CS 798-004 Bonus Question 1

Amit Levi

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The following problem came up during the course of an actual research project I was part of during the last year.

In class you have seen the definition of an influence of a variable. We would like to generalize this definition to a set of variables as follows.

Definition. *Given two disjoint sets $S, T \subseteq [n]$ and two partial assignments $x \in \{\pm 1\}^S$ and $y \in \{\pm 1\}^T$, we let $x \sqcup y \in \{\pm 1\}^{S \cup T}$ be the partial assignment whose i-th coordinate is x_i if $i \in S$ and y_i if $i \in T$. For a Boolean function $f \colon \{\pm 1\}^n \to \{\pm 1\}$, the* set-influence *of a set $S \subseteq [n]$ is defined as*

$$\mathbf{Inf}_f(S) = 2 \cdot \Pr_{x,u,v}\left[f(x \sqcup u) \neq f(x \sqcup v)\right] ,$$

where $x \sim \{\pm 1\}^{[n] \setminus S}$, and $u, v \sim \{\pm 1\}^S$ are all picked uniformly at random and independently. (Think why there is an extra 2 factor ?)

We say that a boolean function $f : \{\pm 1\}^n \to \{\pm 1\}$ is symmetric if its value depends only on the number of TRUE entries in the input x.

Show that if $f : \{\pm 1\}^n \to \{\pm 1\}$ is symmetric, then the influence $\mathbf{Inf}_f(\cdot)$, has the following symmetry property. For any two sets S and S' such that $|S| = |S'|$, it holds that $\mathbf{Inf}_f(S) = \mathbf{Inf}_f(S')$. Is the converse true as well? If so, prove it. Otherwise, give a counter-example.