

Bit complexity for computing one point in each connected component of a smooth real algebraic set

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Abstract

We analyze the bit complexity of an algorithm for the computation of at least one point in each connected component of a smooth real algebraic set. This work is a continuation of our analysis of the hypersurface case (*On the bit complexity of finding points in connected components of a smooth real hypersurface*, ISSAC'20). In this paper, we extend the analysis to more general cases.

Let $F = (f_1, \dots, f_p)$ in $\mathbb{Z}[X_1, \dots, X_n]^p$ be a sequence of polynomials with $V = V(F) \subset \mathbb{C}^n$ a smooth and equidimensional variety and $\langle F \rangle \subset \mathbb{C}[X_1, \dots, X_n]$ a radical ideal. To compute at least one point in each connected component of $V \cap \mathbb{R}^n$, our starting point is an algorithm by Safey El Din and Schost (*Polar varieties and computation of one point in each connected component of a smooth real algebraic set*, ISSAC'03). This algorithm uses random changes of variables that are proven to generically ensure certain desirable geometric properties. The cost of the algorithm was given in an algebraic complexity model; here, we analyze the bit complexity and the error probability, and we provide a quantitative analysis of the genericity statements. In particular, we are led to use Lagrange systems to describe polar varieties, as they make it simpler to rely on techniques such as weak transversality and an effective Nullstellensatz.

Keywords— Real algebraic geometry; weak transversality; Noether position; complexity

1 Introduction

Background and problem statement. Computing one point in each connected component of a real algebraic set S is a basic subroutine in real algebraic and semi-algebraic geometry; it is also useful in its own right, since it allows one to decide if S is empty or not.

We consider the case where S is given as $S = V \cap \mathbb{R}^n$, where $V = V(F) \subset \mathbb{C}^n$ is a complex algebraic set defined by a sequence of polynomials $F = (f_1, \dots, f_p)$ in $\mathbb{Z}[X_1, \dots, X_n]^p$. Algorithms for this task have been known for decades, and their complexity is to some extent well understood. Suppose that all f_i 's have degree at most d , and coefficients of bit-size at most b . Without making any assumption on these polynomials, the algorithm given in [8, Section 13.1] solves our problem using $d^{O(n)}$ operations in \mathbb{Q} ; in addition, the output of the

algorithm is represented by polynomials of degree $d^{O(n)}$, with coefficients of bit-size $hd^{O(n)}$. The key idea behind this algorithm goes back to [21]: sample points are found through the computation of critical points of well-chosen functions on V .

The number of connected components of V admits the lower bound $d^{\Omega(n)}$, so up to polynomial factors this result is optimal. However, due to the generality of the algorithm, the constant hidden in the exponent $O(n)$ in its runtime turns out to be rather large: the algorithm relies on infinitesimal deformations, that affect runtime non-trivially.

In this paper, we will work under the additional assumption that $V = V(f_1, \dots, f_p)$ is a *smooth* complex algebraic set, equidimensional of dimension $\delta = n - p$, and that f_1, \dots, f_p generate a radical ideal (we explain these terms in the next section). We place ourselves in the continuation of the line of work initiated by [4]: that reference deals with cases where V is a smooth hypersurface and $V \cap \mathbb{R}^n$ is compact, pointing out how *polar varieties* (that were introduced in the 1930's in order to define characteristic classes [29, 37]) can play a role in effective real geometry. This paper was extended in several directions: to V being a smooth complete intersection, still with $V \cap \mathbb{R}^n$ compact [5], then without the compactness assumption [31, 6]; the smoothness assumption was then partly dropped in [2, 3].

Our starting point is the algorithm in [31], whose assumptions are slightly more general than ours (V is not required to have dimension $\delta = n - p$). In the cases we consider in this paper, its runtime is $\binom{n}{p}^{2+o(1)} d^{(4+o(1))n}$ operations in \mathbb{Q} . As with many results in this vein, the algorithm is randomized, as we need to assume that we are in generic coordinates; this is done by applying a random change of coordinates prior to all computations. In addition, the algorithm relies on procedures for solving systems of polynomial equations that are themselves randomized. Altogether, we choose $n^{O(1)}$ random vectors, each of them in an affine space of dimension $n^{O(1)}$; every time a choice is made, there exists a hypersurface of the parameter space that one has to avoid in order to guarantee success. In this paper, we revisit this algorithm, modify it in part, and give a complete analysis of its probability of success and its bit complexity.

This work is a continuation of the analysis of the hypersurface case that we gave in [16] (that is, the case $p = 1$). A very useful property in the hypersurface case is that polar varieties can be described by straightforward equations (the partial derivatives of the input polynomial) that form a regular sequence, at least in generic coordinates. In higher codimension, this is not the case anymore: the natural description of polar varieties now involves minors of the Jacobian matrix of the input equations (this is the approach used in [31]). The resulting equations are in general not a complete intersection anymore, which makes it impossible to extend directly several arguments we used in [16].

Our solution is to use a description of polar varieties by means of so-called Lagrange equations. These equations are complete intersections (in generic coordinates), but they involve more variables. As such, they describe algebraic sets that cover polar varieties; we will discuss in detail the relationship between these two presentations, using in particular several results from [7, 33].

Data structures. The output of the algorithm is a finite set in $\overline{\mathbb{Q}}^n$. To represent it, we rely on a widely used data structure based on univariate polynomials [26, 27, 17, 20, 1, 18, 19, 30]. For a zero-dimensional algebraic set $S \subset \mathbb{C}^n$ defined over \mathbb{Q} , a *zero-dimensional parameterization* $\mathcal{Q} = ((q, v_1, \dots, v_n), \lambda)$ of S consists in polynomials (q, v_1, \dots, v_n) , such that $q \in \mathbb{Q}[T]$ is monic and squarefree, all v_i 's are in $\mathbb{Q}[T]$ and satisfy $\deg(v_i) < \deg(q)$, and in a \mathbb{Q} -linear form λ in variables X_1, \dots, X_n , such that

- $\lambda(v_1, \dots, v_n) = Tq' \bmod q$;
- we have the equality $S = \left\{ \left(\frac{v_1(\tau)}{q'(\tau)}, \dots, \frac{v_n(\tau)}{q'(\tau)} \right) \mid q(\tau) = 0 \right\}$.

The constraint on λ says that the roots of q are the values taken by λ on S . The parameterization of the coordinates by rational functions having q' as a denominator goes back to [26, 27]: as pointed out in [1], it allows one to control precisely the size of the coefficients of v_1, \dots, v_n .

Main result. To state our main result, we need to define the *height* of a rational number, and of a polynomial with rational coefficients.

The *height* of a non-zero $a = u/v \in \mathbb{Q}$ is the maximum of $\ln(|u|)$ and $\ln(v)$, where $u \in \mathbb{Z}$ and $v \in \mathbb{N}$ are coprime. For a polynomial f with rational coefficients, if $v \in \mathbb{N}$ is the minimal common denominator of all non-zero coefficients of f , then the *height* $\text{ht}(f)$ of f is defined as the maximum of the logarithms of v and of the absolute values of the coefficients of vf .

Theorem 1.1. *Let $F = (f_1, \dots, f_p) \in \mathbb{Z}[X_1, \dots, X_n]^p$ be a sequence of polynomials with $\deg(f_i) \leq d$ and $\text{ht}(f_i) \leq b$. Suppose that the ideal generated by f_1, \dots, f_p is radical and that $V = V(F) \subset \mathbb{C}^n$ is smooth and equidimensional of dimension $n - p$. Also suppose that $0 < \epsilon < 1$.*

There exists a randomized algorithm that takes F and ϵ as input and produces $n - p + 1$ zero-dimensional parameterizations, the union of whose zeros includes at least one point in each connected component of $V(F) \cap \mathbb{R}^n$, with probability at least $1 - \epsilon$. Otherwise, the algorithm either returns a proper subset of the points, or FAIL. In any case, the algorithm uses

$$O^\sim(d^{3n+2p+1} \log(1/\epsilon)(b + \log(1/\epsilon)))$$

bit operations. The polynomials in the output have degree at most d^{n+p} , and height

$$O^\sim(d^{n+p+1}(b + \log(1/\epsilon))).$$

Here we assume that F is given as a sequence of polynomials in dense representation. Following references such as [20, 18, 19, 4, 31], it would be possible to refine the runtime estimate by assuming that F is given by a *straight-line program* (that is, a sequence of operations $+$, $-$, \times that takes as input X_1, \dots, X_n and evaluates F). Any polynomial of degree d in n variables can be computed by a straight-line program that does $O(d^n)$ operations: evaluate all monomials of degree up to d in n variables, multiply them by their respective coefficients

and sum the results. However, some inputs may be given by a shorter straight-line program, and the algorithm would actually benefit from this.

The algorithm itself is rather simple. To describe it, we need to define *polar varieties*, which will play a crucial role in this paper. Let $V = V(F)$, for $F = (f_1, \dots, f_p)$ as in the theorem. For $i \in \{1, \dots, n-1\}$, denote by $\pi_i : \mathbb{C}^n \rightarrow \mathbb{C}^i$ the projection $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_i)$. The i -th *polar variety*

$$W(i, F) := \{\mathbf{x} \in V \mid \dim \pi_i(T_{\mathbf{x}}V) < i\}$$

is the set of critical points of π_i on V . We will recall below that it is defined by the vanishing of all p -minors $M_{i,1}, \dots, M_{i,S_i}$ of the last $n-i$ columns of the Jacobian matrix of F , together with the equations F themselves (here, S_i is simply the binomial number $\binom{n-i}{p}$).

In general, we cannot say much about the geometry of $W(i, F)$, but if we apply a generic change of coordinates \mathbf{A} to F , then $W(i, F)$ is known to be equidimensional of dimension $(i-1)$ or empty [4, 7, 33], and to be in so-called *Noether position* [31] (background notions in algebraic geometry are in [28, 36, 15]; we will recall key definitions). If this is the case, the algorithm in [31] chooses arbitrary $\sigma_1, \dots, \sigma_{n-1}$ in \mathbb{Q} and solves the systems defined by

$$X_1 - \sigma_1 = \dots = X_{i-1} - \sigma_{i-1} = f_1 = \dots = f_p = M_{i,1} = \dots = M_{i,S_i} = 0 \quad (1)$$

for $i = 1, \dots, n-p+1$. They all admit finitely many solutions, and Theorem 2 in [31] proves that the union of their solution sets contains one point on each connected component of $V \cap \mathbb{R}^n$.

One of our contributions is to analyze precisely what conditions on the change of coordinates \mathbf{A} guarantee success. This is done by revisiting the key ingredients in the proofs given in [7, 31], and giving quantitative versions of these results, bounding the degrees of the hypersurfaces we have to avoid.

We actually do not solve the equations (1), since the (large) number of minors S_i makes this analysis difficult. Instead, we replace (1) by equations involving Lagrange multipliers. Proving correctness requires us to guarantee further genericity properties, but once this is done, we can rely on the algorithm in [34] to solve these equations, for which a complete bit complexity analysis is available.

Further work. This paper is an extension of [16], where the analysis was done for the hypersurface case. In addition, this work should also be seen as a step toward the analysis of further randomized algorithms in real algebraic geometry. In particular, randomized algorithms for deciding *connectivity queries* on smooth, compact algebraic sets have been developed in a series of papers [32, 35], and could be revisited using the techniques introduced here. The techniques would apply to algorithms in real algebraic geometry where transversality or Noether position are required geometric properties established by a random change of coordinates.

Outline. The next section summarizes the main concepts from algebraic geometry needed in this paper. In Section 3, we compare the descriptions of determinantal varieties by the

vanishing of matrix minors, and through the use of Lagrange multipliers; these results, while rather simple, are used throughout. A first application is in Section 4, where we give a quantitative form of Thom’s “weak transversality lemma”.

Section 5 introduces polar varieties and discusses the algorithm sketched above and the genericity conditions required for it to succeed. These conditions are studied in detail in Sections 6, 7 and 8; this allows us to complete the analysis of the algorithm in Section 9, thereby proving Theorem 1.1.

2 Preliminaries

In this section, we gather several basic definitions and properties of algebraic sets and locally closed sets. General references for this material are [28, 36, 15].

Algebraic sets. An algebraic set $V \subset \mathbb{C}^n$ is the set of common zeros of an ideal I in $\mathbb{C}[X_1, \dots, X_n]$. Conversely, the ideal of a subset V of \mathbb{C}^n , that is, the set of polynomials in $\mathbb{C}[X_1, \dots, X_n]$ that vanish at all points of V , is called the *ideal* of V ; this is a radical ideal, which we write $I(V)$.

The smallest algebraic set containing an arbitrary set Y is called the *Zariski closure* of Y and written \overline{Y} .

Irreducible decomposition. An algebraic set $V \subset \mathbb{C}^n$ is *irreducible* when $V = V_1 \cup V_2$, with V_1, V_2 algebraic sets, implies $V = V_1$ or $V = V_2$; this is the case if and only if $I(V)$ is prime. An algebraic set $V \subset \mathbb{C}^n$ can be decomposed into a finite union of irreducible algebraic sets

$$V = V_1 \cup V_2 \cup \dots \cup V_r,$$

with $V_i \not\subset V_j$ for all $i \neq j$. The sets V_1, \dots, V_r are called the *irreducible components* of V ; they are uniquely defined, up to order. In terms of ideals, $I(V)$ being radical, it admits a decomposition as an irredundant intersection of prime ideals I_1, \dots, I_r ; the irreducible algebraic sets $V(I_1), \dots, V(I_r)$ are the irreducible components of V .

Dimension. The *dimension* of an algebraic set $V \subset \mathbb{C}^n$, denoted $\dim(V)$, can be defined as the unique integer d such that $V \cap H_1 \cap \dots \cap H_d$ is finite, but not empty, for a generic choice of hyperplanes H_1, \dots, H_d . The *codimension* of V is $n - \dim(V)$.

An algebraic set V is *equidimensional* if each of its irreducible components has the same dimension; if each component has dimension d then we say that V is d -equidimensional.

Degree. We use the definition of degree from [22]: the *degree* $\deg(V)$ of an irreducible algebraic set V is the number of intersection points between itself and $\dim(V)$ generic hyperplanes, and the degree of an arbitrary algebraic set is defined as the sum of the degrees of its irreducible components.

The degree of a hypersurface defined by a squarefree polynomial f is $\deg(f)$. We particularly care about algebraic sets of dimension zero; by definition, these sets are finite and their degree is equal to their cardinality.

We will often apply the Bézout bound from [22, Theorem 1], which says that $\deg(V \cap V') \leq \deg(V) \deg(V')$ holds for all algebraic sets V, V' . A last useful property is that for any linear mapping $\psi : \mathbb{C}^n \rightarrow \mathbb{C}^m$, $\deg(\overline{\psi(V)}) \leq \deg(V)$.

Noether position. Suppose that the ambient dimension n is fixed. For i in $\{1, \dots, n\}$, let π_i denote the projection

$$\begin{aligned} \mathbb{C}^n &\rightarrow \mathbb{C}^i \\ (x_1, \dots, x_n) &\mapsto (x_1, \dots, x_i). \end{aligned}$$

A d -equidimensional algebraic set $V \subset \mathbb{C}^n$ is in *Noether position* for the projection π_d when the extension

$$\mathbb{C}[X_1, \dots, X_d] \rightarrow \mathbb{C}[X_1, \dots, X_n]/I(V)$$

is injective and integral; here, $I(V) \subset \mathbb{C}[X_1, \dots, X_n]$ is the defining ideal of V . It is then a consequence that for any \mathbf{x} in \mathbb{C}^d , the fiber $V \cap \pi_d^{-1}(\mathbf{x})$ has dimension zero and is thus finite and not empty.

Gradient vectors and Jacobian matrices. The gradient vector of a polynomial $f \in \mathbb{C}[X_1, \dots, X_n]$ is written $\text{grad}(f) \in \mathbb{C}[X_1, \dots, X_n]^{1 \times n}$ (so this is a row vector). Most of the time, the variables with respect to which we differentiate are clear from the context, but we may write $\text{grad}_{\mathbf{X}}(f)$ for clarity, with $\mathbf{X} = X_1, \dots, X_n$.

The Jacobian matrix of polynomials $F = f_1, \dots, f_s$ is the $s \times n$ matrix $\text{jac}(F)$, with $\partial f_i / \partial X_j$ at entry (i, j) , for $1 \leq i \leq s$ and $1 \leq j \leq n$. As we do for gradients, we will write $\text{jac}_{\mathbf{X}}(F)$ if we want to highlight what variables we differentiate with respect to.

Given \mathbf{x} in \mathbb{C}^n , we then write $\text{grad}(f, \mathbf{x})$, resp. $\text{jac}(F, \mathbf{x})$, for the evaluation of respectively $\text{grad}(f)$ and $\text{jac}(F)$ at \mathbf{x} .

Tangent spaces, regular and singular points. Assume that $V \subset \mathbb{C}^n$ is a d -equidimensional algebraic set. The *Zariski-tangent space* to V at $\mathbf{x} \in V$ is the vector space $T_{\mathbf{x}}V \subset \mathbb{C}^n$ defined by the equations

$$\text{grad}(g, \mathbf{x}) \cdot \mathbf{v} = 0 \text{ for all } g \in I(V), \quad \mathbf{v} \in \mathbb{C}^{n \times 1}.$$

Then, the point $\mathbf{x} \in V$ is a *regular point* (or non-singular) if $\dim(T_{\mathbf{x}}V) = d$; otherwise, \mathbf{x} is a *singular point*. We let $\text{reg}(V)$ and $\text{sing}(V)$ respectively denote the regular and singular points of V ; when the latter is empty, we say that V is *smooth*. If $I(V)$ is generated by polynomials $G = (g_1, \dots, g_s) \in \mathbb{C}[X_1, \dots, X_n]^s$, then at any point \mathbf{x} of $\text{reg}(V)$, the Jacobian matrix $\text{jac}(G, \mathbf{x})$ has rank $n - d$ and the right kernel of $\text{jac}(G, \mathbf{x})$ is $T_{\mathbf{x}}V$.

Changes of variables. For a matrix \mathbf{A} in $\mathbb{C}^{n \times n}$ and a polynomial g in $\mathbb{C}[X_1, \dots, X_n]$, we write

$$g^{\mathbf{A}} := g(\mathbf{A}\mathbf{X}) \in \mathbb{C}[X_1, \dots, X_n],$$

where \mathbf{X} is the column vector with entries X_1, \dots, X_n . Similarly, for a sequence of polynomials $G = (g_1, \dots, g_s)$ in $\mathbb{C}[X_1, \dots, X_n]^s$, we write $G^{\mathbf{A}} = (g_1^{\mathbf{A}}, \dots, g_s^{\mathbf{A}})$. For an algebraic set $V \subset \mathbb{C}^n$ and a matrix $\mathbf{A} \in \text{GL}(n)$, we define $V^{\mathbf{A}}$ as the image of V by the map $\phi_{\mathbf{A}} : \mathbf{x} \mapsto \mathbf{A}^{-1}\mathbf{x}$. Notice in particular that $V(G^{\mathbf{A}}) = \phi_{\mathbf{A}}(V(G)) = V(G)^{\mathbf{A}}$.

Locally closed sets. We will also need to work with *locally closed* sets: we say that $Y \subset \mathbb{C}^n$ is locally closed if we can write it as $Y = V - V'$, for some algebraic sets V, V' .

The notions of dimension and equidimensionality carry over to this context (they are defined through the Zariski closure of Y), as does that of tangent space: for \mathbf{x} in Y , we set $T_{\mathbf{x}}Y = T_{\mathbf{x}}V$ (this is independent of the choice of V, V' in the definition above). If Y is equidimensional, as we did for algebraic sets, we can then define the *regular points* (or non-singular points) of Y as those points at which the tangent space has dimension d , and we say that Y is smooth if all its points are regular.

Open sets are locally closed. As another example, for any d -equidimensional algebraic set V , $\text{reg}(V)$ is a smooth d -equidimensional locally closed set.

3 Describing determinantal varieties

In this section, we work with polynomials in $\mathbb{C}[Y_1, \dots, Y_N]$, for some positive integer N . Given a matrix \mathbf{A} in $\mathbb{C}[Y_1, \dots, Y_N]^{q \times r}$, with $q \leq r$, together with some equations $B = (b_1, \dots, b_s)$ in $\mathbb{C}[Y_1, \dots, Y_N]$, we consider the locus S defined as

$$S = \{\mathbf{y} \in \mathbb{C}^N \mid b_1(\mathbf{y}) = \dots = b_s(\mathbf{y}) = 0 \text{ and } \text{rank}(\mathbf{A}(\mathbf{y})) < q\}.$$

One of our goals here is to give a degree bound for S ; this will be used twice, in the next section for our discussion of the weak transversality lemma (in a slightly more general context where we work in an open subset of \mathbb{C}^N), then also to control the degrees of the systems of equations we will solve.

Consider the polynomials

$$\mathfrak{J}(\mathbf{A}, B) = (b_1, \dots, b_s, M_1, \dots, M_P),$$

where M_1, \dots, M_P are the q -minors of \mathbf{A} , with $P = \binom{r}{q}$. Since we have $V(\mathfrak{J}(\mathbf{A}, B)) = S$, we may derive a degree bound on S using the Bézout inequality. However, even the refined form given in [23, Proposition 2.3] involves an exponential dependency in either the ambient dimension N or the number of minors P . This might be acceptable in some contexts (such as when estimating the degrees of polar varieties), but is way beyond our target bound in the context of weak transversality, for instance.

Instead, we use Lagrange systems. We let L_1, \dots, L_q be new variables, thought of as Lagrange multipliers, and consider the ‘‘Lagrange polynomials’’ given as the r entries of $[L_1 \cdots L_q] \cdot \mathbf{A}$. We denote by $Z \subset \mathbb{C}^{N+q}$ the algebraic set defined by the vanishing of

$$(b_1, \dots, b_s, [L_1 \cdots L_q] \cdot \mathbf{A})$$

and by Z' the algebraic set

$$Z' := \overline{Z - \{(\mathbf{y}, 0, \dots, 0) \in \mathbb{C}^{N+q} \mid (\mathbf{y}, 0, \dots, 0) \in Z\}},$$

where the bar denotes Zariski closure (we have to remove such points, since $L_1 = \cdots = L_q = 0$ is always a trivial solution to the Lagrange equations). Finally, consider the projection

$$\begin{aligned} \mu : \mathbb{C}^{N+q} &\rightarrow \mathbb{C}^N \\ (\mathbf{y}, \boldsymbol{\ell}) &\mapsto \mathbf{y}. \end{aligned}$$

It is then possible to prove that S is the Zariski closure of $\mu(Z')$, and derive degree bounds using the equations defining Z . However, while introducing Z' is convenient, computing defining equations for it is non-trivial, as it involves saturation; besides, in several contexts, it will be advantageous to work with equations in complete intersection, which the following construction will guarantee in certain cases. For $\mathbf{u} = (u_1, \dots, u_q) \in \mathbb{C}^q$, consider the equations

$$\mathfrak{L}(\mathbf{A}, B, \mathbf{u}) = (b_1, \dots, b_s, [L_1 \cdots L_q] \cdot \mathbf{A}, u_1 L_1 + \cdots + u_q L_q - 1),$$

and let $Z_{\mathbf{u}} \subset \mathbb{C}^{N+q}$ be its zero-set. Using the linear equation $u_1 L_1 + \cdots + u_q L_q - 1$ allows us to discard solutions where $L_1 = \cdots = L_q = 0$, but unlucky choices of \mathbf{u} may discard other components as well. The following proposition makes this more precise.

Proposition 3.1. *For any \mathbf{u} in \mathbb{C}^q , we have the inclusion $\mu(Z_{\mathbf{u}}) \subset S$. There exists a non-empty open set $\mathcal{O} \subset \mathbb{C}^q$ such that for \mathbf{u} in \mathcal{O} , we have the inclusion $S \subset \overline{\mu(Z_{\mathbf{u}})}$, and thus the equalities $S = \overline{\mu(Z_{\mathbf{u}})}$ and*

$$\sqrt{\langle \mathfrak{L}(\mathbf{A}, B, \mathbf{u}) \rangle \cap \mathbb{C}[Y_1, \dots, Y_N]} = \sqrt{\langle \mathfrak{J}(\mathbf{A}, B) \rangle}.$$

The set \mathcal{O} is the complement of at most $\deg(S)$ hyperplanes.

Proof. If $(\mathbf{y}, \boldsymbol{\ell})$ cancels all polynomials in $\mathfrak{L}(\mathbf{A}, B, \mathbf{u})$, then $\boldsymbol{\ell}$ is non-zero, so that $\mathbf{A}(\mathbf{y})$ is rank-deficient. As a consequence, \mathbf{y} is in S . This proves the first assertion.

For the second one, let S_1, \dots, S_K be the irreducible components of S . For any given k in $\{1, \dots, K\}$, since all q -minors of \mathbf{A} vanish on S_k , they vanish in the function field $\mathbb{C}(S_k)$, so \mathbf{A} has rank less than q as a matrix over $\mathbb{C}(S_k)$. Thus, there exists a non-zero vector of rational functions

$$\boldsymbol{\ell}_k = (\ell_{k,1}, \dots, \ell_{k,q}) = \left(\frac{N_{k,1}}{D_k}, \dots, \frac{N_{k,q}}{D_k} \right) \in \mathbb{C}(S_k)^q,$$

such that $\ell_k \cdot \mathbf{A} = 0$ in $\mathbb{C}(S_k)^r$ (here, we see ℓ_k in $\mathbb{C}(S_k)^{1 \times q}$). For definiteness, assume that $N_{k,t_k} \neq 0$. Then, in particular, $S'_k = S_k - V(D_k N_{k,t_k})$ is dense in S_k ; for \mathbf{y} in S'_k , $\ell_k(\mathbf{y})$ is well-defined, non-zero, and still satisfies $\ell_k(\mathbf{y}) \cdot \mathbf{A}(\mathbf{y}) = 0$.

Then, pick a point \mathbf{y}_k in S'_k , so that $\ell_k(\mathbf{y}_k)$ is a well-defined, non-zero vector in \mathbb{C}^q . This allows us to define a non-empty Zariski open set $\mathcal{O}_k \subset \mathbb{C}^q$ by the condition

$$\mathcal{O}_k := \{\mathbf{u} \in \mathbb{C}^q \mid \ell_k(\mathbf{y}_k) \cdot \mathbf{u} \neq 0\},$$

where in the dot product we take $\ell_k(\mathbf{y}_k)$ in $\mathbb{C}^{1 \times q}$ and \mathbf{u} in $\mathbb{C}^{q \times 1}$. Finally, we let $\mathcal{O} := \bigcap_{1 \leq k \leq K} \mathcal{O}_k$, which is open, non-empty, and defined as the complement of $K \leq \deg(S)$ hyperplanes, as claimed. We now prove that for \mathbf{u} in \mathcal{O} , the inclusion $S \subset \overline{\mu(Z_{\mathbf{u}})}$ holds.

For this, we take k as above, and we prove that S_k is contained in $\overline{\mu(Z_{\mathbf{u}})}$. Consider the rational mapping

$$\begin{aligned} S'_k &\rightarrow \mathbb{C} \\ \mathbf{y} &\mapsto \ell_k(\mathbf{y}) \cdot \mathbf{u} = \frac{u_1 N_{k,1}(\mathbf{y}) + \cdots + u_q N_{k,q}(\mathbf{y})}{D_k(\mathbf{y})}. \end{aligned}$$

Put $S''_k = S'_k - V(u_1 N_{k,1} + \cdots + u_q N_{k,q})$; this is again an open subset of S_k , and the fact that $\ell_k(\mathbf{y}_k) \cdot \mathbf{u}$ is non-zero, with \mathbf{y}_k in S'_k , shows that S''_k is not empty. In particular, it is dense in S_k .

Take \mathbf{y} in S''_k . Then, $\alpha := \ell_k(\mathbf{y}) \cdot \mathbf{u}$ is non-zero, so we can define $\ell' := 1/\alpha \ell_k(\mathbf{y})$. Then, ℓ' is still in the left nullspace of $\mathbf{A}(\mathbf{y})$, and by construction $\ell' \cdot \mathbf{u} = 1$, so that (\mathbf{y}, ℓ') is in $Z_{\mathbf{u}}$. In other words, S''_k is contained in $\mu(Z_{\mathbf{u}})$. Taking the Zariski closure, we obtain that S_k is contained in $\overline{\mu(Z_{\mathbf{u}})}$, as claimed. The equality $S = \overline{\mu(Z_{\mathbf{u}})}$ follows, as does the claimed equality between ideals. \square

Corollary 3.2. *If all polynomials b_1, \dots, b_s have respective degrees at most d_1, \dots, d_s , and all entries of \mathbf{A} have degree at most d' , then the degree of S is at most $d_1 \cdots d_s (d' + 1)^r$.*

Proof. Choose \mathbf{u} in the set \mathcal{O} of the previous lemma. The algebraic set $Z_{\mathbf{u}}$ is defined by s equations of respective degrees at most d_1, \dots, d_s , r equations of degree at most $d' + 1$ and a linear equation. It follows from Bézout's Theorem [22] that $\deg(Z_{\mathbf{u}}) \leq d_1 \cdots d_s (d' + 1)^r$. Degree does not increase through projection, so the conclusion follows from the previous lemma. \square

Remark 3.3. *In the next section, we will consider the following slight variant of the problem considered here, where we are interested in the locally closed set*

$$S' = \{\mathbf{y} \in \Omega \mid b_1(\mathbf{y}) = \cdots = b_s(\mathbf{y}) = 0 \text{ and } \text{rank}(\mathbf{A}(\mathbf{y})) < q\},$$

for some Zariski open set $\Omega \subset \mathbb{C}^N$. The Zariski closure $\overline{S'}$ is the union of certain irreducible components of the set S defined above, so the degree bound of Corollary 3.2 still holds for $\overline{S'}$.

We note that in some cases, sharper bounds are known for the degrees of determinantal varieties: for instance, when X is finite, and defined as the set of critical points on a smooth algebraic set [39], or when we want to take into account differences in the degrees of the rows and columns of \mathbf{A} [38, 40, 41].

4 Weak transversality

Several of the generic properties of polar varieties are consequences of *weak transversality*, which is an important extension of Sard’s lemma due to Thom (this observation goes back to work of Giusti, Heintz and collaborators [4, 3]). In this section, we develop a quantitative extension of Thom’s weak transversality theorem, specialized to the particular case of transversality to a point. In the sequel, we will apply this result to bound the degree of particular hypersurfaces our algorithm needs to avoid to guarantee success.

4.1 Definitions and statement of the result

In its differential version, Sard’s lemma states that the set of critical values of a smooth function $\mathbb{R}^n \rightarrow \mathbb{R}^m$ has measure zero; extensions exist to smooth mapping between differential manifolds. In our algebraic context, we will use the following definitions.

Consider a polynomial mapping $\Phi : Y \rightarrow \mathbb{C}^m$ from a smooth n -equidimensional locally closed set Y to \mathbb{C}^m , with $m \leq n$. A *critical point* of Φ is a point $\mathbf{y} \in Y$ for which the image of the tangent space $T_{\mathbf{y}}Y$ by the Jacobian matrix $\text{jac}(\Phi, \mathbf{y})$ has dimension less than m . For instance, the case that will interest us in this section is when Y is Zariski open in \mathbb{C}^n , in which case we have $T_{\mathbf{y}}Y = \mathbb{C}^n$ for all \mathbf{y} in Y , and the condition is equivalent to the Jacobian of Φ having rank less than m at \mathbf{y} . *Critical values* are the images by Φ of critical points; the complement of this set are the *regular values*. Notice then, a regular value is not necessarily in the image of Φ .

One can then give “algebraic” versions of Sard’s lemma: for instance, [28, (3.7)] shows that for Y an irreducible algebraic set and Φ dominant, the critical values of Φ are contained in a strict algebraic subset of \mathbb{C}^m ; below, we will rely on a straightforward generalization given in [33]. See also [10, Chapter 9] for the semi-algebraic case.

Thom’s weak transversality lemma, as given for instance in [13], generalizes Sard’s lemma. In this section, we consider a particular case of this result (transversality to a point), and establish a quantitative version of it.

Let n, s , and m be positive integers, with $m \leq n$ as before, let \mathcal{O} be a Zariski open subset of \mathbb{C}^n , and denote by $\Phi : \mathcal{O} \times \mathbb{C}^s \rightarrow \mathbb{C}^m$ a mapping given by polynomials in $n + s$ indeterminates $X_1, \dots, X_n, \Theta_1, \dots, \Theta_s$ (the latter should be thought of as parameters). For ϑ in \mathbb{C}^s , we let $\Phi_{\vartheta} : \mathcal{O} \rightarrow \mathbb{C}^m$ be the induced mapping $\mathbf{x} \mapsto \Phi(\mathbf{x}, \vartheta)$. Thom’s weak transversality lemma tells us that if 0 is a regular value of the mapping Φ , then 0 remains a regular value of the induced mapping Φ_{ϑ} for a generic ϑ . (Here, we are dealing with the particular case of transversality to a point, which can be rephrased entirely in terms of regular and critical values.) Our quantitative version of this result is the following.

Proposition 4.1 (Weak transversality). *Let all notation be as before, and suppose that Φ is defined by m polynomials of degree at most d . If 0 is a regular value of Φ , there exists a non-zero polynomial $\Gamma \in \mathbb{C}[\Theta_1, \dots, \Theta_s]$ of degree at most d^{m+n} such that for ϑ in \mathbb{C}^s , if $\Gamma(\vartheta) \neq 0$, then 0 is a regular value of Φ_{ϑ} .*

Example 4.2. Consider a squarefree polynomial f in $\mathbb{C}[X_1, X_2]$, with degree at most d , defining a smooth curve $V(f)$ in \mathbb{C}^2 , and let the mapping $\Phi : \mathbb{C}^2 \times \mathbb{C} \rightarrow \mathbb{C}^2$ be defined by $\Phi(X_1, X_2, \Theta) = (f(X_1, X_2), X_1 - \Theta)$ (so $m = n = 2$ and $s = 1$). One checks that the Jacobian of Φ with respect to (X_1, X_2, Θ) has full rank two at any point in $\Phi^{-1}(0)$, so that 0 is a regular value of Φ and therefore the assumptions of the proposition apply.

We then deduce that a non-zero polynomial $\Gamma \in \mathbb{C}[\Theta]$ exists, with degree at most d^4 with the property that, if ϑ in \mathbb{C} does not cancel Γ then 0 is a regular value of the induced mapping Φ_ϑ . In particular, for all ϑ in \mathbb{C} except at most d^4 values, the ideal $(f(X_1, X_2), X_1 - \vartheta)$ is radical in $\mathbb{C}[X_1, X_2]$; equivalently, $f(\vartheta, X_2)$ is squarefree.

In this example, we could of course obtain the same result (with a sharper degree bound) by considering the discriminant of f with respect to X_2 , but the construction above will be useful later on, in a generalized form. (In this example, the bound d^4 could be sharpened by utilizing the fact that only one of the polynomials defining Φ has degree d , whereas the other one is linear.)

The rest of the section is devoted to the proof of the proposition. The proof of [33, Theorem B.3] already shows the existence of Γ ; it is essentially the classical proof for smooth mappings [13, Section 3.7], written in an algebraic context. In what follows, we revisit this proof, establishing a bound on the degree of Γ .

4.2 Proof of the proposition

In what follows, we use the notation of Proposition 4.1, so that we consider m polynomials Φ that depend on variables X_1, \dots, X_n and $\Theta_1, \dots, \Theta_s$, with $m \leq n$, and an open set $\mathcal{O} \subset \mathbb{C}^n$.

In the context of Thom's weak transversality, the "bad" parameter values show up as the critical values of a certain projection. Put $Y = \Phi^{-1}(0) \cap (\mathcal{O} \times \mathbb{C}^s)$, and let V be the Zariski closure of Y . If Y is empty, there is nothing to do, since all values ϑ in \mathbb{C}^s satisfy the conclusion of the proposition. We therefore assume that Y is not empty. Take (\mathbf{x}, ϑ) in Y ; then by assumption, $\text{jac}(\Phi, (\mathbf{x}, \vartheta))$ has full rank m . Since in a neighbourhood of (\mathbf{x}, ϑ) , V coincides with $Y = \Phi^{-1}(0) \cap (\mathcal{O} \times \mathbb{C}^s)$, the Jacobian criterion [15, Corollary 16.20] implies that there is a unique irreducible component $V_{(\mathbf{x}, \vartheta)}$ of V that contains (\mathbf{x}, ϑ) , that (\mathbf{x}, ϑ) is regular on this component and that $\dim V_{(\mathbf{x}, \vartheta)} = n + s - m$. This implies that Y is a smooth, $(n + s - m)$ -equidimensional locally closed set. Now, consider the projection

$$\begin{aligned} \pi : \mathbb{C}^{n+s} &\rightarrow \mathbb{C}^s \\ (\mathbf{x}, \vartheta) &\mapsto \vartheta, \end{aligned}$$

and let Z be the set of critical points of the restriction $\pi|_Y$ of π to Y ; that is,

$$Z := \{(\mathbf{x}, \vartheta) \in Y \mid \dim(\pi(T_{\mathbf{x}, \vartheta} Y)) < s\}.$$

The projection $\pi(Z) \subset \mathbb{C}^s$ is thus the set of critical values of $\pi|_Y$.

Lemma 4.3. *The Zariski closure $\overline{\pi(Z)}$ is a strict subset of \mathbb{C}^s .*

Proof. The discussion above implies that $\text{reg}(V)$ is a smooth, $(n + s - m)$ -equidimensional locally closed set containing Y . Let then Z' be the critical points of $\pi|_{\text{reg}(V)}$; by the algebraic form of Sard's lemma as given in [28, Theorem 3.7] (for irreducible V) and [33, Proposition B.2] (for general V), the Zariski closure $\overline{\pi(Z')}$ is a strict closed subset of \mathbb{C}^s . Now, at any point $(\mathbf{x}, \boldsymbol{\vartheta})$ of Y , the tangent spaces $T_{(\mathbf{x}, \boldsymbol{\vartheta})}Y$ and $T_{(\mathbf{x}, \boldsymbol{\vartheta})}\text{reg}(V)$ coincide. As a result, Z is contained in Z' , and the claim follows. \square

We can now explain how $\boldsymbol{\vartheta}$ being a regular value of $\pi|_Y$ relates to 0 being a regular value of $\Phi_{\boldsymbol{\vartheta}}$. In what follows, we write our indeterminates as blocks of variables, with $\mathbf{X} = X_1, \dots, X_n$ and $\boldsymbol{\Theta} = \Theta_1, \dots, \Theta_s$. When not explicitly mentioned, Jacobian matrices involve derivatives with respect to both \mathbf{X} and $\boldsymbol{\Theta}$.

Lemma 4.4. *For $(\mathbf{x}, \boldsymbol{\vartheta})$ in Y , $(\mathbf{x}, \boldsymbol{\vartheta})$ is in Z if and only if $\text{jac}_{\mathbf{X}}(\Phi, (\mathbf{x}, \boldsymbol{\vartheta}))$ has rank less than m .*

Proof. Let \mathbf{M} denote the $(s + m) \times (s + n)$ Jacobian matrix of π and Φ with respect to X_1, \dots, X_n and $\Theta_1, \dots, \Theta_s$, that is,

$$\mathbf{M} = \begin{bmatrix} \text{jac}(\pi) \\ \text{jac}(\Phi) \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{s \times n} & \mathbf{I}_s \\ \text{jac}(\Phi) & \end{bmatrix}.$$

Take $(\mathbf{x}, \boldsymbol{\vartheta})$ on Y . Then, the rank of $\mathbf{M}(\mathbf{x}, \boldsymbol{\vartheta})$ can be written as $\text{rank}(\text{jac}(\Phi, (\mathbf{x}, \boldsymbol{\vartheta}))) + \text{rank}([\mathbf{0}_{s \times n} \ \mathbf{I}_s] | \ker \text{jac}(\Phi, (\mathbf{x}, \boldsymbol{\vartheta})))$, where the latter is the rank of the restriction of $[\mathbf{0}_{s \times n} \ \mathbf{I}_s]$ to the nullspace of $\text{jac}(\Phi, (\mathbf{x}, \boldsymbol{\vartheta}))$.

Since $(\mathbf{x}, \boldsymbol{\vartheta})$ is in Y and since 0 is a regular value of Φ , $\text{jac}(\Phi, (\mathbf{x}, \boldsymbol{\vartheta}))$ has full rank m . On the other hand, the nullspace of that matrix is the tangent space $T_{\mathbf{x}, \boldsymbol{\vartheta}}Y$, and $\text{rank}([\mathbf{0}_{s \times n} \ \mathbf{I}_s] | \ker \text{jac}(\Phi, (\mathbf{x}, \boldsymbol{\vartheta})))$ is the dimension of $\pi(T_{\mathbf{x}, \boldsymbol{\vartheta}}Y)$. In other words, the rank of $\mathbf{M}(\mathbf{x}, \boldsymbol{\vartheta})$ is equal to $m + \dim(\pi(T_{\mathbf{x}, \boldsymbol{\vartheta}}Y))$.

This proves that for $(\mathbf{x}, \boldsymbol{\vartheta})$ in Y , $(\mathbf{x}, \boldsymbol{\vartheta})$ is in Z if and only if the matrix \mathbf{M} has rank less than $s + m$ at $(\mathbf{x}, \boldsymbol{\vartheta})$. Now, notice that

$$\mathbf{M}(\mathbf{x}, \boldsymbol{\vartheta}) = \begin{bmatrix} \mathbf{0}_{s \times n} & \mathbf{I}_s \\ \text{jac}_{\mathbf{X}}(\Phi, (\mathbf{x}, \boldsymbol{\vartheta})) & \text{jac}_{\boldsymbol{\Theta}}(\Phi, (\mathbf{x}, \boldsymbol{\vartheta})) \end{bmatrix}.$$

This shows that the rank of $\mathbf{M}(\mathbf{x}, \boldsymbol{\vartheta})$ equals $s + \text{rank}(\text{jac}_{\mathbf{X}}(\Phi, (\mathbf{x}, \boldsymbol{\vartheta})))$, and the lemma follows. \square

As a result, suppose we take $\boldsymbol{\vartheta}$ in $\mathbb{C}^s - \pi(Z)$. Then for all \mathbf{x} in $\Phi_{\boldsymbol{\vartheta}}^{-1}(0) \cap \mathcal{O}$, $(\mathbf{x}, \boldsymbol{\vartheta})$ is in Y , so it is not in Z ; the previous lemma then implies that the Jacobian matrix of $\Phi_{\boldsymbol{\vartheta}}$, which is $\text{jac}_{\mathbf{X}}(\Phi, (\mathbf{X}, \boldsymbol{\vartheta}))$, has full rank m at \mathbf{x} . In other words, 0 is a regular value of $\Phi_{\boldsymbol{\vartheta}}$ in the open set \mathcal{O} . To prove Proposition 4.1, it is thus enough to establish the existence of a non-zero polynomial of degree at most d^{m+n} that vanishes on $\overline{\pi(Z)}$. We already established that $\overline{\pi(Z)}$ is a strict subset of \mathbb{C}^s , so the only missing ingredient is to prove that it has degree at most d^{m+n} .

We start by bounding above the degree of \overline{Z} . The previous lemma shows the equality

$$Z = \{(\mathbf{x}, \boldsymbol{\vartheta}) \in \Omega \times \mathbb{C}^s \mid \Phi(\mathbf{x}, \boldsymbol{\vartheta}) = 0 \text{ and } \text{rank}(\text{jac}_{\mathbf{X}}(\Phi, (\mathbf{x}, \boldsymbol{\vartheta}))) < m\}.$$

Since all polynomials in Φ have degree at most d , and all entries of $\text{jac}_{\mathbf{X}}(\Phi)$ at most $d-1$, we can apply Corollary 3.2, so as to deduce that $\deg(\overline{Z}) \leq d^{m+n}$. This implies that $\pi(\overline{Z})$ has degree at most d^{m+n} , and the equality $\overline{\pi(\overline{Z})} = \overline{\pi(Z)}$ allows us to conclude the proof.

5 Overview of the main algorithm

Let $F = (f_1, \dots, f_p)$ be a sequence of polynomials in $\mathbb{C}[X_1, \dots, X_n]$. Suppose that the ideal $\langle F \rangle \subset \mathbb{C}[X_1, \dots, X_n]$ is radical and that $V(F)$ is smooth and equidimensional of dimension $\delta = n - p$.

In this section, we give a high-level description of an algorithm from [31] that computes at least one point in each connected component of $V(F) \cap \mathbb{R}^n$. Correctness of this algorithm was established in [31] provided we are in generic coordinates: the algorithm solves a family of systems of equations that describe points on the *polar varieties* of $V(F)$, and being in generic coordinates ensures several desirable properties for these polar varieties.

After a brief review of the basic properties of polar varieties, we sketch the main algorithm and highlight what properties are needed for its correctness (the next sections will give quantitative statements regarding the genericity of these properties). In that, we mainly follow [31], but we also introduce requirements related to Lagrange systems, as introduced in Section 3, as they will be of help in further sections.

5.1 Polar varieties

Let F be as in the preamble and let $V = V(F)$. Recall that, for $i \in \{1, \dots, n\}$, we denote by π_i the projection

$$\begin{aligned} \mathbb{C}^n &\rightarrow \mathbb{C}^i \\ (x_1, \dots, x_n) &\mapsto (x_1, \dots, x_i). \end{aligned}$$

For $i \leq \delta$, the i -th *polar variety* $W(i, F)$ is the set of critical points of the restriction of π_i to V , that is,

$$W(i, F) := \{\mathbf{x} \in V \mid \dim \pi_i(T_{\mathbf{x}}V) < i\}.$$

We naturally extend this definition to $i = \delta + 1$, by setting $W(\delta + 1, F) = V$.

For $1 \leq i \leq \delta + 1$, let $\text{jac}(F)$, resp. $\text{jac}(F, i)$, denote the Jacobian matrix of $F = (f_1, \dots, f_p)$ with respect to (X_1, \dots, X_n) , resp. to (X_{i+1}, \dots, X_n) :

$$\text{jac}(F) = \begin{bmatrix} \frac{\partial f_1}{\partial X_1} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \vdots & & \vdots \\ \frac{\partial f_p}{\partial X_1} & \cdots & \frac{\partial f_p}{\partial X_n} \end{bmatrix}, \quad \text{jac}(F, i) = \begin{bmatrix} \frac{\partial f_1}{\partial X_{i+1}} & \cdots & \frac{\partial f_1}{\partial X_n} \\ \vdots & & \vdots \\ \frac{\partial f_p}{\partial X_{i+1}} & \cdots & \frac{\partial f_p}{\partial X_n} \end{bmatrix}.$$

Since F generates a radical ideal, for any \mathbf{x} in V , the tangent space $T_{\mathbf{x}}(V)$ is the kernel of $\text{jac}(F, \mathbf{x})$; the assumption that V be δ -equidimensional and smooth implies that this kernel has dimension $\delta = n - p$ at all such \mathbf{x} . It follows that we can rephrase the definition of $W(i, F)$ as

$$W(i, F) = \{\mathbf{x} \in \mathbb{C}^n \mid f_1(\mathbf{x}) = \dots = f_p(\mathbf{x}) = 0 \text{ and } \text{rank}(\text{jac}(F, i, \mathbf{x})) < p\}.$$

Let $P_i = \binom{n-i}{p}$ be the number of p -minors in $\text{jac}(F, i)$, and let $M_{i,1}, \dots, M_{i,P_i}$ be these minors (for $i = \delta + 1$, $P_{\delta+1} = 0$ since $\text{jac}(F, \delta + 1)$ has size $p \times (p - 1)$). Then, as in Section 3, we deduce that $W(i, F)$ is defined by the polynomials

$$\mathfrak{J}(i, F) = (f_1, \dots, f_p, M_{i,1}, \dots, M_{i,P_i}). \quad (2)$$

The downside to defining polar varieties using minors of the truncated Jacobian matrix is that these equations are in general not complete intersection, due to the relations between minors of a matrix (the hypersurface case is an exception, since in this case only partial derivatives are used to define polar varieties). For both the polynomial system algorithm we will use below, and an application we will make of an effective Nullstellensatz, it will be necessary to have equations without such relations. To make this possible, we use an alternative modeling of polar varieties that uses Lagrange variables, as in Section 3. We may thus consider the zero-set of the polynomials

$$(F, [L_1 \ \dots \ L_p] \cdot \text{jac}(F, i)) \in \mathbb{C}[X_1, \dots, X_n, L_1, \dots, L_p]^{p+n-i},$$

but as before, we will want to discard from the zero-set of these equations in \mathbb{C}^{n+p} those components where all L_i 's vanish identically. We pointed out that the saturation needed to remove such components is unlikely to yield convenient sets of generators, so we will again introduce a single additional equation, of the form $u_1 L_1 + \dots + u_p L_p - 1$, for a certain $\mathbf{u} = (u_1, \dots, u_p)$ in \mathbb{C}^p . Thus, for such a vector \mathbf{u} , we define the following polynomials:

$$\mathfrak{L}(i, F, \mathbf{u}) = (F, [L_1 \ \dots \ L_p] \cdot \text{jac}(F, i), u_1 L_1 + \dots + u_p L_p - 1) \in \mathbb{C}[X_1, \dots, X_n, L_1, \dots, L_p]^{p+n-i+1}. \quad (3)$$

Introducing the last equation discards all solutions with $L_1 = \dots = L_p = 0$, but other components of interest may be removed as well. However, Proposition 3.1 shows that for a *generic* vector \mathbf{u} , the Zariski closure of the projection of the zero-set of these equations on the X_1, \dots, X_n -space is indeed $W(i, F)$. In the algorithm, we will use random u_i 's; the former proposition will allow us to quantify bad choices.

5.2 The algorithm

All notation being as before, we can now give the outline of Safey El Din and Schost's algorithm for computing at least one point in each connected component of $V(F) \cap \mathbb{R}^n$. To ensure its correctness, we will need certain genericity assumptions, which will be discussed in detail in the next sections.

After applying a randomly chosen change of variables \mathbf{A} , we further choose random $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_\delta)$ in \mathbb{C}^δ , with $\delta = n - p$. Then, for $i = 1, \dots, \delta + 1$, we compute (in the new coordinates) the points $\mathbf{x} = (x_1, \dots, x_n)$ satisfying

$$x_1 = \sigma_1, \dots, x_{i-1} = \sigma_{i-1}, f_1(\mathbf{x}) = \dots = f_p(\mathbf{x}) = 0, \text{rank}(\text{jac}(F, i, \mathbf{x})) < p. \quad (4)$$

In geometric terms, this means that we compute the intersection of $W(i, F)$ with the fiber $\pi_i^{-1}(\sigma_1, \dots, \sigma_{i-1})$. Then, we return the union of all these sets.

Departing from [31], and following the discussion in the previous subsection, we will avoid solving the system generated by $F = (f_1, \dots, f_p)$ and the p -minors of $\text{jac}(F, i)$: to control costs, it will be beneficial to use the Lagrange system of (3) instead. Hence, some of our genericity assumptions will concern these equations. For $i = 1, \dots, \delta + 1$, we define the following properties:

$\mathbf{H}_i(1)$: $W(i, F)$ is either empty or $(i - 1)$ -equidimensional;

$\mathbf{H}_i(2)$: 0 is a regular value of the $n + p - i$ polynomials $F, [L_1 \ \dots \ L_p] \cdot \text{jac}(F, i)$ in the open set defined by $(L_1, \dots, L_p) \neq (0, \dots, 0)$;

$\mathbf{H}_i(3)$: assuming $\mathbf{H}_i(1)$ holds, $W(i, F)$ is either empty or in Noether position for π_{i-1} .

As we will see, these properties hold after applying a generic change of variables. Properties $\mathbf{H}_i(1)$ and $\mathbf{H}_i(3)$ ensure that Eq. (4) defines a finite set (as a consequence of the definition of Noether position), and guarantee that the output of the algorithm contains at least one point in each connected component of $V(F) \cap \mathbb{R}^n$ (this is proved in [31, Theorem 2]). The second one will be used to establish that assumption \mathbf{H}'_i defined below holds generically.

Indeed, assuming (possibly after applying a change of variables) that F satisfies \mathbf{H}_i , we define our second genericity property:

\mathbf{H}'_i : $\boldsymbol{\sigma}$ is such that 0 is a regular value of the $n + p - 1$ polynomials

$$X_1 - \sigma_1, \dots, X_{i-1} - \sigma_{i-1}, F, [L_1 \ \dots \ L_p] \cdot \text{jac}(F, i),$$

in the open set defined by $(L_1, \dots, L_p) \neq (0, \dots, 0)$.

Again, we will see that this property holds for a generic choice of $\boldsymbol{\sigma}$ and that as a consequence, 0 is a regular value of the $n + p$ polynomials

$$X_1 - \sigma_1, \dots, X_{i-1} - \sigma_{i-1}, F, [L_1 \ \dots \ L_p] \cdot \text{jac}(F, i), u_1 L_1 + \dots + u_p L_p - 1. \quad (5)$$

In particular, these equations admit finitely many solutions.

Suppose that for some i in $\{1, \dots, \delta + 1\}$, F satisfies \mathbf{H}_i and $\boldsymbol{\sigma}$ satisfies \mathbf{H}'_i ; then, we know that both systems (4) and (5) have finitely many solutions. In order to find the solutions of (4), we will compute those of (5) and project them on the X_1, \dots, X_n -space; we choose to solve equations (5), since for this input, we can use the algorithm in [34], for which a complete bit complexity analysis is available. To guarantee success of this approach, we will rely on our last genericity property:

\mathbf{H}_i'' : \mathbf{u} is such that the projections of the solutions of (5) on the X_1, \dots, X_n -space are the solutions of (1).

Applying Proposition 3.1 to the polynomials in (5) shows that this property holds for a generic choice of \mathbf{u} (notice that since (5) has finitely many solutions, taking the Zariski closure, as done in the proposition, is not necessary in this case). If this is the case, the previous discussion shows that solving the systems (5), for $i = 1, \dots, \delta + 1$, and projecting their solutions on the X_1, \dots, X_n -space, solves our problem.

The next three sections prove the claims made above on the genericity of these properties: $\mathbf{H}_i(1)$ and $\mathbf{H}_i(2)$ in Section 6, as a first application of weak transversality; $\mathbf{H}_i(3)$ in Section 7, as an application of an effective Nullstellensatz; and \mathbf{H}_i' in Section 8, as another first of weak transversality (essentially, Sard's lemma). In all cases, we gave quantitative form of these genericity statements. As we pointed out above, Proposition 3.1 is enough to prove that \mathbf{H}_i'' holds for generic \mathbf{u} , and already gives a quantitative statement.

6 Genericity of $\mathbf{H}_i(1)$ and $\mathbf{H}_i(2)$

Notation in this section is as before: we let $F = (f_1, \dots, f_p) \in \mathbb{Z}[X_1, \dots, X_n]^p$ be a sequence of polynomials defining a radical ideal, and where the degree of each polynomial is at most d ; we also assume that the zero-set $V(F) \subset \mathbb{C}^n$ is smooth and δ -equidimensional, with $\delta = n - p$.

Consider an $n \times n$ matrix \mathfrak{A} with indeterminates with entries $(\mathfrak{A}_{j,k})_{1 \leq j,k \leq n}$. In this section, we prove the following proposition.

Proposition 6.1. *For $i = 1, \dots, \delta + 1$, there exists a non-zero polynomial $\Delta_{i,1}$ in $\mathbb{C}[(\mathfrak{A}_{j,k})_{1 \leq j,k \leq n}]$ of degree at most $n(d^{5n} + 1)$ and with the following property. For \mathbf{A} in $\mathbb{C}^{n \times n}$, if \mathbf{A} does not cancel $\Delta_{i,1}$, then $F^{\mathbf{A}}$ satisfies $\mathbf{H}_i(1)$ and $\mathbf{H}_i(2)$.*

The rest of this section is devoted to the proof of the proposition; it is based on a construction introduced by Giusti, Heintz *et al.* (see for instance [7]). In all that follows, i is fixed in $1, \dots, \delta + 1$; we then let $\mathfrak{A}_{\leq i}$ denote the in indeterminates $(\mathfrak{A}_{j,k})_{1 \leq j \leq i, 1 \leq k \leq n}$. Writing $\mathbf{X} = X_1, \dots, X_n$, we let $\mathbf{K}_i(\mathbf{X}, \mathfrak{A}_{\leq i})$ denote the $(p + i) \times n$ matrix

$$\mathbf{K}_i(\mathbf{X}, \mathfrak{A}_{\leq i}) = \begin{bmatrix} \text{jac}(F) \\ \mathfrak{A}_{1,1} & \cdots & \mathfrak{A}_{1,n} \\ \vdots & & \vdots \\ \mathfrak{A}_{i,1} & \cdots & \mathfrak{A}_{i,n} \end{bmatrix}.$$

Consider elements $\mathbf{a} \in \mathbb{C}^{in}$ as vectors of length i of the form $\mathbf{a} = (\mathbf{a}_1, \dots, \mathbf{a}_i)$ with $\mathbf{a}_i \in \mathbb{C}^n$; we say that \mathbf{a} has rank i when \mathbf{a} is a sequence of linearly independent vectors. Then for such an \mathbf{a} , $\mathbf{K}_i(\mathbf{X}, \mathbf{a})$ is naturally defined with the indeterminates $\mathfrak{A}_{\leq i}$ evaluated at \mathbf{a} .

Let $\Phi : \mathbb{C}^{n+p+i} \times \mathbb{C}^{in} \rightarrow \mathbb{C}^{n+p}$ be the polynomial mapping in indeterminates $\mathbf{X} = X_1, \dots, X_n$, $\mathbf{L} = L_1, \dots, L_p$, $\mathbf{T} = T_1, \dots, T_p$ and $\mathfrak{A}_{\leq i}$ defined as

$$\Phi = (F, [L_1 \cdots L_p \ T_1 \cdots T_p] \cdot \mathbf{K}_i),$$

and for \mathbf{a} in \mathbb{C}^{ni} , let $\Phi_{\mathbf{a}} : \mathbb{C}^{n+p+i} \rightarrow \mathbb{C}^{n+p}$ be the induced mapping $\Phi_{\mathbf{a}} = \Phi(\mathbf{X}, \mathbf{L}, \mathbf{T}, \mathbf{a})$ in variables \mathbf{X} , \mathbf{L} and \mathbf{T} .

Let further $\mathcal{A} \subset \mathbb{C}^{n+p+i}$ be the open set defined by the condition $(L_1, \dots, L_p) \neq (0, \dots, 0)$. In [7, Section 3.2], it is shown that, for any $(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\vartheta}, \mathbf{a})$ in $\mathcal{A} \times \mathbb{C}^{in}$, the Jacobian matrix $\text{jac}(\Phi)$, taken with respect to all indeterminates $\mathbf{X}, \mathbf{L}, \mathbf{T}, \mathfrak{A}_{\leq i}$, has full rank $n+p$ at $(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\vartheta}, \mathbf{a})$. In particular, this is true for $(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\vartheta}, \mathbf{a})$ in $\Phi^{-1}(0)$, so that 0 is a regular value of Φ on $\mathcal{A} \times \mathbb{C}^{in}$. It therefore follows by Proposition 4.1 that there exists a non-zero polynomial $\Gamma_i \in \mathbb{C}[\mathfrak{A}_{1,1}, \dots, \mathfrak{A}_{i,n}]$ of degree at most

$$d^{(n+p+i)+(n+p)} \leq d^{5n},$$

such that if $\mathbf{a} \in \mathbb{C}^{in}$ does not cancel Γ_i , then 0 is a regular value of $\Phi_{\mathbf{a}}$ on \mathcal{A} . That is, for $(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\vartheta}) \in \mathcal{A} \cap \Phi_{\mathbf{a}}^{-1}(0)$, the Jacobian matrix $\text{jac}(\Phi_{\mathbf{a}})$ has full rank $n+p$ at $(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\vartheta})$.

Let $\mathfrak{B} = \mathfrak{A}^{-1}$ in $\mathbb{C}((\mathfrak{A}_{j,k})_{1 \leq j, k \leq n})^{n \times n}$ and let $\mathfrak{B}_1 = [\mathfrak{B}_{1,1}, \dots, \mathfrak{B}_{1,n}], \dots, \mathfrak{B}_n = [\mathfrak{B}_{n,1}, \dots, \mathfrak{B}_{n,n}]$ denote the rows of \mathfrak{B} . Set

$$\Delta_{i,1} := \Gamma_i(\mathfrak{B}_1, \dots, \mathfrak{B}_i) \cdot (\det(\mathfrak{A}))^{\deg(\Gamma_i)+1}.$$

By multiplying through by $(\det(\mathfrak{A}))^{\deg(\Gamma_i)+1}$, we cancel all denominators and make $\Delta_{i,1}$ a polynomial multiple of $\det(\mathfrak{A})$.

Lemma 6.2. *The degree of $\Delta_{i,1}$ is at most $n(d^{5n} + 1)$.*

Proof. Assume that

$$\mathfrak{B}_{s,t} = \mathfrak{N}_{s,t} / \det(\mathfrak{A}) \quad \text{with} \quad \mathfrak{N}_{s,t}, \det(\mathfrak{A}) \in \mathbb{C}[(\mathfrak{A}_{j,k})_{1 \leq j, k \leq n}],$$

for $1 \leq s, t \leq n$. Then, by Cramer's formulas, we have $\deg(\mathfrak{N}_{s,t}), \deg(\det(\mathfrak{A})) \leq n$, and since we have cleared all denominators by multiplying through with $(\det(\mathfrak{A}))^{\deg(\Gamma_i)+1}$, and guaranteed the presence of an extra factor $\det(\mathfrak{A})$, we therefore obtain

$$\deg(\Delta_{i,1}) \leq n \deg(\Gamma_i) + n \leq n(d^{5n} + 1). \quad \square$$

We first prove that $\Delta_{i,1}$ allows us to control when $F^{\mathbf{A}}$ satisfies $\mathbf{H}_i(1)$. The main ingredients in the proof of the following lemma are taken from [33], with no modification; this reference itself follows previous work such as [7].

Lemma 6.3. *For \mathbf{A} in $\mathbb{C}^{n \times n}$, if \mathbf{A} does not cancel $\Delta_{i,1}$, then \mathbf{A} is invertible and the polar variety $W(i, F^{\mathbf{A}})$ is either empty or $(i-1)$ -equidimensional.*

Proof. Consider $\mathbf{A} \in \mathbb{C}^{n \times n}$ that does not cancel $\Delta_{i,1}$. Since $\det(\mathfrak{A})$ divides $\Delta_{i,1}$, \mathbf{A} is invertible, and by construction the first i rows \mathbf{b} of \mathbf{A}^{-1} do not cancel Γ_i . We put

$$Y := \{\mathbf{x} \in V(F) \mid \text{rank}(\mathbf{K}_i(\mathbf{x}, \mathbf{b})) < p+i\}.$$

Lemma B.5 from [33] shows that all irreducible components of Y have dimension at least $i-1$; this is essentially Eagon and Northcott's result on determinantal varieties [14], and does

not depend on our choice of \mathbf{b} . On the other hand, our assumption on \mathbf{b} allows us to apply Lemma B.11 from [33], which shows that all irreducible components of Y have dimension at most $i - 1$. Therefore, Y is either empty or $(i - 1)$ -equidimensional. To conclude the proof, we use the equality

$$Y^{\mathbf{A}} = W(i, F^{\mathbf{A}}),$$

established in the same reference immediately before Lemma B.10. \square

We conclude this section with the second property, $\mathbf{H}_i(2)$.

Lemma 6.4. *For \mathbf{A} in $\mathbb{C}^{n \times n}$, if \mathbf{A} does not cancel $\Delta_{i,1}$, then 0 is a regular value of the $n+p-i$ polynomials $F^{\mathbf{A}}$, $[L_1 \cdots L_p] \cdot \text{jac}(F^{\mathbf{A}}, i)$ in the open set defined by $(L_1, \dots, L_p) \neq (0, \dots, 0)$.*

Proof. Take \mathbf{A} in $\mathbb{C}^{n \times n}$ so that $\Delta_{i,1}(\mathbf{A}) \neq 0$, and let $(\mathbf{x}, \boldsymbol{\ell}) \in \mathbb{C}^{n+p}$ be a zero of the $n+p-i$ polynomials $F^{\mathbf{A}}$ and $[L_1 \cdots L_p] \cdot \text{jac}(F^{\mathbf{A}}, i)$, with $\boldsymbol{\ell}$ non-zero. We have to show that the Jacobian matrix of these polynomials has full rank $n+p-i$ at $(\mathbf{x}, \boldsymbol{\ell})$.

We define a vector $\boldsymbol{\vartheta} = [\vartheta_1 \cdots \vartheta_i] \in \mathbb{C}^i$ by writing $\boldsymbol{\ell} \cdot \text{jac}(F^{\mathbf{A}}) = [-\vartheta_1 \cdots -\vartheta_i \ 0 \cdots 0]$ (the trailing zeros result from our assumption on \mathbf{x} and $\boldsymbol{\ell}$). It follows that $(\mathbf{x}, \boldsymbol{\ell}, \boldsymbol{\vartheta})$ cancels the equations

$$F^{\mathbf{A}}, \quad [L_1 \cdots L_p \ T_1 \cdots T_i] \begin{bmatrix} \text{jac}(F^{\mathbf{A}}) \\ \mathbf{I}_i \ \mathbf{0}_{i \times (n-i)} \end{bmatrix}, \quad (6)$$

where the Jacobian matrix of F is taken with respect to the variables $\mathbf{X} = X_1, \dots, X_n$. We then post-multiply the right-hand matrix by \mathbf{A}^{-1} , and use the fact that $\text{jac}(F^{\mathbf{A}}) = \text{jac}(F)^{\mathbf{A}} \mathbf{A}$. This shows that $(\mathbf{x}, \boldsymbol{\ell}, \boldsymbol{\vartheta})$ also cancels the polynomials

$$F^{\mathbf{A}}, \quad [L_1 \cdots L_p \ T_1 \cdots T_i] \begin{bmatrix} \text{jac}(F)^{\mathbf{A}} \\ \mathbf{b} \end{bmatrix}, \quad (7)$$

where again \mathbf{b} denotes the first i rows of \mathbf{A}^{-1} . Setting $\mathbf{x}' = \mathbf{A}^{-1}\mathbf{x}$, we deduce that the point $(\mathbf{x}', \boldsymbol{\ell}, \boldsymbol{\vartheta})$ cancels

$$F, \quad [L_1 \cdots L_p \ T_1 \cdots T_i] \begin{bmatrix} \text{jac}(F) \\ \mathbf{b} \end{bmatrix}, \quad (8)$$

that is, the polynomials $\Phi_{\mathbf{b}}$ defined in the preamble. The assumption on \mathbf{A} shows that 0 is a regular value of this mapping in the open set defined by $(L_1, \dots, L_p) \neq (0, \dots, 0)$. Since $\boldsymbol{\ell}$ is by definition non-zero, this implies that the Jacobian matrix of the polynomials in Eq. (8) has full rank $n+p$ at $(\mathbf{x}', \boldsymbol{\ell}, \boldsymbol{\vartheta})$. Back in the original coordinates, we deduce that the Jacobian matrix of the polynomials in Eq. (7) has full rank $n+p$ at $(\mathbf{x}, \boldsymbol{\ell}, \boldsymbol{\vartheta})$. Right multiplication by \mathbf{A}^{-1} in (8) amounts to performing a linear combination of the equations; hence, the Jacobian matrix of the polynomials in Eq. (6) has full rank $n+p$ at $(\mathbf{x}, \boldsymbol{\ell}, \boldsymbol{\vartheta})$ as well.

The Jacobian matrix of these polynomials taken with respect to the variables $X_1, \dots, X_n, L_1, \dots, L_p$ and T_1, \dots, T_i is equal to

$$\begin{bmatrix} \text{jac}(F^{\mathbf{A}}) & \mathbf{0}_{p \times p} & \mathbf{0}_{p \times i} \\ \text{jac}_{\mathbf{X}, \mathbf{L}} \left([L, T] \cdot \begin{bmatrix} \text{jac}(F^{\mathbf{A}}) \\ \mathbf{I}_i \ \mathbf{0}_{i \times (n-i)} \end{bmatrix} \right) & \mathbf{I}_i & \mathbf{0}_{(n-i) \times i} \end{bmatrix} = \begin{bmatrix} \text{jac}(F^{\mathbf{A}}) & \mathbf{0}_{p \times p} & \mathbf{0}_{p \times i} \\ * & * & * \\ \text{jac}_{\mathbf{X}, \mathbf{L}} (\mathbf{L} \cdot \text{jac}(F^{\mathbf{A}}, i)) & \mathbf{I}_i & \mathbf{0}_{(n-i) \times i} \end{bmatrix}.$$

Therefore, after removing i rows and columns, we can see that the submatrix

$$\begin{bmatrix} \text{jac}(F^{\mathbf{A}}) & \mathbf{0}_{p \times p} \\ \text{jac}_{\mathbf{x}, \mathbf{L}}(\mathbf{L} \cdot \text{jac}(F^{\mathbf{A}}, i)) \end{bmatrix} \quad (9)$$

has full rank $n + p - i$ at (\mathbf{x}, ℓ) . \square

7 Genericity of $\mathbf{H}_i(3)$

Notation being as before, we now discuss the last genericity property that depends on our choice of coordinates. We already showed that in generic coordinates, the polar variety $W(i, F)$ is either empty or $(i - 1)$ -equidimensional. It remains to do the same for $\mathbf{H}_i(3)$, that is, to prove that if it is not empty, $W(i, F)$ is generically in Noether position for π_{i-1} . We will prove the following, where $\Delta_{i,1}$ is from Proposition 6.1.

Proposition 7.1. *For $i = 1, \dots, \delta + 1$, there exists a non-zero polynomial $\Delta_{i,2}$ in $\mathbb{C}[(\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}]$ of degree at most $4n^2(2d)^{4n}$ such that if \mathbf{A} does not cancel $\Delta_{i,1}\Delta_{i,2}$, then $F^{\mathbf{A}}$ satisfies $\mathbf{H}_i(1)$, $\mathbf{H}_i(2)$ and $\mathbf{H}_i(3)$.*

Some results in a similar vein appear in the literature. For instance, Lemma 5 in [24] and Proposition 4.5 in [25] are quantitative Noether position statements. However, our results do not follow from these previous references, as these previous works analyze the probability that for a *fixed* algebraic set V , $V^{\mathbf{A}}$ be in Noether position. This does not solve our question, since $W(i, F^{\mathbf{A}})$, which we are interested in, is in general different from $W(i, F)^{\mathbf{A}}$.

Instead, we will rely on the proof given in [31] that $W(i, F^{\mathbf{A}})$ is in Noether position for a generic \mathbf{A} . However, we will not directly analyze the constructions used in that reference, since they involve e.g. primary decomposition in $\mathbb{C}((\mathfrak{A}_{j,k})_{1 \leq j, k \leq n})[X_1, \dots, X_n]$, and the resulting degree bounds would be way beyond our target. We will instead combine results from [31] with an effective form of the Nullstellensatz given in [12]; as a result, we have to use Lagrange systems to describe polar varieties, since systems of minors do not satisfy assumptions needed to apply this effective Nullstellensatz.

The rest of this section is devoted to the proof of this proposition. From now on, we fix i in $0, \dots, \delta + 1$.

7.1 Preliminaries

Property $\mathbf{H}_i(2)$ states that 0 is a regular value of the $n + p - i$ polynomials $F, [L_1 \cdots L_p] \cdot \text{jac}(F, i)$ in the open set defined by $(L_1, \dots, L_p) \neq (0, \dots, 0)$; we saw that it holds in generic coordinates. We start by establishing some consequences of this fact for the polynomials $\mathfrak{L}(i, F, \mathbf{u})$ of Eq. (3).

Lemma 7.2. *Suppose that 0 is a regular value of the $n + p - i$ polynomials $F, [L_1 \cdots L_p] \cdot \text{jac}(F, i)$ in the open set defined by $(L_1, \dots, L_p) \neq (0, \dots, 0)$. Then, for any $\mathbf{u} = (u_1, \dots, u_p)$ in \mathbb{C}^p , the $n + p - i + 1$ polynomials*

$$\mathfrak{L}(i, F, \mathbf{u}) = F, [L_1 \cdots L_p] \cdot \text{jac}(F, i), u_1 L_1 + \cdots + u_p L_p - 1$$

define a radical ideal, either trivial or $(i - 1)$ -equidimensional.

Proof. Take $(\mathbf{x}, \boldsymbol{\ell})$ in \mathbb{C}^{n+p} that cancels the $n + p - i + 1$ polynomials in (3). We prove that the Jacobian matrix of these equations has full rank $n + p - i + 1$ at $(\mathbf{x}, \boldsymbol{\ell})$; the conclusion then follows from the Jacobian criterion.

Since $\boldsymbol{\ell}$ cannot be zero, our assumption implies that the Jacobian of the polynomials F and $[L_1 \cdots L_p] \cdot \text{jac}(F, i)$ has full rank $n + p - i$ at $(\mathbf{x}, \boldsymbol{\ell})$. The conclusion therefore holds if $\text{grad}(u_1 L_1 + \cdots + u_p L_p - 1) = [\mathbf{0}_{1 \times n} \ u_1 \ \cdots \ u_p]$ is not in the row space of this matrix at $(\mathbf{x}, \boldsymbol{\ell})$. The Jacobian matrix of F and $[L_1 \cdots L_p] \cdot \text{jac}(F, i)$ is equal to

$$\begin{bmatrix} \text{jac}(F) & \mathbf{0}_{p \times p} \\ * * * & \text{jac}(F, i)^T \end{bmatrix}.$$

Suppose that $[\mathbf{0}_{1 \times n} \ u_1 \ \cdots \ u_p]$ is in the row-space of this matrix. Considering the last p columns gives us an equality $[u_1 \ \cdots \ u_p] = \boldsymbol{\mu} \text{jac}(F, i)^T$, for some $\boldsymbol{\mu}$ in $\mathbb{C}^{1 \times (n-i)}$. Right-multiplying by $\boldsymbol{\ell}^T \in \mathbb{C}^{p \times 1}$, we obtain $1 = 0$, a contradiction. \square

The result carries over to our original polynomials in generic coordinates. In what follows, just as we defined $F^{\mathbf{A}}$ for \mathbf{A} in $\mathbb{C}^{n \times n}$, we define $F^{\mathfrak{A}} = (f_1^{\mathfrak{A}}, \dots, f_p^{\mathfrak{A}})$ as

$$(f_1(\mathfrak{A}\mathbf{X}), \dots, f_p(\mathfrak{A}\mathbf{X})) \in \mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n})[X_1, \dots, X_n]^p.$$

Corollary 7.3. *For any $\mathbf{u} = (u_1, \dots, u_p)$ in \mathbb{C}^p , the $n + p - i + 1$ polynomials*

$$\mathfrak{L}(i, F^{\mathfrak{A}}, \mathbf{u}) = F^{\mathfrak{A}}, [L_1 \cdots L_p] \cdot \text{jac}(F^{\mathfrak{A}}, i), u_1 L_1 + \cdots + u_p L_p - 1 \quad (10)$$

define a radical ideal in $\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n})[X_1, \dots, X_n, L_1, \dots, L_p]$.

Proof. Proposition 6.1 and the previous lemma show that for \mathbf{A} in a Zariski-dense subset of $\mathbb{C}^{n \times n}$, $\mathfrak{L}(i, F^{\mathbf{A}}, \mathbf{u})$ is radical in $\mathbb{C}[X_1, \dots, X_n, L_1, \dots, L_p]$; as a result, this must also be the case for the ideal $\mathfrak{L}(i, F^{\mathfrak{A}}, \mathbf{u})$ in $\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n})[X_1, \dots, X_n, L_1, \dots, L_p]$. \square

7.2 Degree bounds for integral dependence relationships

The results in Section 6 imply that $F^{\mathfrak{A}}$ satisfies $\mathbf{H}_i(1)$, so that $W(i, F^{\mathfrak{A}})$ is either empty or equidimensional of dimension $i - 1$. We now point out that $F^{\mathfrak{A}}$ also satisfies $\mathbf{H}_i(3)$. In what follows, as in Eq. (2), we let $\mathfrak{J}(i, F^{\mathfrak{A}})$ be the polynomials consisting of $F^{\mathfrak{A}}$ and all p -minors of $\text{jac}(F^{\mathfrak{A}}, i)$ in $\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n})[X_1, \dots, X_n]$, and we let \mathfrak{K} be the ideal they generate in $\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n})[X_1, \dots, X_n]$. In particular, the defining ideal of $W(i, F^{\mathfrak{A}})$ is $\sqrt{\mathfrak{K}}$.

Our first lemma simply recalls results from [31]. In the following two lemmas, membership statements are all considered in $\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n})[X_1, \dots, X_n]$; however, in the course of the proof of Lemma 7.5, we will work with the same polynomials, but seen in other polynomial rings.

Lemma 7.4. *For $j = i, \dots, n$, there exists Q_j in $\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n})[X_1, \dots, X_{i-1}, X_j]$, monic in X_j and with Q_j in $\sqrt{\mathfrak{K}}$. Furthermore, for any prime component \mathfrak{P} of the ideal $\sqrt{\mathfrak{K}}$ in $\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n})[X_1, \dots, X_n]$, we have $\mathfrak{P} \cap \mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n})[X_1, \dots, X_{i-1}] = \{0\}$.*

Proof. If $W(i, F^{\mathfrak{A}})$ is empty, then $\sqrt{\mathcal{K}}$ is the trivial ideal, so we simply take $Q_j = 1$ for all j ; the second statement is vacuously true.

Otherwise, let $(\mathfrak{P}_\ell)_{1 \leq \ell \leq L}$ be the prime components of $\sqrt{\mathcal{K}}$. By assumption, $L \geq 1$ and all \mathfrak{P}_ℓ have dimension $i - 1$. By [31, Proposition 1], for all ℓ ,

$$\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n})[X_1, \dots, X_{i-1}] \rightarrow \mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n})[X_1, \dots, X_n]/\mathfrak{P}_\ell$$

is injective and integral. In particular, this means that \mathfrak{P}_ℓ contains no non-trivial polynomial in $\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n})[X_1, \dots, X_{i-1}]$, as claimed. Also, it proves that polynomials $q_{\ell,j} \in \mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n})[X_1, \dots, X_{i-1}, X_j]$ exist, all monic in X_j , with $q_{\ell,j} \in \mathfrak{P}_\ell$ for each j in $\{i, \dots, n\}$. Thence,

$$Q_j := \prod_{1 \leq \ell \leq L} q_{\ell,j}$$

is monic in X_j and satisfies $Q_j \in \sqrt{\mathcal{K}}$, for each $j \in \{i, \dots, n\}$. \square

The former lemma does not directly give us degree bounds on the polynomials Q_j . This is the objective of the next step, where we control degree with respect to all unknowns involved, X_1, \dots, X_n as well as $\mathfrak{A}_{1,1}, \dots, \mathfrak{A}_{n,n}$. In this respect, if P is any polynomial in $\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n})[X_1, \dots, X_n]$, we will let $D \in \mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n})$ be the minimal common denominator of all its coefficients (defined up to a non-zero constant in \mathbb{C}), and we will write $\overline{P} := DP$, so that \overline{P} is in $\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}, X_1, \dots, X_n)$.

Lemma 7.5. *For $j = i, \dots, n$, there exists P_j in $\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n})[X_1, \dots, X_{i-1}, X_j]$, monic in X_j , with P_j in $\sqrt{\mathcal{K}}$ and $\deg(\overline{P}_j) \leq (2d)^{2n}$.*

Proof. Consider the following ideals: they all have for generators the polynomials $\mathfrak{J}(i, F^{\mathfrak{A}})$, that is, $F^{\mathfrak{A}}$ and the p -minors of $\text{jac}(F^{\mathfrak{A}}, i)$, but they lie in different polynomial rings.

- \mathcal{I} is the ideal generated by $\mathfrak{J}(i, F^{\mathfrak{A}})$ in the polynomial ring $\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}, X_1, \dots, X_n)$ in $n^2 + n$ indeterminates;
- \mathcal{K} , which we already saw is generated by the polynomials $\mathfrak{J}(i, F^{\mathfrak{A}})$ in the polynomial ring $\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n})[X_1, \dots, X_n]$ in n indeterminates (this is the ideal we are mainly interested in);
- \mathcal{M} is the ideal defined by the same polynomials, but this time in the polynomial ring $\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}, X_1, \dots, X_{i-1})[X_i, \dots, X_n]$ in $n - i + 1$ indeterminates.

Step 0: Excluding a trivial case. Suppose that $W(i, F^{\mathfrak{A}})$ is empty, or equivalently that \mathcal{K} is the trivial ideal. In this case, we take $P_j = 1$ for all j , and we are done. Henceforth, we assume that we are not in this situation.

Step 1: Defining the minimal polynomial P_j . The previous lemma shows that every irreducible component of the zero-set $W(i, F^{\mathfrak{A}})$ of \mathcal{K} has dimension $i - 1$, and that its image

by the projection π_i is onto. As a result, the extended ideal \mathcal{M} has dimension zero, and the ring extension

$$\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}, X_1, \dots, X_{i-1}) \rightarrow \mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}, X_1, \dots, X_{i-1})[X_i, \dots, X_n]/\sqrt{\mathcal{M}}$$

is a product of finite field extensions. For $j = i, \dots, n$, let then P_j be the minimal of X_j in this extension. Then, P_j is in $\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}, X_1, \dots, X_{i-1})[X_j]$ and is monic in X_j .

Step 2: P_j is polynomial in X_1, \dots, X_{i-1} . For j as above, the polynomial Q_j also belongs to $\sqrt{\mathcal{M}}$, so that P_j divides Q_j in $\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}, X_1, \dots, X_{i-1})[X_j]$. We can therefore write

$$Q_j = P_j R_j, \quad P_j, R_j \in \mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}, X_1, \dots, X_{i-1})[X_j].$$

It then follows by Gauss's lemma that we can write

$$Q_j = p_j r_j, \quad p_j, r_j \in \mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n})[X_1, \dots, X_{i-1}, X_j],$$

such that $\mu_j \in \mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}, X_1, \dots, X_{i-1})$ exists with

$$P_j = \mu_j p_j, \quad R_j = \mu_j^{-1} r_j.$$

Since Q_j is monic in X_j , p_j and r_j must also be monic in X_j , so μ_j must be the coefficient of the highest degree term of P_j in X_j . Since P_j is monic in X_j , $\mu_j = 1$ and hence

$$P_j = 1 \cdot p_j = p_j \in \mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n})[X_1, \dots, X_{i-1}, X_j].$$

Step 3: P_j is in $\sqrt{\mathcal{H}}$. It follows that P_j belongs to $\sqrt{\mathcal{M}} \cap \mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n})[X_1, \dots, X_{i-1}, X_j]$. Equivalently, there exists a non-negative exponent s such that P_j^s is in the intersection $\mathcal{M} \cap \mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n})[X_1, \dots, X_{i-1}, X_j]$. Clearing denominators in the membership equality in \mathcal{M} , this means that there exists D non-zero in $\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}, X_1, \dots, X_{i-1})$ such that DP_j^s is in \mathcal{H} , and thus in $\sqrt{\mathcal{H}}$.

By the previous lemma, no prime component of $\sqrt{\mathcal{H}}$ contains any non-zero polynomial in $\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}, X_1, \dots, X_{i-1})$. As a consequence, P_j^s is in $\sqrt{\mathcal{H}}$, and thus so is P_j itself.

Step 4: Degree of $\overline{P_j}$. For the last step of the proof, the ideal \mathcal{J} is used. Let indeed Z be its zero-set in \mathbb{C}^{n^2+n} . Since all polynomials in $F^{\mathfrak{A}}$, resp. $\text{jac}(F^{\mathfrak{A}}, i)$, have respective degrees at most $2d$ in $(\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}, X_1, \dots, X_n$, resp. $2d-1$, Corollary 3.2 shows that Z has degree at most $(2d)^{2n}$.

Let further Z' be obtained by removing from Z all those irreducible components whose projection on the space of coordinates $(\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}, X_1, \dots, X_{i-1}$ is not dense, and let \mathcal{J}' be its defining ideal. It is a routine verification that the extension of \mathcal{J}' in the polynomial ring $\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}, X_1, \dots, X_{i-1})[X_i, \dots, X_n]$ is the radical of \mathcal{M} . As a result, Theorem 2 in [11] implies that the total degree of $\overline{P_j}$ is bounded above by $\deg(Z)$; this finishes the proof. \square

7.3 Applying the effective Nullstellensatz

The previous lemma could allow us to give a quantitative proof that $\mathbf{H}_i(3)$ holds generically, if we were able to bound the degree in $(\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}$ of the corresponding membership equality in $\sqrt{\mathcal{K}}$. However, we are not aware of a suitable effective Nullstellensatz. The best suited one, due to D'Andrea, Krick and Sombra [12], requires that the number of generators in the ideal we consider be no more than the ambient dimension; this is in general not the case for the polynomials $\mathfrak{J}(i, F^{\mathfrak{A}})$.

As a result, we will use Lagrange systems instead. For \mathbf{u} in \mathbb{C}^p , recall the definition of the Lagrange system $\mathfrak{L}(i, F^{\mathfrak{A}}, \mathbf{u})$ given in Eq. (10); let further $\mathcal{L}_{\mathbf{u}}$ be the ideal these polynomials generate in $\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n})[X_1, \dots, X_n, L_1, \dots, L_p]$. Proposition 3.1 gives the inclusion $\sqrt{\mathcal{K}} \subset \sqrt{\mathcal{L}_{\mathbf{u}}}$, and Corollary 7.3 shows that $\mathcal{L}_{\mathbf{u}}$ is a radical ideal, so that we have $\sqrt{\mathcal{K}} \subset \mathcal{L}_{\mathbf{u}}$. As a consequence, the polynomials P_j , and thus \overline{P}_j as well, are in $\mathcal{L}_{\mathbf{u}}$ for any \mathbf{u} in \mathbb{C}^p .

Let then $\mathbf{u}_1, \dots, \mathbf{u}_p$ be new indeterminates, and consider the ideal $\mathcal{L}_{\mathbf{u}}$ generated by the polynomials $\mathfrak{L}(i, F^{\mathfrak{A}}, \mathbf{u})$ in the ring of polynomials in $X_1, \dots, X_n, L_1, \dots, L_p$ over the field of coefficients $\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}, \mathbf{u}_1, \dots, \mathbf{u}_p)$; the only difference with the previous setting is that the linear form involved in these equations is now $\mathbf{u}_1 L_1 + \dots + \mathbf{u}_p L_p - 1$. The previous discussion implies that all polynomials \overline{P}_j belong to $\mathcal{L}_{\mathbf{u}}$; we are now going to apply an effective Nullstellensatz to these membership equalities.

Let T be a new variable, and let $G_1, \dots, G_{n+p-i+1}$ be the $n+p-i+1$ polynomials in the Lagrange system $\mathfrak{L}(i, F^{\mathfrak{A}}, \mathbf{u})$. For $j = i, \dots, n$ applying the Nullstellensatz in $\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}, \mathbf{u}_1, \dots, \mathbf{u}_p)[X_1, \dots, X_n, L_1, \dots, L_p, T]$, and clearing denominators, we obtain the existence of A_j in $\mathbb{C}[(\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}, \mathbf{u}_1, \dots, \mathbf{u}_p] - \{0\}$ and of polynomial coefficients $C_{j,1}, \dots, C_{j,n+p-i+1}, B_j$ in $\mathbb{C}[(\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}, \mathbf{u}_1, \dots, \mathbf{u}_p][X_1, \dots, X_n, L_1, \dots, L_p, T]$, such that

$$A_j = \sum_{\ell=1}^{n+p-i+1} C_{j,\ell} G_{\ell} + B_j (1 - \overline{P}_j T). \quad (11)$$

Let us then see A_j as a polynomial in $\mathbf{u}_1, \dots, \mathbf{u}_p$ with non-zero coefficients in $\mathbb{C}[(\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}]$, and let α_j be one of these coefficients, arbitrarily chosen. We can then define

$$\Delta_{i,2} := \alpha_i \cdots \alpha_n \in \mathbb{C}[(\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}] - \{0\}.$$

With this definition of $\Delta_{i,2}$, we prove the following lemma. It almost completes the proof of Proposition 7.1, except for the degree bound.

Lemma 7.6. *If $\mathbf{A} \in \mathbb{C}^{n \times n}$ does not cancel $\Delta_{i,1} \Delta_{i,2}$, then $F^{\mathbf{A}}$ satisfies $\mathbf{H}_i(1)$, $\mathbf{H}_i(2)$ and $\mathbf{H}_i(3)$.*

Proof. Let us take such a matrix \mathbf{A} . The non-vanishing of $\Delta_{i,1}(\mathbf{A})$ already guarantees that $F^{\mathbf{A}}$ satisfies $\mathbf{H}_i(1)$ and $\mathbf{H}_i(2)$. It remains to establish that $\mathbf{H}_i(3)$ holds, that is, that if it is not empty, $W(i, F^{\mathbf{A}})$ is in Noether position for π_{i-1} . In what follows, we assume that $W(i, F^{\mathbf{A}})$ is not empty; by $\mathbf{H}_i(1)$, it is $(i-1)$ -equidimensional.

Fix j in i, \dots, n . Because $\alpha_j(\mathbf{A})$ is non-zero, the polynomial $a_j := A_j(\mathbf{A}, \mathbf{u}_1, \dots, \mathbf{u}_p)$ is non-zero in $\mathbb{C}[\mathbf{u}_1, \dots, \mathbf{u}_p]$. We choose $\mathbf{u} = (u_1, \dots, u_p)$ in \mathbb{C}^p such that $a_j(u_1, \dots, u_p)$ does not vanish, and such that \mathbf{u} lies in the open set \mathcal{O} associated by Proposition 3.1 to the set $W(i, F^{\mathbf{A}})$.

Let $g_1, \dots, g_{n+p-i+1}$ be the polynomials in $\mathfrak{L}(i, F^{\mathbf{A}}, \mathbf{u})$. Evaluating $(\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}$ at the entries of \mathbf{A} and $\mathbf{u}_1, \dots, \mathbf{u}_p$ at u_1, \dots, u_p in (11) gives a relation of the form

$$\tilde{a}_j = \sum_{\ell=1}^{n+p-i+1} c_{j,\ell} g_\ell + b_j(1 - p_j T),$$

with $\tilde{a}_j = a_j(u_1, \dots, u_p) \in \mathbb{C} - \{0\}$, polynomials $c_{j,\ell}$ and b_j in $\mathbb{C}[X_1, \dots, X_n, L_1, \dots, L_p, T]$ and p_j in $\mathbb{C}[X_1, \dots, X_{i-1}, X_j]$, monic in X_j .

The next step is routine. Replace T by $1/p_j$ in the previous equality; after clearing denominators, this gives a membership equality of the form

$$p_j^k \in \langle \mathfrak{L}(i, F^{\mathbf{A}}, \mathbf{u}) \rangle$$

for some integer $k \geq 1$ (we cannot have $k = 0$, since we assumed that $W(i, F^{\mathbf{A}})$ is not empty). Using our assumption on \mathbf{u} , Proposition 3.1 then shows that p_j^k is in the ideal generated by $\mathfrak{J}(i, F^{\mathbf{A}})$, that is, by $F^{\mathbf{A}}$ and the p -minors of $\text{jac}(F^{\mathbf{A}}, i)$. In other words, p_j is in the defining ideal of the polar variety $W(i, F^{\mathbf{A}})$. Repeating this for all $j = i, \dots, n$ proves that $W(i, F^{\mathbf{A}})$ is in Noether position for π_{i-1} . \square

To estimate the degree of $\Delta_{i,2}$, what remains is to give an upper bound on the degrees of $\alpha_i, \dots, \alpha_n$. This will come as an application of the effective Nullstellensatz given in [12] over the function field $\mathbb{C}((\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}, \mathbf{u}_1, \dots, \mathbf{u}_p)$. For this, we first need to determine degree bounds, separately in the actual indeterminates $X_1, \dots, X_n, L_1, \dots, L_p, T$ and in the ‘‘constants’’ $(\mathfrak{A}_{k,m})_{1 \leq k, m \leq n}, \mathbf{u}_1, \dots, \mathbf{u}_p$, of the polynomials in the membership relationship; we denote the former by $\deg_{\mathbf{X}, \mathbf{L}, T}$ and the latter by $\deg_{\mathfrak{A}, \mathbf{u}}$. Then, we have

$$\deg_{\mathbf{X}, \mathbf{L}, T} \{G_1, \dots, G_{n+p-i+1}, \mathbf{u}_1 L_1 + \dots + \mathbf{u}_p L_p - 1\} \leq d, \quad \deg_{\mathbf{X}, \mathbf{L}, T} (1 - T \overline{P}_j) \leq (2d)^{2n} + 1,$$

and

$$\deg_{\mathfrak{A}, \mathbf{u}} \{G_1, \dots, G_{n+p-i+1}, \mathbf{u}_1 L_1 + \dots + \mathbf{u}_p L_p - 1\} \leq d \text{ and } \deg_{\mathfrak{A}, \mathbf{u}} (1 - T \overline{P}_j) \leq (2d)^{2n}.$$

For each $j \in \{i, \dots, n-p+1\}$, since the number of equations in the ideal we consider is less than or equal to the ambient dimension $n+p+1$, it follows from [12, Theorem 0.5] that

$$\deg(A_j) \leq (2n+2)d^{2n+1}((2d)^{2n} + 1);$$

we will use the slightly less precise bound

$$\deg(A_j) \leq 4n(2d)^{4n}.$$

In particular, the same bound holds for the degree of α_j , and this gives

$$\deg(\Delta_{i,2}) \leq 4n^2(2d)^{4n}.$$

This concludes the proof of Proposition 7.1.

8 Genericity of \mathbf{H}'_i and consequences

We still consider a sequence of polynomials $F = (f_1, \dots, f_p) \in \mathbb{C}[X_1, \dots, X_n]^p$ as before; in particular, recall we write $\delta = n - p$ and that d is an upper bound on the degrees of f_1, \dots, f_p . Besides, we now also assume that F satisfies all assumptions \mathbf{H}_i (for instance, because we have already applied a generic change of coordinates), and we prove the following.

Proposition 8.1. *For $i = 1, \dots, \delta + 1$, if F satisfies \mathbf{H}_i , there exists a non-zero polynomial $\Xi_i \in \mathbb{C}[S_1, \dots, S_{i-1}]$ of degree at most d^{4n} such that if $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_{i-1}) \in \mathbb{C}^{i-1}$ does not cancel Ξ_i , then $\boldsymbol{\sigma}$ satisfies assumption \mathbf{H}'_i , that is, 0 is a regular value of the $n + p - 1$ polynomials*

$$X_1 - \sigma_1, \dots, X_{i-1} - \sigma_{i-1}, F, [L_1 \ \cdots \ L_p] \cdot \text{jac}(F, i)$$

in the open set defined by $(L_1, \dots, L_p) \neq (0, \dots, 0)$.

Proof. Let $\Psi : \mathbb{C}^{n+p} \times \mathbb{C}^{i-1} \rightarrow \mathbb{C}^{n+p-1}$ be the mapping defined by the $n + p - 1$ polynomials

$$X_1 - S_1, \dots, X_{i-1} - S_{i-1}, F, [L_1 \ \cdots \ L_p] \cdot \text{jac}(F, i)$$

in indeterminates $X_1, \dots, X_n, L_1, \dots, L_p, S_1, \dots, S_{i-1}$. We claim that 0 is a regular value of Ψ in the open set $\Omega \times \mathbb{C}^{i-1} \subset \mathbb{C}^{n+p} \times \mathbb{C}^{i-1}$, here $\Omega \subset \mathbb{C}^{n+p}$ is defined by $(L_1, \dots, L_p) \neq (0, \dots, 0)$.

Consider a zero $(\mathbf{x}, \boldsymbol{\ell}, \boldsymbol{\sigma})$ of Ψ , with $\boldsymbol{\ell}$ non-zero. Indexing columns by

$$X_1, \dots, X_n, L_1, \dots, L_p, S_1, \dots, S_{i-1},$$

the Jacobian matrix of Ψ is equal to

$$\begin{bmatrix} \mathbf{I}_{i-1} & \mathbf{0}_{(i-1) \times (n+p-i+1)} & -\mathbf{I}_{i-1} \\ \text{jac}_{\mathbf{X}, \mathbf{L}}(F, \mathbf{L} \cdot \text{jac}(F, i)) & \mathbf{0}_{(n+p-i) \times (i-1)} & \end{bmatrix}.$$

Because $\boldsymbol{\ell}$ is non-zero, $\mathbf{H}_i(2)$ shows that the Jacobian matrix $\text{jac}_{(\mathbf{X}, \mathbf{L})}(F, [L_1 \ \cdots \ L_p] \cdot \text{jac}(F, i))$ has full rank $n + p - i$ at $(\mathbf{x}, \boldsymbol{\ell})$. Hence, the entire matrix must have full rank $n + p - 1$ at $(\mathbf{x}, \boldsymbol{\ell}, \boldsymbol{\sigma})$, and 0 is a regular value of Ψ .

Since all polynomials defining Ψ have degree at most d , it follows by Proposition 4.1 that there exists a non-zero polynomial Ξ_i in $\mathbb{C}[S_1, \dots, S_{i-1}]$ of degree at most $d^{(n+p)+(n+p-1)} \leq d^{4n}$, with the property that, if $\Xi_i(\boldsymbol{\sigma}) \neq 0$ then at any root $(\mathbf{x}, \boldsymbol{\ell})$ of the induced mapping $\psi_{\boldsymbol{\sigma}}$ given by

$$X_1 - \sigma_1, \dots, X_{i-1} - \sigma_{i-1}, F, [L_1 \ \cdots \ L_p] \cdot \text{jac}(F, i), \tag{12}$$

if $\boldsymbol{\ell}$ non-zero, then the Jacobian matrix of these equations has full rank $n + p - 1$ at $(\mathbf{x}, \boldsymbol{\ell})$. The proposition is proved. \square

We will use this property through the following corollary, which we already mentioned in Subsection 5.2.

Corollary 8.2. For $i = 1, \dots, \delta + 1$, if F satisfies \mathbf{H}_i and $\boldsymbol{\sigma}$ satisfies \mathbf{H}'_i , then for any $\mathbf{u} = (u_1, \dots, u_p)$ in \mathbb{C}^p , 0 is a regular value of the $n + p$ polynomials

$$X_1 - \sigma_1, \dots, X_{i-1} - \sigma_{i-1}, F, [L_1 \ \cdots \ L_p] \cdot \text{jac}(F, i), u_1 L_1 + \cdots + u_p L_p - 1.$$

Proof. The proof is similar to that of Lemma 7.2. Suppose that $\Xi_i(\boldsymbol{\sigma})$ is non-zero, let $\mathbf{u} = (u_1, \dots, u_p)$ be arbitrary in \mathbb{C}^p and take $(\mathbf{x}, \boldsymbol{\ell})$ in \mathbb{C}^{n+p} that cancels the $n + p$ polynomials

$$X_1 - \sigma_1, \dots, X_{i-1} - \sigma_{i-1}, F, [L_1 \ \cdots \ L_p] \cdot \text{jac}(F, i), u_1 L_1 + \cdots + u_p L_p - 1. \quad (13)$$

Since $\boldsymbol{\ell}$ is necessarily non-zero, the previous discussion implies that the Jacobian of the polynomials in Eq. (12) has full rank $n + p - 1$ at $(\mathbf{x}, \boldsymbol{\ell})$. This Jacobian matrix is equal to

$$\begin{bmatrix} \mathbf{I}_{i-1} & \mathbf{0}_{(i-1) \times (n-i+1)} & \mathbf{0}_{(i-1) \times p} \\ & \text{jac}(F) & \mathbf{0}_{p \times p} \\ & * * * & \text{jac}(F, i)^T \end{bmatrix}.$$

As in the proof of Lemma 7.2, if we suppose that $[\mathbf{0}_{1 \times n} \ u_1 \ \cdots \ u_p]$ is in the row-space of this matrix, considering the last p columns and multiplying by $\boldsymbol{\ell}^T \in \mathbb{C}^{p \times 1}$ leads us to a contradiction. This proves that the Jacobian matrix of the equations in Eq. (13) has full rank $n + p$ at $(\mathbf{x}, \boldsymbol{\ell})$, as claimed. \square

9 Analysis of the main algorithm

We conclude this paper by revisiting the algorithm sketched in Subsection 5.2. The probability analysis is based on the quantitative genericity results we established in the previous sections, using the DeMillo-Lipton-Schwartz-Zippel lemma. *In order to simplify some big- O estimates, we assume that the bound d on the degrees of the input polynomials satisfies $d \geq 2$, since the case of linear polynomials is trivial.*

9.1 Description of the pseudocode

The algorithm is randomized and takes as input a parameter $\epsilon \in (0, 1)$; the choices made in the algorithm guarantee that the probability of success is at least $1 - \epsilon$.

Randomness occurs in part due to the various choices we make (change of variables \mathbf{A} , parameter $\boldsymbol{\sigma}$, parameter \mathbf{u}). Besides, at Step 4, we use a minor modification of [34, Algorithm 2] to solve the system

$$X_1 - \sigma_1, \dots, X_{i-1} - \sigma_{i-1}, F^{\mathbf{A}}, [L_1 \ \cdots \ L_p] \cdot \text{jac}(F^{\mathbf{A}}, i), u_1 L_1 + \cdots + u_p L_p - 1.$$

of $n + p$ equations in $n + p$ unknowns $X_1, \dots, X_n, L_1, \dots, L_p$. This subroutine is randomized as well; in order to guarantee a higher probability of success, we repeat the calculation k times, for a well-chosen parameter k , and keep the output with the largest cardinality (we discuss this in our probability analysis below). Upon success, we have obtained a zero-dimensional

parameterization $\mathcal{Q}_i = ((q_i, v_{i,1}, \dots, v_{i,n+p}), \lambda_i)$ of the solutions Z_i of these equations, but we are only interested in the projection Z'_i of these points on the X_1, \dots, X_n -space. Recall that \mathcal{Q}_i is such that

- $\lambda_i(v_{i,1}, \dots, v_{i,n+p}) = Tq'_i \bmod q_i$, with λ_i a \mathbb{Q} -linear form in $X_1, \dots, X_n, L_1, \dots, L_p$;
- we have the equality $Z_i = \left\{ \left(\frac{v_{i,1}(\tau)}{q'(\tau)}, \dots, \frac{v_{i,n+p}(\tau)}{q'(\tau)} \right) \mid q(\tau) = 0 \right\}$.

The only constraint on λ_i is that it take pairwise distinct values on the points of Z_i . Now, since the equations defining Z_i are linear in L_1, \dots, L_p , the projection $Z_i \rightarrow Z'_i$ is one-to-one; this means that we can take λ_i depending on X_1, \dots, X_n only. This constraint can be enforced at no extra cost in the algorithm of [34]; if this is the case, then $\mathcal{Q}'_i = ((q_i, v_{i,1}, \dots, v_{i,n}), \lambda_i)$ is a zero-dimensional parameterization of Z'_i .

The algorithm of [34] also requires that the input system be given by a straight-line program. We build it (at Step 3) in the straightforward manner already suggested in the introduction: given $F = (f_1, \dots, f_p)$ in $\mathbb{C}[X_1, \dots, X_n]^p$, we can build a straight-line program that evaluates each f_i in $O(d^n)$ operations, by computing all monomials of degree up to d , multiplying them by the corresponding coefficients in f_i , and adding results. To obtain a straight-line program for $f_i^{\mathbf{A}}$, we add $O(n^2)$ steps corresponding to the application of the change of variables \mathbf{A} . The number of operations here is thus

$$O(nd^n + n^3) = O^\sim(d^n);$$

note that here, we use the assumption $d \geq 2$. From this, we can compute and evaluate the required partial derivatives in the Jacobian of $F^{\mathbf{A}}$ in

$$O(nd^n) = O^\sim(d^n)$$

operations [9]. Then, the matrix vector product with the vector of Lagrange multipliers adds a cost that is polynomial in n and which we can therefore neglect in the soft-O notation. Finally, we add the linear equations $X_1 - \sigma_1, \dots, X_{i-1} - \sigma_{i-1}$; this gives the straight-line program Γ_i , whose length is $O^\sim(d^n)$.

As we already pointed out in Subsection 5.2, if $F^{\mathbf{A}}$ satisfies \mathbf{H}_i , $\boldsymbol{\sigma}$ satisfies \mathbf{H}'_i and \mathbf{u} satisfies \mathbf{H}''_i , and if \mathcal{Q}_i is a zero-dimensional parametrization of the solutions of the equations (14) at Step 3 (for all $i \in \{1, \dots, n - p + 1\}$), Theorem 2 in [31] establishes that the output returned in Step 7 will contain one point in each connected component of $V \cap \mathbb{R}^n$. (The claim made in Subsection 5.2 relied on an assertion that has since been proved, in Corollary 8.2.)

9.2 Bit operation cost

The following lists the costs for each step of Algorithm 1, assuming that the input polynomials have degree d and integer coefficients of height at most b .

Algorithm 1: Main Algorithm

Input: $F = (f_1, \dots, f_p) \in \mathbb{Z}[X_1, \dots, X_n]^p$ with $\deg(f_i) \leq d$ and $\text{ht}(f_i) \leq b$, and $0 < \epsilon < 1$. Assume that $d \geq 2$.

Output: $n - p + 1$ zero-dimensional parameterizations, the union of whose zeros includes at least one point in each connected component of $V(F) \cap \mathbb{R}^n$, with probability at least $1 - \epsilon$.

1 Construct

$$S := \{1, 2, \dots, \lceil 4\epsilon^{-1}5n^3(2d)^{5n} \rceil\},$$

$$T := \{1, 2, \dots, \lceil 4\epsilon^{-1}nd^{4n} \rceil\},$$

$$R := \{1, 2, \dots, \lceil 4\epsilon^{-1}nd^{2n} \rceil\},$$

and choose $\mathbf{A} \in S^{n^2}$, $\boldsymbol{\sigma} \in T^{n-1}$ and $\mathbf{u} \in R^p$ uniformly at random;

2 for $i \leftarrow 1$ to $n - p + 1$ do

3 Build a straight-line program Γ_i that computes the equations

$$X_1 - \sigma_1, \dots, X_{i-1} - \sigma_{i-1}, F^{\mathbf{A}}, [L_1 \cdots L_p] \cdot \text{jac}(F^{\mathbf{A}}, i), u_1 L_1 + \cdots + u_p L_p - 1; \quad (14)$$

4 Run [34, Algorithm 2] $k \geq \log_2(4n/\epsilon)$ times with input Γ_i ;

5 Let $\mathcal{Q}_i = ((q_i, v_{i,1}, \dots, v_{i,n+p}), \lambda_i)$ be the highest cardinality zero-dimensional parameterization returned in Step 4;

6 Denote by $\mathcal{Q}'_i = ((q_i, v_{i,1}, \dots, v_{i,n}), \lambda_i)$ the parameterization of the projection of \mathcal{Q}_i onto the X_1, \dots, X_n -space;

7 return $[\mathcal{Q}'_1, \dots, \mathcal{Q}'_{n-p+1}]$.

(1) We defined $S := \{1, 2, \dots, \lceil 4\epsilon^{-1}5n^3(2d)^{5n} \rceil\}$ and therefore the height of any $a_{i,j} \in S$ is at most

$$\log(4/\epsilon) + \log(5n^3(2d)^{5n}) \in O^\sim(\log(1/\epsilon) + n \log d).$$

Since $|R|, |T| \leq |S|$, we also have that the height of any $\sigma_k \in T$ and $u_\ell \in R$ admits the same upper bound.

(3) After computing the partial derivatives in the Jacobian matrix, the height grows by at most another factor of $\log d$. Thus, all polynomials in the system considered at Step 3 have height

$$O^\sim(b + d(\log(1/\epsilon) + n \log(d))) = O^\sim(b + d \log(1/\epsilon) + dn).$$

All integer coefficients appearing in the straight-line program Γ_i satisfy the same bound.

(4) As a result, after applying [34, Algorithm 2] k times for each index i , with $k = O(\log(n) + \log(1/\epsilon))$, the total boolean cost of the algorithm is

$$O^\sim(d^{3n+2p+1} \log(1/\epsilon)(b + \log(1/\epsilon)))$$

where the polynomials in the output have degree at most d^{n+p} , and height at most

$$O^\sim(d^{n+p+1}(b + \log(1/\epsilon))).$$

This proves the runtime estimate, as well as our bounds on the height of the output.

9.3 Probability of success

Let $\Delta_{i,1}$ and $\Delta_{i,2} \in \mathbb{C}[(\mathfrak{A}_{j,k})_{1 \leq j,k \leq n}]$ be the polynomials from Propositions 6.1 and 7.1. Denote by $\Delta := \prod_{i=1}^{n-p+1} \Delta_{i,1} \Delta_{i,2}$, and note that

$$\deg(\Delta) = \sum_{i=1}^{n-p+1} \deg(\Delta_{i,1}) + \deg(\Delta_{i,2}) \leq 5n^3(2d)^{5n}. \quad (15)$$

If $\mathbf{A} \in \mathbb{C}^{n \times n}$ does not cancel Δ , then \mathbf{A} is invertible and $F^{\mathbf{A}}$ satisfies \mathbf{H}_i for all i in $\{1, \dots, n-p+1\}$. Now, assuming that \mathbf{A} is such a matrix, let $\Xi_i \in \mathbb{C}[S_1, \dots, S_{i-1}]$ be the polynomials from Proposition 8.1 applied to $F^{\mathbf{A}}$. Denote by $\Xi := \prod_{i=1}^{n-p+1} \Xi_i$, and note that

$$\deg(\Xi) = \sum_{i=1}^{n-p+1} \deg(\Xi_i) \leq nd^{4n}. \quad (16)$$

If $\sigma \in \mathbb{C}^{i-1}$ does not cancel Ξ , then it satisfies \mathbf{H}'_i for all $i \in \{1, \dots, n-p+1\}$. Assume that this is the case.

As argued in Subsection 5.2, the last condition on our parameters is that \mathbf{u} satisfy \mathbf{H}''_i for all i . For a given index i , Proposition 3.1 shows the existence of a non-zero polynomial Υ_i in $\mathbb{C}[U_1, \dots, U_p]$ such that if $\Upsilon_i(u_1, \dots, u_p)$ is non-zero, \mathbf{H}''_i holds; in addition, that proposition and Corollary 3.2 give an upper bound of d^{n+p} for the degree of Υ_i . We denote by $\Upsilon := \prod_{i=1}^{n-p+1} \Upsilon_i$, and note that

$$\deg(\Upsilon) = \sum_{i=1}^{n-p+1} \deg(\Upsilon_i) \leq nd^{2n}. \quad (17)$$

If $\mathbf{u} \in \mathbb{C}^p$ does not cancel Υ , then \mathbf{u} satisfies \mathbf{H}''_i for all $i \in \{1, \dots, n-p+1\}$.

Then, the algorithm is guaranteed to succeed, as long as our calls to Algorithm 2 in [34] succeed in solving the equations at Step 3. That reference establishes that by repeating the calculation k times, and keeping the output of highest degree among those k results, we succeed with probability at least $1 - (1/2)^k$. When Algorithm 2 does not succeed, it either returns a proper subset of the solutions, or FAIL. Note that Algorithm 2 is shown to succeed in a single run with probability at least $1 - 11/32$, and we bound the probability of success with $1 - 1/2$ for simplicity. Now, recall that we choose \mathbf{A} in S^{n^2} , σ in T^{n-1} and \mathbf{u} in R^p uniformly at random, with

$$\begin{aligned} S &= \{1, 2, \dots, \lceil 4\epsilon^{-1}5n^3(2d)^{5n} \rceil\}, \\ T &= \{1, 2, \dots, \lceil 4\epsilon^{-1}nd^{4n} \rceil\}, \\ R &= \{1, 2, \dots, \lceil 4\epsilon^{-1}nd^{2n} \rceil\}. \end{aligned}$$

Using the DeMillo-Lipton-Schwartz-Zippel lemma, we obtain

$$\mathbb{P}[\Delta(\mathbf{A}) = 0] \leq \frac{\deg \Delta}{|S|} = \epsilon/4.$$

If this is the case, then

$$\mathbb{P}[\Xi(\boldsymbol{\sigma}) = 0] \leq \frac{\deg \Xi}{|T|} = \epsilon/4,$$

and if this is the case, then

$$\mathbb{P}[\Upsilon(\mathbf{u}) = 0] \leq \frac{\deg \Upsilon}{|R|} = \epsilon/4.$$

When all this holds, for a given index i , Step 4 succeeds with probability at least $1 - 1/2^k$, so the probability that all indices i succeed is at least $(1 - 1/2^k)^n$; our choice of the parameter k at Step 4 ensures that this probability is at least $\epsilon/4$ as well. Therefore, the overall probability of success is at least

$$(1 - \epsilon/4)^4 \geq 1 - \epsilon.$$

This finishes the proof of Theorem 1.1.

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