Question 1.3

Consider any fixed $S \subseteq [n]$. By the definition of the Fourier coefficient:

$$\hat{f}(S) = \mathbb{E}_{x \in \{\pm 1\}^n} [f(x)\chi_S(x)] = \frac{1}{2^n} \left[ \sum_{x : \chi_S(x) = 1} f(x) + \sum_{x : \chi_S(x) = -1} f(x) \right] = \frac{1}{2^n} \sum_{x : \chi_S(x) = 1} f(x) - \sum_{x : \chi_S(x) = -1} f(x) .$$

By the fact that the number of inputs $x$ such that $f(x) = -1$ is odd and the fact that $f$ is a boolean function, we get that the expression above is never 0.

Question 1.7

We start with computing the expectation: fix any set $S \subseteq [n]$

$$\mathbb{E}_f[\hat{f}(S)] = \mathbb{E}_f \left[ \mathbb{E}_{x \in \{\pm 1\}^n} [f(x)\chi_S(x)] \right] = \mathbb{E}_f \left[ \frac{1}{2^n} \sum_x f(x)\chi_S(x) \right] = \frac{1}{2^n} \sum_x \mathbb{E}_f[f(x)]\chi_S(x) = 0 .$$

For the variance, we need to analyze $\mathbb{E}_f[\hat{f}(S)^2]$. There are two ways to do it. The first is by expending the sums, and exchanging sums with expectation. However, there is a shortcut. We note that since $f : \{\pm 1\}^n \rightarrow \{\pm 1\}$,

$$\mathbb{E}_S \left[ \mathbb{E}_f[\hat{f}(S)^2] \right] = \frac{1}{2^n} .$$

Now, consider any $S, T \subseteq [n]$. We will show that $\mathbb{E}_f[\hat{f}(S)^2] = \mathbb{E}_f[\hat{f}(T)^2]$. Consider the function $g = f \cdot \chi_{S \triangle T}$. We note that since $f$ is a random function then $g$ is also a random function. Thus,

$$\mathbb{E}_f[\hat{f}(S)^2] = \mathbb{E}_{g = f \chi_{S \triangle T}} \left[ \hat{f}(S)^2 \right] = \mathbb{E}_f \left[ \mathbb{E}_x \left[ f(x) \chi_{S \triangle T}(x) \chi_T(x) \right]^2 \right] = \mathbb{E}_f[\hat{f}(T)^2] .$$

Since the expectation of all the coefficients are equal, we get the desired result.
3 Question 1.21

- Suppose that there exists \( f : \{\pm 1\} \to \{\pm 1\} \) such that \( f \) has only two non-zero coefficients, Say \( S_1, S_2 \subseteq [n] \). By the fact that \( f \) is \( \pm 1 \) valued function we have that \( \hat{f}(S_1)^2 + \hat{f}(S_2)^2 = 1 \). On the other hand, we have that

\[
1 = \left( \hat{f}(S_1)\chi_{S_1} + \hat{f}(S_2)\chi_{S_2} \right)^2 = \hat{f}(S_1)^2 + \hat{f}(S_2)^2 + 2\hat{f}(S_2)\chi_{S_2}\hat{f}(S_1)\chi_{S_1} = 1 + 2\hat{f}(S_2)\chi_{S_2}\hat{f}(S_1)\chi_{S_1},
\]

which implies that either \( \hat{f}(S_1) = 0 \) or \( \hat{f}(S_2) = 0 \), which is a contradiction.

- As before, we assume that there is a function \( f : \{\pm 1\} \to \{\pm 1\} \) such that \( f \) has only 3 non-zero coefficients, Say \( S_1, S_2, S_3 \subseteq [n] \).

Consider any \( x \in \{\pm 1\}^n \). Assume that there exists an index \( i \in [n] \) such that \( i \) appears in exactly one of the three sets \( S_1, S_2, S_3 \), and assume without the loss of generality that \( i \in S_1 \). Consider the following input \( x^{(i)} \) generated from \( x \) by flipping the \( i \)-th bit. Namely, \( x^{(i)} = (x_1, \ldots, -x_i, \ldots, x_n) \). Then,

\[
f(x^{(i)}) = \hat{f}(S_1)\chi_{S_1}(x^{(i)}) + \hat{f}(S_2)\chi_{S_2}(x^{(i)}) + \hat{f}(S_3)\chi_{S_3}(x^{(i)})
= -\hat{f}(S_1)\chi_{S_1}(x) + \hat{f}(S_2)\chi_{S_2}(x) + \hat{f}(S_3)\chi_{S_3}(x).
\]

So, \( f(x) - f(x^{(i)}) = 2\hat{f}(S_1)\chi_{S_1}(x) \), which implies that \( \hat{f}(S_2) = \hat{f}(S_3) = 0 \) which is a contradiction. Now, assume that there is an index \( i \) such that \( i \) appears in exactly 2 sets, say \( S_1, S_2 \). Similarly to the above, we get that \( f(x) + f(x^{(i)}) = 2\hat{f}(S_3)\chi_{S_3}(x) \), which implies that \( \hat{f}(S_1) = \hat{f}(S_2) = 0 \), which is a contradiction.

4 Question 2.23

Since \( f : \{\pm 1\}^n \to \{\pm 1\} \) is monotone, we have that \( I(f) = \sum_{i \in [n]} \hat{f}(\{i\}) \). Therefore, by using Cauchy–Schwarz and Parseval,

\[
I(f)^2 = \left( \sum_i \hat{f}(i) \right)^2 \leq \sum_i 1^2 \cdot \sum_i \hat{f}(i)^2 \leq n \cdot \sum_{S \subseteq [n]} \hat{f}(S)^2 \leq n.
\]

Concluding the proof. We proceed to the second part of the question. Recall that a function \( f \) is unate if it is monotone in each direction \( i \in [n] \). Fix some \( i \in [n] \). There can be two cases: (1) \( \forall x \) we have that \( f(x^{(i \to 1)}) \geq f(x^{(i \to -1)}) \) or (2) \( \forall x \) we have that \( f(x^{(i \to 1)}) \leq f(x^{(i \to -1)}) \).

1. For the first case we have that \( D_i(f) \in \{0, 1\} \), and therefore:

\[
\inf_{i \cdot} f = \mathbb{E}_x[D_i(f)^2] = \mathbb{E}_x[D_i(f)] = \mathbb{E}_x[f(x^{(i \to 1)})/2] - \mathbb{E}_x[f(x^{(i \to -1)})/2].
\]

Note that,

\[
\mathbb{E}_x[f^{(i \to 1)}] = \mathbb{E}_x \left[ \sum_{S \subseteq [n]} \hat{f}(S)\chi_{S}(x^{(i \to 1)}) \right] = \sum_{S} \hat{f}(S)\mathbb{E}_x[\chi_{S}(x^{(i \to 1)})] = \sum_{S} \hat{f}(S)\mathbb{E}_x[\chi_{S \setminus i}(x)] = \hat{f}(i).
\]

Similarly, we have that \( \mathbb{E}_x[f^{(i \to -1)}] = -\hat{f}(i) \), and therefore, \( \inf_{i \cdot} f = \hat{f}(i) \).
2. For the second case, we note that $D_i(f) \in \{0, -1\}$, and similar to the above, we get that $	ext{Inf}_i[f] = -\hat{f}(i)$. Thus, if $f$ is unate, we have that for any $i \in [n]$, $	ext{Inf}_i[f] \in \{\hat{f}(i), -\hat{f}(i)\}$. Hence,

$$I[f]^2 \leq \left(\sum_i |\hat{f}(i)|\right)^2 \leq n \sum_S \hat{f}(S)^2 \leq n.$$  

And we are done.

5 Question 2.56

1. We wish to maximize the following: $\Pr_{y^{(1)}, y^{(2)}}[f_1(y^{(1)}) = f_2(y^{(2)})] = \mathbb{E}[f_1(y^{(1)}) \cdot f_2(y^{(2)})] = \sum_{S, T} \hat{f}_1(S) \hat{f}_2(T) \mathbb{E}[\chi_S(y^{(1)}) \chi_T(y^{(2)})]$. Note that,

$$\mathbb{E}[\chi_S(y^{(1)}) \chi_T(y^{(2)})] = \mathbb{E}_x[\mathbb{E}_y^{(1)}[\chi_S(y^{(1)}) | x] \cdot \mathbb{E}_y^{(2)}[\chi_T(y^{(2)}) | x]] = (\ast),$$

and since for any $j \in \{1, 2\}$ and any set $R \subseteq [n],

$$\mathbb{E}_{y^{(j)}}[\chi_R(y^{(j)}) | x] = \prod_{i \in R} \left(\frac{1}{2} + \frac{1}{2} \rho \right) x_i - \left(\frac{1}{2} - \frac{1}{2} \rho \right)x_i = \prod_{i \in R} \rho x_i = \rho^{|R|} \chi_R(x)$$

We get that,

$$(\ast) = \rho^{|S|+|T|} \mathbb{E}_x[\chi_S(x) \chi_T(x)] = \begin{cases} \rho^{|S|+|T|} & \text{if } |S| = |T| \\ 0 & \text{otherwise} \end{cases}$$

Hence, the expression we wish to maximize is:

$$\sum_S \rho^{|S|} \hat{f}_1(S) \hat{f}_2(S) = \sum_{S \neq \emptyset} \rho^{|S|} \hat{f}_1(S) \hat{f}_2(S) \leq \sum_{S \neq \emptyset} \rho^2 \hat{f}_1(S) \hat{f}_2(S)$$

Where the first equality follows from the requirement that the functions $f^{(1)}, f^{(2)}$ has mean zero. by using Cauchy–Schwarz and Parseval,

$$\sum_{S \neq \emptyset} \rho^2 \hat{f}_1(S) \hat{f}_2(S) \leq \rho^2 \sqrt{\left(\sum_{S \neq \emptyset} \hat{f}_1(S)^2\right) \left(\sum_{S \neq \emptyset} \hat{f}_2(S)^2\right)} \leq \rho^2.$$

When we pick $f_1 = f_2 = \pm \chi_i$ we get,

$$\mathbb{E}[f_1(y^{(1)}) \cdot f_2(y^{(2)})] = \hat{f}_1(i) \hat{f}_2(i) \rho^2 = \rho^2,$$

which is the maximum possible.
2. We want to maximize the probability that the three functions \( f_1, f_2, f_3 \) agree. This is equivalent to maximizing the following expression.

\[
E\left[ f_1(y^{(1)}) \cdot f_2(y^{(2)}) + f_1(y^{(1)}) \cdot f_3(y^{(3)}) + f_3(y^{(3)}) \cdot f_2(y^{(2)}) \right] = (**).
\]

By using linearity of expectation and the previous item,

\[
(**) = \sum_{S \neq \emptyset} \rho^2 \hat{f}_1(S) \hat{f}_2(S) + \sum_{S \neq \emptyset} \rho^2 \hat{f}_1(S) \hat{f}_3(S) + \sum_{S \neq \emptyset} \rho^2 \hat{f}_3(S) \hat{f}_2(S) \leq 3 \rho^2.
\]

So by setting \( f_1 = f_2 = f_3 = \pm \chi_i \), we achieve the maximum.