\varepsilon$. Since the input curve segment is well-parametrized it follows from Proposition 8.7 that in order to compute for any $x_1 \in (c_0, c_N)$, $G(x_1)$ it suffices to compute $\lim_\varepsilon \gamma_\varepsilon(x_1)$. The proof of correctness of the algorithm is then similar to the proof of correctness of Algorithm 3 (Limit of a Bounded Point).

□

Complexity analysis. Let D be a bound on the degrees of all polynomials appearing in the input. We first bound the degrees in the various variables, $\varepsilon, T, X', X_{r+1}, U, V$ of the polynomials computed in various steps of the algorithm. In Step 1, the degrees of the polynomials in $\mathcal{U}_\varepsilon$ are bounded as follows. The degrees in ε, U are bounded by $D^{O(r)}$ by the complexity analysis of [**?**, Algorithm 12.18 (Parametrized Bounded Algebraic Sampling)] and the degrees in the T_i are bounded by D, because of the pseudo-reduction. Moreover, the complexity of this step is bounded by $D^{O(m+r)}$ from the complexity of [**?**, Algorithm 12.18 (Parametrized Bounded Algebraic Sampling)] and the complexity of pseudo-reduction (see Definition 4.2).

The degrees in ε, T_i, X', U in the output of Step 2 are all bounded by $D^{O(1)}$ and the complexity of Step 2 is bounded by

$$(k-r)^{O(1)} D^{O(m+r)} = k^{O(1)} D^{O(m+r)}$$

using the complexity analysis of Algorithm 8 (Reparametrization of a Curve).

The degrees of the polynomials in Step 3 a are bounded as follows. In the output of the call to [**?**, Algorithm 12.18 (Parametrized Bounded Algebraic Sampling)], the degrees in ε, U are bounded by $D^{O(r)}$, and the degrees in the T_i are bounded by D. Now, from the complexity analysis of Algorithm 3 (Limit of a Bounded Point) it follows that the degrees in the T_i of the polynomials output are bounded by D and those in ε, U are bounded by $D^{O(r)}$. Moreover, the complexity of Step 3 a is bounded by $D^{O(m+r)}$ from the complexity of [**?**, Algorithm 12.18 (Parametrized Bounded Algebraic Sampling)], the complexity of Algorithm 3 (Limit of a Bounded Point) and the complexity of pseudo-reduction (see Proposition 8.4).

The degrees of the polynomials in Step 3 b are bounded as follows. In the output of the call to [**?**, Algorithm 12.18 (Parametrized Bounded Algebraic Sampling)], the degrees in ε, X_{r+1}, V are bounded by $D^{O(r)}$, and the degrees in the T_i are bounded by D. The complexity of Step 3 b is bounded by $D^{O(m+r)}$ from the complexity of [**?**, Algorithm 12.18 (Parametrized Bounded Algebraic Sampling)], and the complexity of pseudo-reduction (see Definition 4.2).

The complexity of Step 3 c is bounded by $D^{O(m+r)}$ using the degree bounds from the complexity analysis of the previous steps and the complexity of [**?**, Algorithm 12.23 (Triangular Sample Points)].

It now follows from the complexity analysis of [**?**, Algorithm 12.19 (Triangular Sign Determination)], [**?**, Algorithm 11.19 (Restricted Elimination)], and the degree estimates proved above that the complexity of the remaining steps are all bounded by $k^{O(1)}D^{O(m+r)}$. Thus, the complexity of the algorithm is bounded by $k^{O(1)}D^{O(m+r)}$. □

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