CS 341: Algorithms

University of Waterloo
Éric Schost
eschost@uwaterloo.ca

Module 4: greedy algorithms

Goals

This module: the greedy paradigm through examples

- job scheduling
- interval scheduling
- more scheduling
- fractional knapsack (if time permits)
- Dijsktra's algorithm
- minimum spanning trees

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- job scheduling
- interval scheduling
- more scheduling
- fractional knapsack (if time permits)
- Dijsktra's algorithm
- minimum spanning trees

Computational model:

- word RAM
- assume all weights, capacities, deadlines, etc, fit in a word

Overview

Greedy algorithms

Context: we are trying to solve a **combinatorial optimization** problem:

- have a **large, but finite,** domain $\mathcal{S}$
- want to find an element E in $\mathcal{S}$ that **minimizes / maximizes** a cost function

Greedy algorithms

Context: we are trying to solve a **combinatorial optimization** problem:

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- want to find an element E in $\mathcal{S}$ that **minimizes / maximizes** a cost function

Greedy strategy:

- build E step-by-step
- don't think ahead, just try to improve as much as you can at every step
- simple algorithms
- but usually, no guarantee to get the optimal
- it is often hard to prove correctness, and easy to prove incorrectness.

Example: Huffman

Review from CS240: the **Huffman tree**

- we are given **frequencies** $f_1, \ldots, f_n$ **for characters** $c_1, \ldots, c_n$
- we build a **binary tree** for the whole code

Example: Huffman

Review from CS240: the **Huffman tree**

- we are given **frequencies** $f_1, \ldots, f_n$ **for characters** $c_1, \ldots, c_n$
- we build a **binary tree** for the whole code

Greedy strategy: we build the tree **bottom up**.

- create many single-letter trees
- define the **frequency** of a tree as the sum of the frequencies of the letters in it
- build the final tree by putting together smaller trees: **join the two trees with the least frequencies**

Claim: this minimizes $\sum_i f_i \times \{\text{length of encoding of } c_i\}$

A job scheduling problem

The problem

Input:

- n jobs, with processing times $[t(1), \ldots, t(n)]$

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Output:

- an ordering of the jobs that minimizes the **sum T of the completions times**
- **completion time:** how long it took **(since the beginning)** to complete a job

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Example:

- $n = 5$, processing times $[2, 8, 1, 10, 5]$
- in this order,
  $T = 2 \ + \ (8+2) \ + \ (1+8+2) \ + \ (10+1+8+2) \ + \ (5+10+1+8+2) = 70$
- in the order $[1, 2, 5, 8, 10]$,
  $T = 1 \ + \ (2+1) \ + \ (5+2+1) \ + \ (8+5+2+1) \ + \ (10+8+5+2+1) = 54$

Greedy algorithm

Algorithm:

- order the jobs in **non-decreasing** processing times

Greedy algorithm

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Correctness by an exchange argument

- let $L = [e_1, \ldots, e_n]$ be a permutation of $[1, \ldots, n]$
- suppose that L is **not** in non-decreasing order of processing times. Can it be optimal?
- assumption there exists i such that $\boldsymbol{t(e_i) > t(e_{i+1})}$
- sum of the completion times of L is $\boldsymbol{nt(e_1) + (n-1)t(e_2) + \cdots + t(e_n)}$
- the contribution of e_i and e_{i+1} is $\boldsymbol{(n-i+1)t(e_i) + (n-i)t(e_{i+1})}$
- now, **switch** e_i **and** e_{i+1} **to get a permutation** L'
- their contribution becomes $\boldsymbol{(n-i+1)t(e_{i+1}) + (n-i)t(e_i)}$
- nothing else changes so $\boldsymbol{T(L') - T(L) = t(e_{i+1}) - t(e_i)} < 0$
- so L **not optimal**

Greedy algorithm

Algorithm:

- order the jobs in **non-decreasing** processing times

Review from CS240

- optimal static order for linked list implementation of dictionaries
- same result (up to reverse), same proof

Interval scheduling

The problem

Input:

- n intervals $I_1 = [s_1, f_1], \ldots, I_n = [s_n, f_n]$ start time, finish time
- also write $s_j = \textbf{start}(I_j)$, $f_j = \textbf{finish}(I_j)$

The problem

Input:

- n intervals $I_1 = [s_1, f_1], \ldots, I_n = [s_n, f_n]$ start time, finish time
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Output:

- a choice T of intervals that **do not overlap** and that has **maximal cardinality**

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Example: A car rental company has the following requests for a given day:

I_1**:** 2pm to 8pm

I_2**:** 3pm to 4pm

I_3**:** 5pm to 6pm

Answer is $T = [I_2, I_3]$.

Template for a greedy algorithm

Greedy($\boldsymbol{I} = [I_1, \ldots, I_n]$)

1. $T \leftarrow [\,]$
2. **while** $\boldsymbol{I}$ is not empty **do**
3. choose an interval I from $\boldsymbol{I}$
4. move I to T
5. remove from $\boldsymbol{I}$ all intervals that overlap with I

Observation: no overlap between the intervals in T

A few attempts

Attempt 1:

- I is the interval in $\boldsymbol{I}$ with the **earliest starting time**

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Attempt 3:

- I is the interval in $\boldsymbol{I}$ with the **fewest overlaps**

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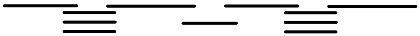

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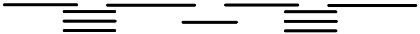

Attempt 4:

- I is the interval in $\boldsymbol{I}$ with the earliest finish time

An $O(n\log(n))$ implementation

Greedy($\boldsymbol{I} = [I_1, \ldots, I_n]$)
1. $T \leftarrow [\,]$
2. sort $\boldsymbol{I}$ by non-decreasing finish time
3. **for** $k = 1, \ldots, n$ **do**
4. if I_k does not overlap the last entry in T
5. append I_k to T

Correctness: greedy stays ahead

Let

- $T = [x_1 < \cdots < x_p]$ be the output of the algorithm,
- $S = [y_1 < \cdots < y_q]$ be any choice of requests without overlaps,
- both sorted by increasing finish time.

Proof that $p \geq q$.

- by induction: for $k = 0, \ldots, q$, $\boldsymbol{p \geq k}$ and $S_k = [x_1 < \cdots < x_k < y_{k+1} < \cdots < y_q]$ has no overlap and is sorted by increasing finish time
- OK for $k = 0$, so we suppose true for some $k < q$, and prove for $k + 1$
- since $[x_1, \ldots, x_k, y_{k+1}]$ is satisfiable, the algorithm didn't stop at x_k. So $\boldsymbol{p \geq k + 1}$.
- by definition of x_{k+1}, **finish**$(x_{k+1}) \leq$ **finish**(y_{k+1}). So we can replace y_{k+1} by x_{k+1} in S_k. We get $S_{k+1} = [x_1 < \cdots < x_{k+1} < y_{k+2} < \cdots < y_q]$, which is still satisfiable and sorted by increasing finish time

Minimizing lateness

The problem

Input:

- jobs $J_1, \ldots, J_n$ with processing times $t(1), \ldots, t(n)$ and deadlines $d(1), \ldots, d(n)$
- can only do one thing at a time

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- jobs $J_1, \ldots, J_n$ with processing times $t(1), \ldots, t(n)$ and deadlines $d(1), \ldots, d(n)$
- can only do one thing at a time

Output:

- a **scheduling** of the jobs which **minimizes maximal lateness**
  - job J_i starts at time $s(i)$ and finishes at $f(i) = s(i) + t(i)$
  - if $f(i) \geq d(i)$, lateness $\ell(i) = f(i) - d(i)$
- maximal lateness $= \max_i \ell(i)$

Example: 3 jobs

- **prepare my slides:** need $t(1) = 4$ hours, deadline $d(1) = 2$ hours
- **write solutions to assignments:** need $t(2) = 6$ hours, deadline $d(2) = 1$ hour
- **finish the midterm:** need $t(3) = 10$ hours, deadline $d(3) = 24$ hours

Example: 3 jobs

- **prepare my slides:** need $t(1) = 4$ hours, deadline $d(1) = 2$ hours
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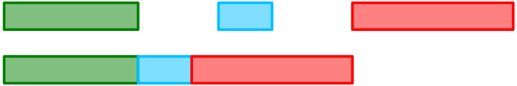
- **1, then 2, then 3:** latenesses $[2, 9, 0]$
- **2, then 1, then 3:** latenesses $[8, 5, 0]$ (optimal)

No breaks

Observation:

- if a scheduling has **idle time**, we can improve it by removing the breaks



- so the optimal has no idle time, and is given by an **ordering** of the jobs

A few attempts

Attempt 1:

- do short jobs first

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- **no**, last example

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- do jobs with little slack first   slack $= d(i) - t(i)$

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slack $= d(i) - t(i)$

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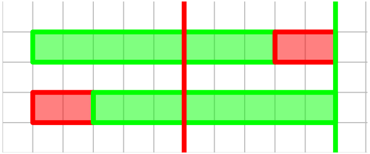

A few attempts

Attempt 1:

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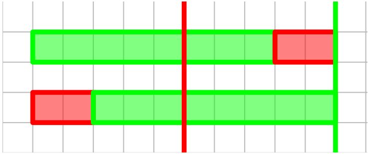


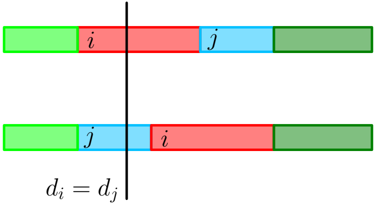
Attempt 3:

- do jobs in non-decreasing deadline order

Non-uniqueness

Observation:

- if $d(i) = d(j)$, the orderings $[\ldots, \boldsymbol{i}, \boldsymbol{j}, \ldots]$ and $[\ldots, \boldsymbol{j}, \boldsymbol{i}, \ldots]$ have the same max-lateness (because the second job is the latest)
- so **all** orderings in non-decreasing deadline order have the same max-lateness



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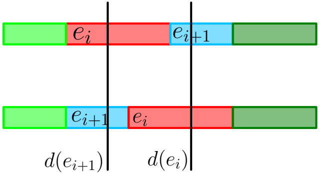
Definition:

- an inversion in $L = [e_1, \dots, e_n]$ is a pair (i, j) with $i < j$ and $d(e_i) > d(e_j)$
- L has no inversion $\iff$ L in non-decreasing deadline order

Correctness: exchange argument

- let $L = [e_1, \ldots, e_n]$ be a solution (as a permutation of $[1, \ldots, n]$)
- suppose that L is **not** in non-decreasing order of deadlines, so there exists i such that $\boldsymbol{d(e_i) > d(e_{i+1})}$
- now, **switch** e_i **and** e_{i+1} **to get a permutation** L'
- the lateness of e_{i+1} cannot increase (because we do e_{i+1} earlier than before), so at most max_lateness(L)
- the **new** lateness of e_i is **at most** the **old** lateness of e_{i+1}, so at most max_lateness(L)



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- the **new** lateness of e_i is **at most** the **old** lateness of e_{i+1}, so at most max_lateness(L)
- nothing else changes, so max_lateness$(L') \leq$ max_lateness(L)
- and we have removed an inversion
- keep going: after at most $n(n-1)/2$ iterations, we have L_{ord} with **no inversion** and such that **max_lateness$(L_{\text{ord}}) \leq$ max_lateness(L)**

Interval coloring

The problem

Input:

- n intervals $I_1 = [s_1, f_1], \ldots, I_n = [s_n, f_n]$ start time, finish time
- also write $s_j = \textbf{start}(I_j)$, $f_j = \textbf{finish}(I_j)$

The problem

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Output:

- assignment of **colors** to each interval
- overlapping intervals get **different colors**
- **minimize** the number of colors used overall

Remarks:

- another version: finding classrooms for lectures
- colors $\leftrightarrow$ numbers $1, 2, \dots$
- $\textbf{finish}(I_j) = \textbf{start}(I_k)$ not an overlap

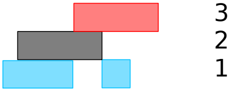
A blueprint for a greedy algorithm

GreedyColoring($\boldsymbol{I} = [I_1, \ldots, I_n]$)
1. sort $\boldsymbol{I}$ somehow
2. **for** $k = 1, \ldots, n$ **do**
3. color I_k with the **minimum** color not used by any of the previous intervals that overlap I_k

A few attempts

Attempt 1:

- sort by non-decreasing finish times
- no



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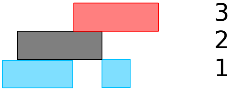
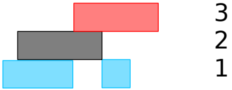


Attempt 2:

- sort from shortest to longest
- no



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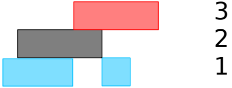

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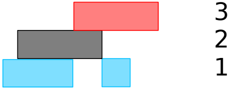

Attempt 3:

- sort by non-decreasing starting times
- maybe

Correctness

Claim

- we suppose the algorithm uses $\boldsymbol{k}$ colors
- we prove that we can't use fewer.

Correctness

Claim

- we suppose the algorithm uses $\boldsymbol{k}$ colors
- we prove that we can't use fewer.

Proof

- suppose we color I_t with color k
- so I_k overlaps with $k-1$ intervals, say $I_{\alpha_1}, \ldots, I_{\alpha_{k-1}}$ seen previously
- so for all j, $\boldsymbol{s_{\alpha_j} \leq s_t < f_{\alpha_j}}$
- so there is a little interval $[s_t, s_t + \varepsilon]$ in all I_{α_j} and I_t
- so we can't do with less than k colors

Exercise

Give an $\boldsymbol{O(n \log(n))}$ implementation.

Fractional knapsack

The problem

Input:

- items $I_1, \ldots, I_n$ with weights $\boldsymbol{w_1, \ldots, w_n}$ and positive values $\boldsymbol{v_1, \ldots, v_n}$
- a capacity W

Output:

- fractions $E = e_1, \ldots, e_n$ such that
  - $0 \leq e_j \leq 1$ for all j
  - $e_1 w_1 + \cdots + e_n w_n \leq W$
  - $e_1 v_1 + \cdots + e_n v_n$ maximal

Example:

- $\boldsymbol{w_1 = 10, v_1 = 60}$, $\boldsymbol{w_2 = 30, v_2 = 90}$, $\boldsymbol{w_3 = 20, v_3 = 100}$
- $\boldsymbol{W = 50}$
- optimal is $\boldsymbol{e_1 = 1}$, $\boldsymbol{e_2 = 2/3}$, $\boldsymbol{e_3 = 1}$, total value **220**

The problem

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  - $0 \leq e_j \leq 1$ for all j
  - $e_1 w_1 + \cdots + e_n w_n \leq W$
  - $e_1 v_1 + \cdots + e_n v_n$ maximal

Remark:

- **0/1-version:** $e_j \in \{0, 1\}$ for all j
- dynamic programming

The knapsack should be full

Remark:

- if $\sum_i w_i < W$, just take all $e_i = 1$
- so assume $\boldsymbol{\sum_i w_i \geq W}$

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- if $\sum_i w_i < W$, just take all $e_i = 1$
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Observation:

- suppose we have an assignment with $\boldsymbol{\sum_i e_i w_i < W}$
- then some e_i must be **less than** 1
- so we can increase the value by non-decreasing this e_i

The knapsack should be full

Remark:

- if $\sum_i w_i < W$, just take all $e_i = 1$
- so assume $\sum_i w_i \geq W$

Observation:

- suppose we have an assignment with $\sum_i e_i w_i < W$
- then some e_i must be **less than** 1
- so we can increase the value by non-decreasing this e_i

Consequence:

- it is enough to consider assignments with $\sum_i e_i w_i = W$

A few attempts

Attempt 1:

- pack with items in **decreasing value** v_i

A few attempts

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- **no**, previous example (we get $[0, 1, 1]$ with total value **190**)

A few attempts

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Attempt 2:

- pack with items in **increasing weight** w_i

A few attempts

Attempt 1:

- pack with items in **decreasing value** v_i
- **no**, previous example (we get $[0, 1, 1]$ with total value **190**)

Attempt 2:

- pack with items in **increasing weight** w_i
- **no**: $\boldsymbol{W = 10}$, $\boldsymbol{w_1 = 10, v_1 = 1}$, $\boldsymbol{w_2 = 5, v_2 = 100}$

A few attempts

Attempt 1:

- pack with items in **decreasing value** v_i
- **no**, previous example (we get $[0, 1, 1]$ with total value $\mathbf{190}$)

Attempt 2:

- pack with items in **increasing weight** w_i
- **no**: $\boldsymbol{W = 10}$, $\boldsymbol{w_1 = 10, v_1 = 1}$, $\boldsymbol{w_2 = 5, v_2 = 100}$

Attempt 3:

- pack with items in **decreasing "value per kilo"** v_i/w_i
- first example $[\mathbf{6, 3, 5}]$, second example $[\mathbf{1/10, 20}]$

Pseudo-code

```
GreedyKnapsack(v, w, W)
    E ← [0, ..., 0]
    sort items by decreasing order of v_i/w_i
    for k = 1, ..., n do
        if w_k < W then
            E[k] ← 1
            W ← W − w_k
        else
            E[k] ← W/w_k
            return
```

Remark: output is $S = [1, \ldots, 1, e_k, 0, \ldots, 0]$

Runtime: $O(n \log(n))$

Correctness: exchange argument

- let $E = [e_1, \ldots, e_n]$ be the optimal assignment, with $\boldsymbol{\sum e_i w_i = W}$
- let $S = [s_1, \ldots, s_n]$ be **any** assignment, with $\boldsymbol{\sum s_i w_i = W}$
- suppose S different from E, and let i be the **first** index with $\boldsymbol{e_i \neq s_i}$
- greedy strategy: $\boldsymbol{e_i > s_i}$
- because their weights are the same, there is $j > i$ with $\boldsymbol{s_j > e_j}$
- set $\boldsymbol{s'_i = s_i + \alpha/w_i}$ and $\boldsymbol{s'_j = s_j - \alpha/w_j}$, for α TBD > 0, all other $s'_k = s_k$
- in any case, $\boldsymbol{\sum s'_i w_i = W}$ and **value$(S') \geq$ value(S)**
- choose α such that **either** $s'_i = e_i$ **or** $s'_j = e_j$

$$\alpha = \min(w_i(e_i - s_i), w_j(s_j - e_j))$$

- so we found S' that has **one more common entry** with E, and which is at least as good as S
- keep going

Dijkstra's algorithm

Conventions

Input:

- a **directed** graph $G = (V, E)$
- with **weights** $w(e)$ on the edges
  $w(\gamma)$ = weight of a path γ = sum of the weights of its edges
- no **loops** = edges $v \to v$
- no **isolated vertices**, with no incoming or outgoing edge $m \geq n/2$

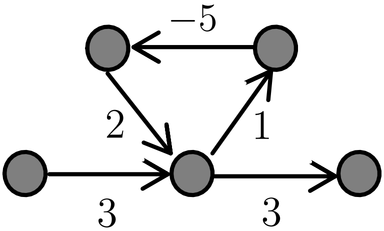
Output:

- the shortest (=minimal weight) paths between a **source** s and **all vertices**
- dynamic programming: shortest paths between **all vertices**

Remark: nothing faster known (to me) for single-source, single-destination

Remarks

1. shortest paths may not exist if there are **negative length cycles**

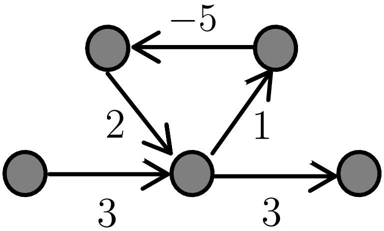


some algorithms can deal with negative edges (and detect negative cycles)

Dijkstra's algorithm needs positive weights

Remarks

1. shortest paths may not exist if there are **negative length cycles**



some algorithms can deal with negative edges (and detect negative cycles)

Dijkstra's algorithm needs positive weights

2. if there exists a shortest path $s \rightsquigarrow t$, write $\boldsymbol{\delta(s,t)}$ for its weight
   - called the **distance** from s to t (but we may not have $\delta(s,t) = \delta(t,s)$)
   - if there is **no path** $s \rightsquigarrow t$, $\delta(s,t) = \infty$

Outlook

Assumption

All weights are non-negative

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Idea of the algorithm:

- starting from s, grow a tree (S, T) rooted at s, together with the **distances** $\delta(s, v)$ for v in S
- at every step, add to S the remaining vertex v **closest to s**
- **no negative weight:** this vertex is on an edge (u, v), u in S, v in $V - S$
- if there is no such edge, we're done (all remaining vertices are unreachable)

greedy algorithm!

Key property

Claim

Let (S, T) be a tree rooted at s and take an edge (u, v) such that

- u is in S, v is in $V - S$
- $\delta(s, u) + w(u, v)$ **minimal** among these edges

Then $\boldsymbol{\delta(s, u) + w(u, v) = \delta(s, v)}$

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Then $\boldsymbol{\delta(s, u) + w(u, v) = \delta(s, v)}$

Proof:

- take a path $\gamma : s \rightsquigarrow v$ and let (x, y) be its first edge $\boldsymbol{S \to V - S}$
- $w(\gamma) = w(\boldsymbol{s \rightsquigarrow x}) + w(x, y) + w(\boldsymbol{y \rightsquigarrow v}) \geq \delta(s, x) + w(x, y) \boldsymbol{+0}$
- so $\boldsymbol{w(\gamma) \geq \delta(s, u) + w(u, v)}$ — choice of u, v
- but also $\boldsymbol{\delta(s, u) + w(u, v) \geq \delta(s, v)}$ — def of distance $s \to v$
- take **shortest** $\boldsymbol{\gamma}$: $w(\gamma) = \delta(s, v)$ so $\boldsymbol{\delta(s, v) \geq \delta(s, u) + w(u, v) \geq \delta(s, v)}$

High-level view of the algorithm

Dijkstra(G, s)

$S \leftarrow \{s\}$
while $S \neq V$ **do**
    choose (u, v) with u in S, v not in S and $\delta(s, u) + w(u, v)$ minimal (the min value gives $\delta(s, v)$)
    add v to S
    if not such (u, v), **stop**

High-level view of the algorithm

```
Dijkstra(G, s)
    S ← {s}
    while S ≠ V do
        choose (u, v) with u in S, v not in S and δ(s, u) + w(u, v) minimal
        (the min value gives δ(s, v))
        add v to S
        if not such (u, v), stop
```

Correctness:

- we find $\delta(s, v)$ for all v in S
- if $S = V$ at the end, OK
- if not, when we stop, the remaining vertices are unreachable

High-level view of the algorithm

Dijkstra(G, s)
1. $S \leftarrow \{s\}$
2. **while** $S \neq V$ **do**
3. choose (u, v) with u in S, v not in S and $\delta(s, u) + w(u, v)$ minimal (the min value gives $\delta(s, v)$)
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5. **if** not such (u, v), **stop**

Correctness:

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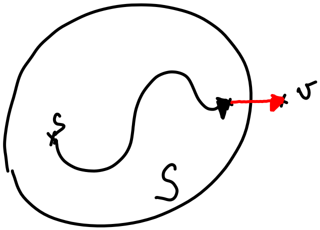
Questions:

- how to find (u, v) efficiently
- probably need a priority queue (heap) of some kind
- good choice: a priority queue of vertices

The min-priority queue

Building P

- contains all vertices in $\boldsymbol{V - S}$ (initially, all V)
- set $\mathsf{priority}[s] = 0$
- for $v \neq s$, we will maintain $\mathsf{priority}[v] = \boldsymbol{\min_{u \in S, (u,v) \in E}(\delta(s,u) + w(u,v))}$ with $\min(\emptyset) = \infty$



- initially $\mathsf{priority}[v] = \infty$ for $v \neq s$
- also store the vertex u that gives the min

The min-priority queue

Updating P

- if v is the vertex with **minimal priority**, then

$$\begin{aligned}\mathsf{priority}[v] &= \min_{v' \in V - S} \mathsf{priority}[v'] \\ &= \min_{v' \in V - S} \min_{u \in S, (u,v') \in E} (\delta(s,u) + w(u,v')) \\ &= \delta(s,v) \qquad \text{(key property)}\end{aligned}$$

we store it in an array d

The min-priority queue

Updating P

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  we store it in an array d

- then for all v' remaining in P, we must set

$$\mathsf{priority}[v'] = \boldsymbol{\min_{u \in S + v, (u, v') \in E} (\delta(s, u) + w(u, v'))}$$

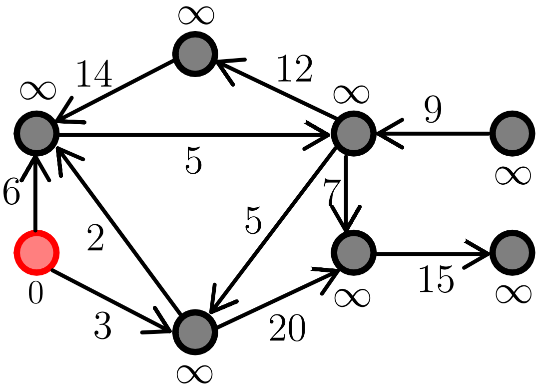
  - if there is no edge (v, v'), $\mathsf{priority}[v']$ unchanged
  - else, the new priority is $\min(\mathsf{priority}[v'], d[v] + w(v, v'))$

Pseudo-code

Dijkstra(G, s)

$P \leftarrow$ **heapify**$([s, 0, s], [v, \infty, \bullet]_{v \neq s})$
while P not empty **do**
    $[v, \ell, u] \leftarrow$ **remove_min**(P)
    $d[v] \leftarrow \ell$
    parent$[v] \leftarrow u$
    for all edges (v, v') **do**
        if $d[v] + w(v, v') <$ priority$[v']$ **then**
            replace $[v', \_, \_]$ by $[v', d[v] + w(v, v'), v]$ in P

Example



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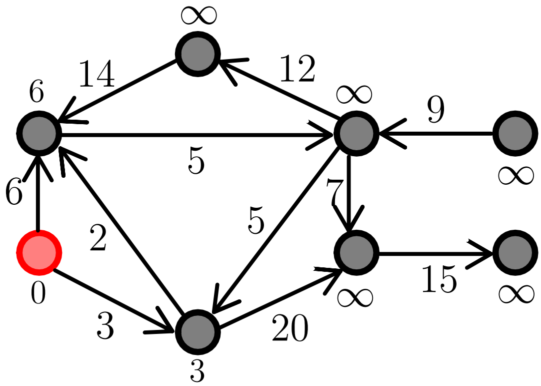
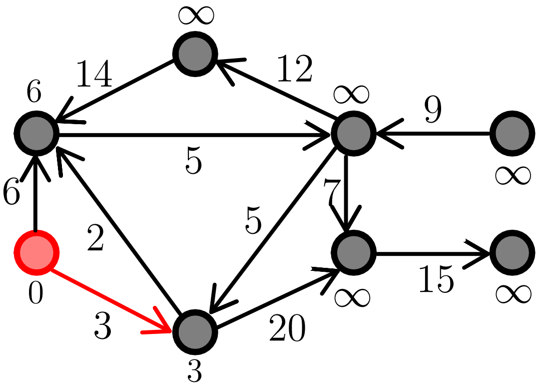


Example



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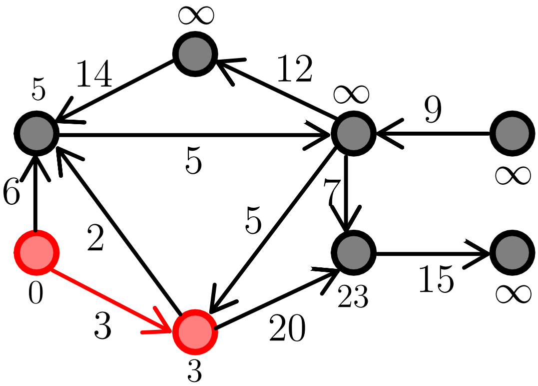

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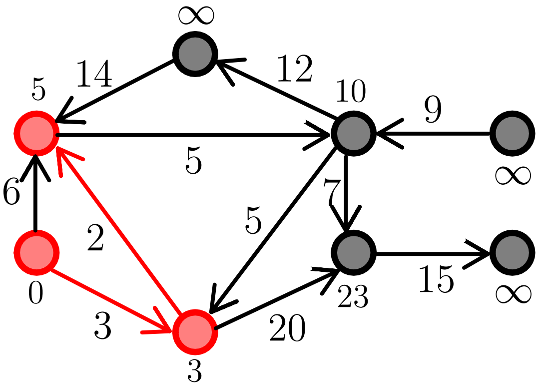

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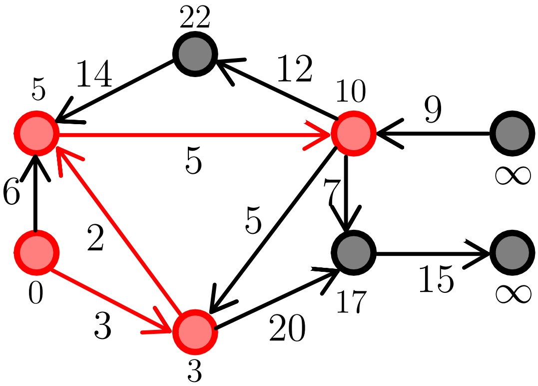

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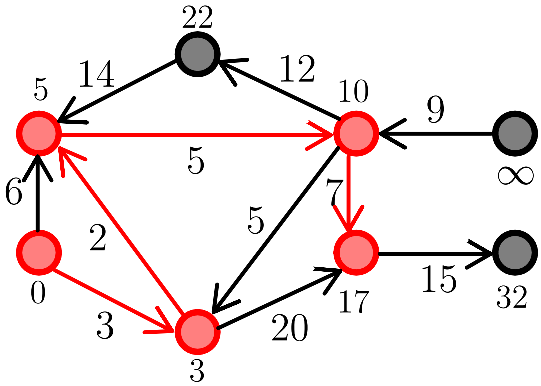

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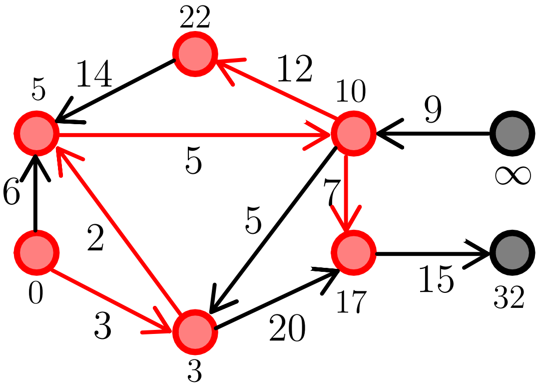

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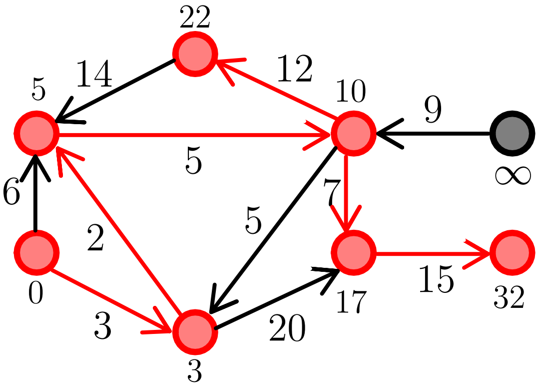

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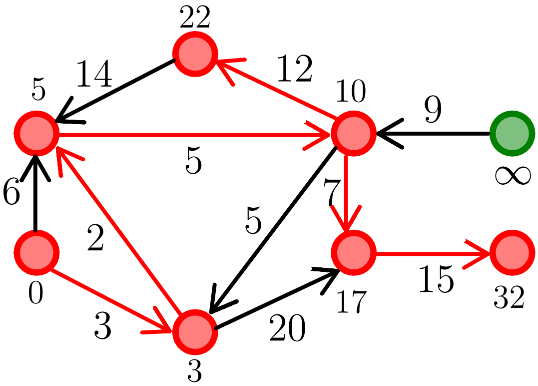


Runtime

Enhanced priority queue

- we need to be able to **change the priority** of a key
- binary heap implementation: $\boldsymbol{O(\log(n))}$ for remove-min and change priority

Total

- n remove min, m change priority $\quad m \geq n/2$
- gives $\boldsymbol{O(m\log(m))}$ $\quad \log(m) \in \Theta(\log(n))$

Remark

- **Fibonacci heaps:** constant amortized time for change priority
- total becomes $\boldsymbol{O(m + n\log(m))}$

Kruskal's algorithm

Spanning trees

Definition:

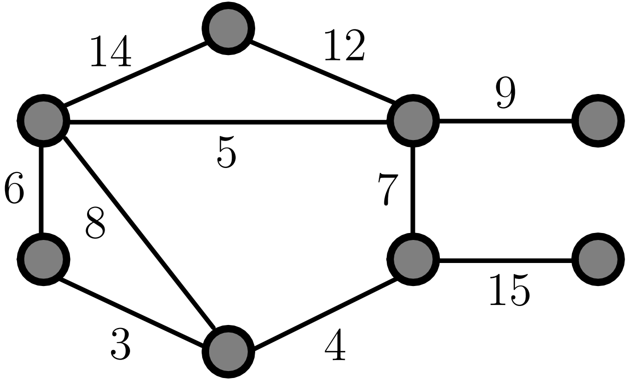
- $G = (V, E)$ is a **connected graph**
- a **spanning tree** in G is a tree of the form (V, T), with T a subset of E
- **in other words:** a tree with edges from E that covers all vertices
- examples: BFS tree, DFS tree

Now, suppose the edges have **weights** $w(e_i)$

Goal:

- a spanning tree with **minimal weight**

Example



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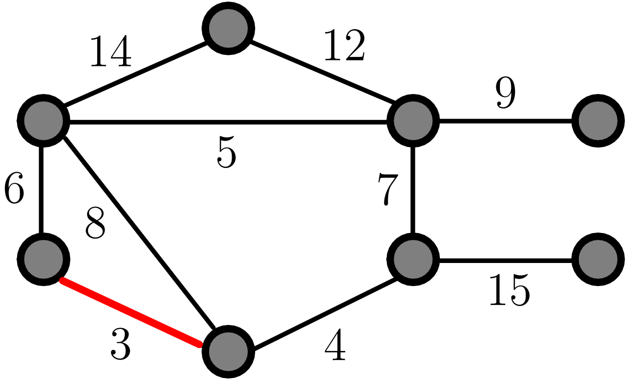

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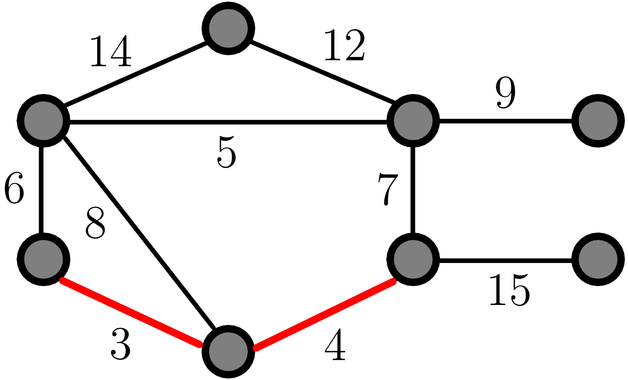

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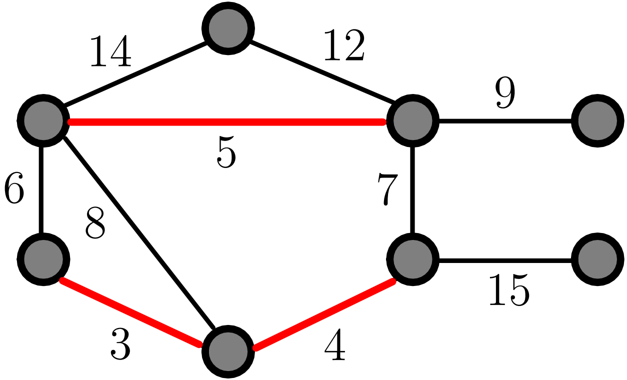

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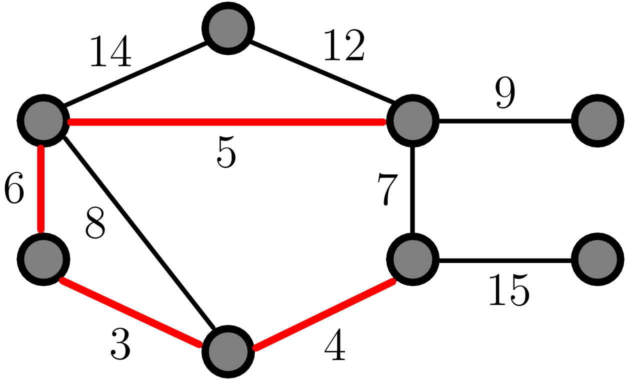

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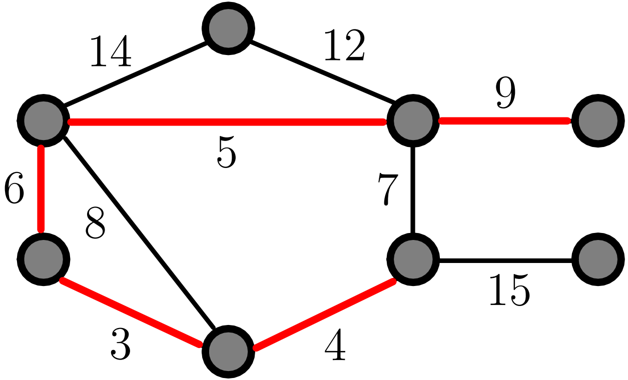

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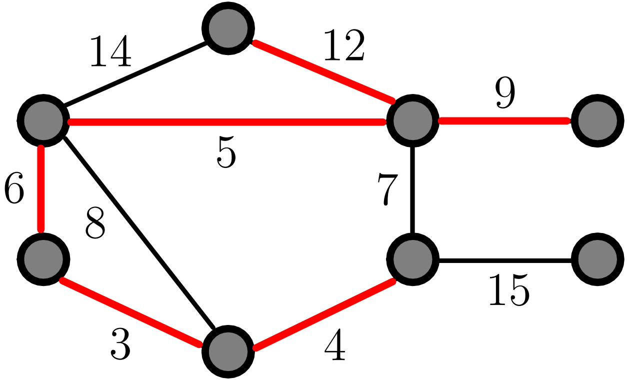

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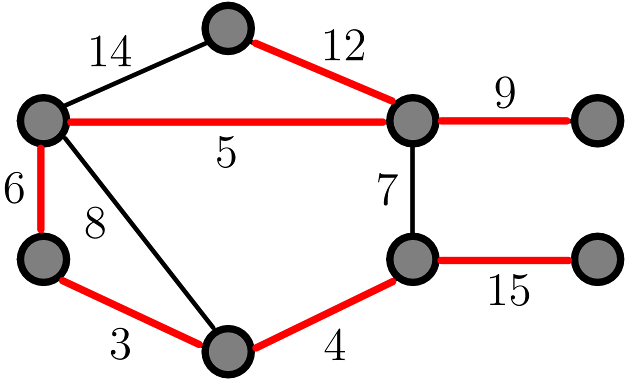


Kruskal's algorithm

GreedyMST(G)

$A \leftarrow [\ ]$
sort edges by non-decreasing weight
for $k = 1, \ldots, m$ **do**
    if e_k does not create a cycle in A **then**
        append e_k to A

Properties of the output

Claim

If the output is $A = [e_1, \ldots, e_r]$, then $\boldsymbol{(V, A)}$ **is a spanning tree** (and so $r = n - 1$)

Proof:

- of course, (V, A) has no cycle: it is a **union of trees**
- suppose (V, A) is **not connected**. Then, there exists an edge e not in A, such that $(V, A \cup \{e\})$ still has no cycle (joining two connected components)
- when we checked e, we did not include it
- means that it created a loop with some edges already in A: **impossible**.

Adding edges to spanning trees

Claim

Let (V, A) be a spanning tree, and let e be an edge not in A.

Then adding e to A creates **a unique cycle**

Proof (bonus)

- let $e = \{v, w\}$.
- **from 239:** in (V, A), there is a **unique simple path** $\boldsymbol{\gamma : v \rightsquigarrow w}$
- adding e creates a cycle
- if it created two different cycles, there would be two paths in (V, A)

Exchanging edges

Claim

Let (V, A) and (V, T) be two spanning trees, and let e be an edge in T but not in A.

- there exists an edge e' in A but not in T such that $(V, T + e' - e)$ is still a spanning tree
- e' is on the cycle that e creates in A.

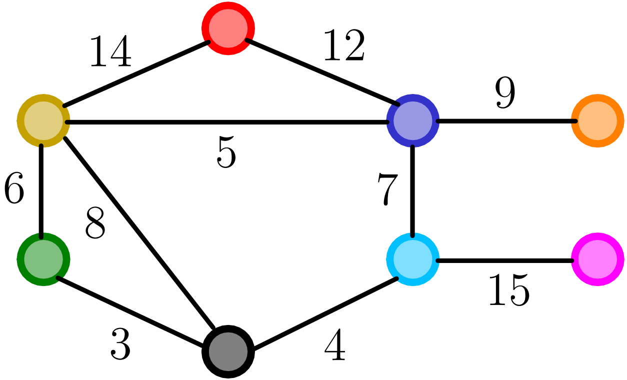
Proof (bonus):

- write $e = \{v, w\}$
- $(V, A + e)$ contains a cycle $\boldsymbol{c = v, w, \ldots, v}$
- removing e from T splits $(V, T - e)$ into two connected components T_1, T_2
- $\boldsymbol{c}$ starts in T_1, crosses over to T_2, so it contains another edge e' between T_2 and T_1
- e' is in A, but not in T
- $(V, T + e' - e)$ is a spanning tree (covers V, $n - 1$ edges, connected)

Correctness: exchange argument

- let A be the output of the algorithm
- let (V, T) be **any** spanning tree
- if $T \neq A$, let e be an edge in T but not in A
- so there is an edge e' in A but not in T such that $\boldsymbol{(V, T + e' - e)}$ is a spanning tree, **and** e' is on the cycle that e creates in A
- during the algorithm, we considered e but rejected it, because it created a cycle in A
- all other elements in this cycle have smaller (or equal) weight
- so $\boldsymbol{w(e') \leq w(e)}$
- so $T' = T + e' - e$ has weight $\leq w(T)$, and **one more common element** with A
- keep going

Merging connected sets of vertices



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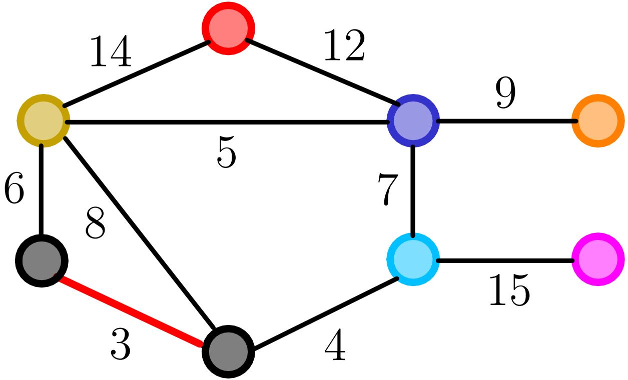

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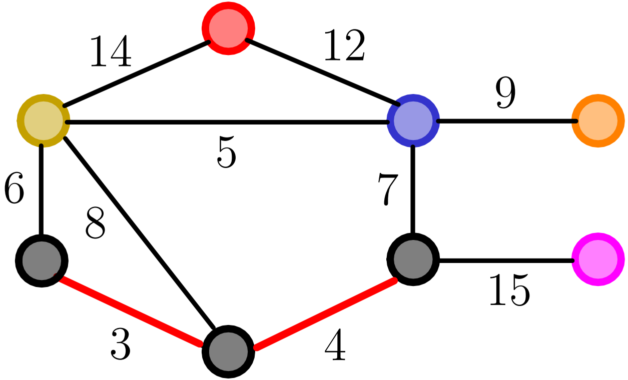

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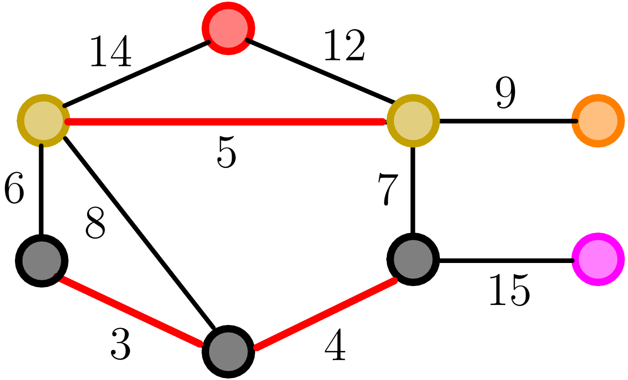

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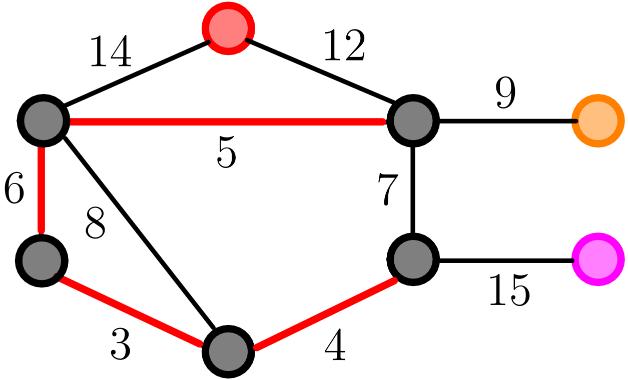

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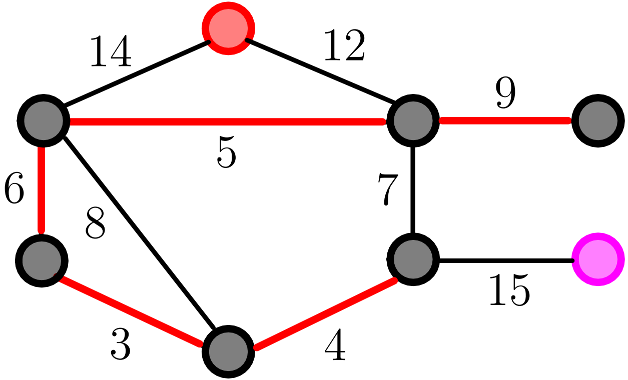

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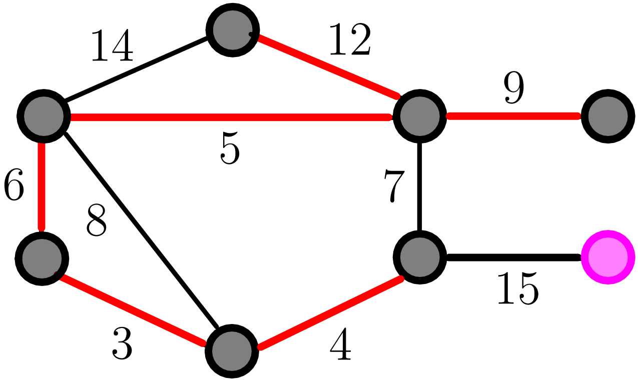

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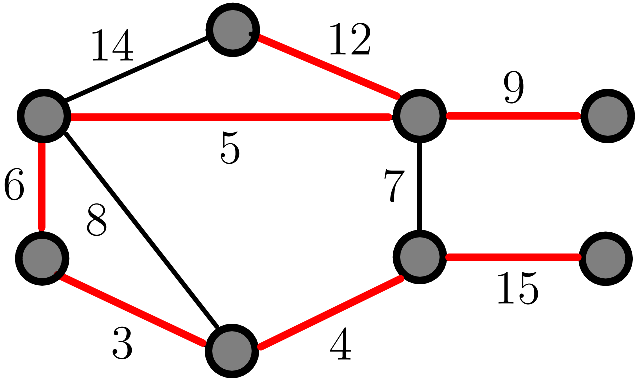


Data structures

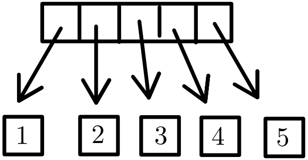
Operations on **disjoint sets of vertices:**

- **Find:** identify which set contains a given vertex
- **Union:** replace two sets by their union

GreedyMST_UnionFind(G)
$T \leftarrow [\ ]$
$U \leftarrow \{\{v_1\}, \ldots, \{v_n\}\}$
sort edges by non-decreasing weight
for $k = 1, \ldots, m$ **do**
  if U.Find(e_k.1) $\neq$ U.Find(e_k.2) **then**
    U.Union(U.Find(e_k.1), U.Find(e_k.2))
    append e_k to T

An OK solution

- U is an **array of linked lists**



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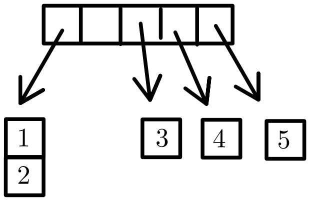

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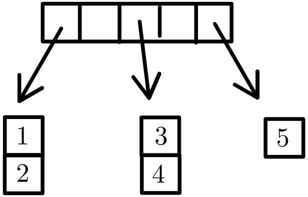

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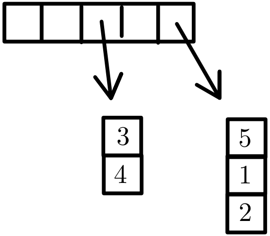

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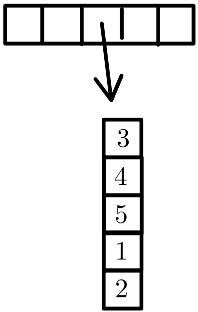

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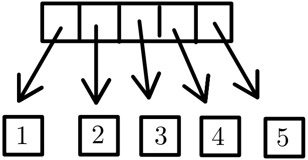
- to do find, add an **array of indices**, $X[i]$ = set that contains i



$$X = [1, 2, 3, 4, 5]$$

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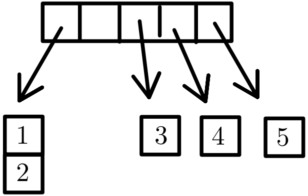

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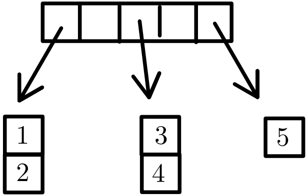

$$X = [1, 1, 3, 3, 5]$$

An OK solution

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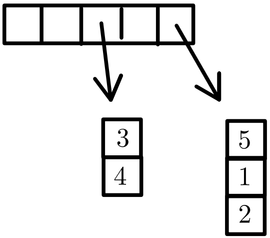
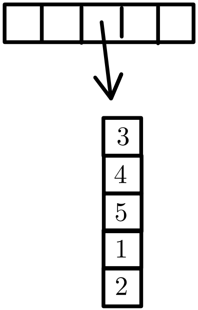


$$X = [5, 5, 3, 3, 5]$$

An OK solution

- U is an **array of linked lists**
- to do find, add an **array of indices**, $X[i]$ = set that contains i



$$X = [3, 3, 3, 3, 3]$$

Analysis

Worst case:

- **Find** is $O(1)$
- **Union** **traverses** one of the linked lists, **updates** corresponding entries of X, concatenates two linked lists. Worst case $\Theta(n)$

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- **Find** is $O(1)$
- **Union traverses** one of the linked lists, **updates** corresponding entries of X, concatenates two linked lists. Worst case $\Theta(n)$

Kruskal's algorithm:

- sorting edges $\boldsymbol{O(m\log(m))}$
- $O(m)$ **Find**
- $O(n)$ **Union**

Worst case $\boldsymbol{O(m\log(m)+n^2)}$

A simple heuristics for Union

Modified Union

- each set in U keeps track of its size
- only traverse the **smaller list**
- also add a pointer to the **tail** of the lists to concatenate in $O(1)$

A simple heuristics for Union

Modified Union

- each set in U keeps track of its size
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Key observation: worst case for **one** union **still** $\Theta(n)$, but better total time.

- for any given vertex v, the size of the set containing V **at least doubles** when we update $X[v]$
- so $X[v]$ updated at most $\log(n)$ times
- so the **total** cost of union per vertex is $O(\log(n))$

A simple heuristics for Union

Modified Union

- each set in U keeps track of its size
- only traverse the **smaller list**
- also add a pointer to the **tail** of the lists to concatenate in $O(1)$

Key observation: worst case for **one** union **still** $\Theta(n)$, but better total time.

- for any given vertex v, the size of the set containing V **at least doubles** when we update $X[v]$
- so $X[v]$ updated at most $\log(n)$ times
- so the **total** cost of union per vertex is $O(\log(n))$

Conclusion: $\boldsymbol{O(n \log(n))}$ for all unions and $\boldsymbol{O(m \log(m))}$ total