Your name
Your student number

CS 466/666: ALGORITHM DESIGN AND ANALYSIS
PROBLEM SET 1

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1. Problem 1: Confidence booster

For this problem, you will show that we can always reduce the probability that one-sided bounded error algorithms with very little overhead.

Definition 1. A randomized algorithm $\mathcal{A}$ computes the function $f : \{0,1\}^n \to \{0,1\}$ with *one-sided bounded error* if

- for every $x \in f^{-1}(1)$,[1] the algorithm $\mathcal{A}$ *always* correctly outputs the value 1; and
- for every $x \in f^{-1}(0)$, the algorithm $\mathcal{A}$ outputs the value 0 with probability at least $\frac{2}{3}$.

Theorem 1. *For any $0 < \delta < \frac{1}{3}$, if there is a randomized algorithm $\mathcal{A}$ with time complexity t that computes $f : \{0,1\}^n \to \{0,1\}$ with one-sided bounded error, then there is also a randomized algorithm $\mathcal{A}'$ with time complexity $O(t \log \frac{1}{\delta})$ that satisfies the following two conditions:*

(1) For every $x \in f^{-1}(1)$, the algorithm $\mathcal{A}'$ always correctly outputs 1; and
(2) For every $x \in f^{-1}(0)$, the algorithm $\mathcal{A}'$ outputs the value 0 with probability at least $1 - \delta$.

Proof. Enter your answer here. □

Hint. Your algorithm $\mathcal{A}'$ should be calling $\mathcal{A}$ as a subroutine.

[1]The set $f^{-1}(1) = \{x \in \{0,1\}^n : f(x) = 1\}$ is the set of inputs $x \in \{0,1\}^n$ for which f takes the value $f(x) = 1$.

2. Problem 2: Another confidence booster

Recall that a randomized algorithm $\mathcal{A}$ computes the function $f : \{0,1\}^n \to \{0,1\}$ with *(two-sided) bounded error* if on every input $x \in \{0,1\}^n$, it outputs the value $f(x)$ with probability at least $\frac{2}{3}$. Show that we can also reduce the probability of error of (two-sided) bounded error algorithms.

Theorem 2. *For any $0 < \delta < \frac{1}{3}$, if there is a bounded error randomized algorithm $\mathcal{A}$ with time complexity t that computes the function $f : \{0,1\}^n \to \{0,1\}$ with bounded error, then there is also a randomized algorithm $\mathcal{A}'$ that computes f with time complexity $O(t \log \frac{1}{\delta})$ such that for every $x \in \{0,1\}^n$, $\mathcal{A}'$ outputs the value $f(x)$ with probability at least $1 - \delta$.*

Proof. Enter your answer here. □

Hint. You may use the following special case of Hoeffding's inequality in your proof.

Lemma 1. *Fix $m \geq 1$ and $\epsilon > 0$. When $X_1, \ldots, X_m \in \{0,1\}$ are independent random variables each satisfying $\Pr[X_i = 1] \leq \frac{1}{3}$, then*

$$\Pr\left[\frac{1}{m}\sum_{i=1}^{m} X_i \geq \frac{1}{2}\right] \leq e^{-m/18}.$$

3. Problem 3: Nearly the same

Consider the slight variant on the Equality problem where we want to know whether two strings $x, y \in \{0,1\}^n$ differ on at most one bit. In this problem, you will show how to solve this problem (asymptotically) as efficiently as the Equality problem.

Definition 2. The function NearEqual : $\{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ is defined by setting

$$\text{NearEqual}(x, y) = \begin{cases} 1 & \text{if } |\{i \in [n] : x_i \neq y_i\}| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

We will consider the NearEqual function in the same setting as we considered Equality in class: Alice has x, Bob has y, they want to design an algorithm to compute NearEqual(x, y), and they want this algorithm to require as little communication as possible.

Theorem 3. *There is a randomized algorithm $\mathcal{A}$ that computes* NearEqual *with bounded error which requires Alice and Bob to exchange only $O(1)$ bits of communication.*

Proof. Enter your answer here. □