# On testing and robust characterizations of convexity


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## Abstract

A body $K \subset \mathbb{R}^n$ is convex if and only if the line segment between any two points in $K$ is completely contained within $K$ or, equivalently, if and only if the convex hull of a set of points in $K$ is contained within $K$. We show that neither of those characterizations of convexity are robust: there are bodies in $\mathbb{R}^n$ that are far from convex—in the sense that the volume of the symmetric difference between the set $K$ and any convex set $C$ is a constant fraction of the volume of $K$—for which a line segment between two randomly chosen points $x, y \in K$ or the convex hull of a random set $X$ of points in $K$ is completely contained within $K$ except with exponentially small probability. These results show that any algorithms for testing convexity based on the natural line segment and convex hull tests have exponential query complexity.



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## 1 Introduction

A *body* is a subset of $\mathbb{R}^n$ that is compact—i.e., closed and bounded—and has a non-empty interior. A body $K \subset \mathbb{R}^n$ is *convex* if for every two points $x, y \in K$ and every parameter $\lambda$ in the range $[0, 1]$, the point $z = \lambda x + (1 - \lambda)y$ is also in $K$. The *geometric convexity testing* problem is a formalization of the following property testing problem:

How efficiently can we distinguish convex bodies from those that are far from convex?

The geometric convexity testing problem was first studied by Rademacher and Vempala [18], who formalized the problem as follows. A body $K \subset \mathbb{R}^n$ is *$\epsilon$-far from convex* for some $\epsilon > 0$ if for every convex body $C \subset \mathbb{R}^n$, the volume of the symmetric difference of $K$ and $C$ is bounded below by $\mathrm{Vol}(K \triangle C) \geq \epsilon \, \mathrm{Vol}(K)$. Following the standard framework of property testing [20, 12], we can then define an *$\epsilon$-tester for convexity* to be a bounded-error randomized algorithm that distinguishes convex bodies from bodies that are $\epsilon$-far from convex. We consider testers that access an unknown body $K \subset \mathbb{R}^n$ via the following two standard oracles:

**Membership oracle.** Given as input a point $x \in \mathbb{R}^n$, the oracle returns "yes" if and only if $x \in K$.

**Random oracle.** The oracle returns a point $x$ drawn uniformly at random from $K$.

The measure of complexity of $\epsilon$-testers of convexity that we examine is the minimum number of queries to either of these oracles that they require, and the main open question is whether there exists an $\epsilon$-tester of convexity of bodies in $\mathbb{R}^n$ that has query complexity that is polynomial in both $n$ and $1/\epsilon$. Three natural testers have been proposed and studied previously with the aim of answering this question.



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### I. Approximation tester.

Rademacher and Vempala [18] showed that there is an $\epsilon$-tester for convexity of bodies in $\mathbb{R}^n$ with query complexity $q = (cn/\epsilon)^n$ for some constant $c > 0$. This tester is obtained via the natural *testing by learning* approach [12]. With $q$ queries to the random oracle for $K$, we obtain a set $S$ of points whose convex hull $C$ satisfies $\mathrm{Vol}(K \triangle C) < \frac{\epsilon}{2} \mathrm{Vol}(K)$ with high probability when $K$ is convex and (by definition) always satisfies $\mathrm{Vol}(K \triangle C) \geq \epsilon \mathrm{Vol}(K)$ when $K$ is $\epsilon$-far from convex. A tester can then distinguish between these two cases with $O(1/\epsilon)$ queries to the random and membership oracles for $K$.

However, any $\epsilon$-tester for convexity that follows the testing by learning approach must have query complexity exponential in $n$, since any algorithm that learns a convex set $C$ which satisfies $\mathrm{Vol}(K \triangle C) \leq \epsilon \mathrm{Vol}(K)$ for some unknown convex body $K \subset \mathbb{R}^n$ must have query complexity $2^{\Omega(\sqrt{n}/\epsilon)}$ [13]. (In fact, a number of queries that is exponential in $n$ is required even just for estimating the volume of $K$ [2, 11].) So a completely different approach is required if we aim to test convexity of high-dimensional bodies more efficiently.

### II. Line segment tester.

The definition of convex bodies immediately suggests a simple *line segment test* for convexity: draw two points $x, y \in K$ using the random oracle for $K$, pick a parameter $\lambda \in [0, 1]$ according to some distribution, and use the membership oracle to determine if the point $z = \lambda x + (1 - \lambda)y$ is in $K$. If $K$ is convex, this test will always pass, and conversely when $K$ is $\epsilon$-far from convex then there must exist some points $x, y, z$ for which this test does not pass.

A natural idea for constructing an $\epsilon$-tester for convexity is to simply run the line segment test multiple times and accept if and only if each test passes. But Rademacher and Vempala [18] showed that the resulting tester cannot have query complexity that is polynomial in *both* $n$ and $1/\epsilon$. More precisely, they showed that there is a body $K \subset \mathbb{R}^n$ which is $\Omega(\frac{1}{n^2})$-far from convex but for which the line segment $\overline{xy}$ joining two points $x$ and $y$ drawn uniformly at random from $K$ satisfies $\Pr[\overline{xy} \not\subseteq K] = 2^{-\Omega(n)}$.

The counter-example of Rademacher and Vempala, however, does not rule out the possibility that there exists an $\epsilon$-tester of convexity with query complexity polynomial in $n$ when $\epsilon$ is a constant. Our first result rules out this possibility as well, showing that the query complexity of testers obtained from the line segment test must be exponential in $n$ for all $\epsilon \leq \frac{1}{8}$.

▶ **Theorem 1.1.** *There exists a body $K \subset \mathbb{R}^n$ that is $\frac{1}{8}$-far from convex for which*

$$\Pr_{x,y \in K}\left[\overline{xy} \not\subseteq K\right] = 2^{-\Omega(n)}. \qquad (1)$$

Theorem 1.1 has another interpretation that is independent of property testing: it says that the line segment characterization of convexity is not *robust*: while it is true that only convex bodies satisfy $\overline{xy} \in K$ for every $x, y \in K$, there are bodies that are far from convex where $\overline{xy} \in K$ still holds for "most" points in $K$.

### III. Convex hull tester

For any $m \geq 1$, the *convex hull* of a set $X = \{x^{(1)}, \ldots, x^{(m)}\}$ of $m$ points in $\mathbb{R}^n$ is

$$\mathrm{conv}\Big(X\Big) := \left\{ y : \exists \lambda_1, \ldots, \lambda_m \geq 0, \sum_{i=1}^{m} \lambda_i = 1 \text{ s.t. } y = \sum_{i=1}^{m} \lambda_i x^{(i)} \right\},$$

the set of points that can be obtained by taking a convex combination of the points in $X$. A natural extension of the line segment test is the *convex hull test*: for some $m \geq 2$, draw a set $X$ of $m$ points from $K$ using the random oracle, draw a point $z$ from $\mathrm{conv}(X)$ according to some distribution, and check whether $z$ is in $K$. When $m = 2$, the convex hull test is equivalent to the line test which, as we have seen above, cannot lead to an efficient tester for convexity. For $m \geq 3$, however, it is possible that it leads to much more efficient testing algorithms. Indeed, Berman, Murzabulatov, and Raskhodnikova [6, 5] showed that in a slightly different property testing model, the convex hull test can be used to test convexity with a number of queries that is polynomial in $1/\epsilon$ in the two-dimensional setting where $K \subset \mathbb{R}^2$. (See also [19, 7] for related results.)

Our next and main result rules out the possibility of obtaining an efficient tester for convexity in the high-dimensional setting using the convex hull test by showing that the convex hull characterization is not robust, even when taking the convex hull of an *exponential* (in the dimension $n$) number of points.

▶ **Theorem 1.2.** *There exist a body $K \subset \mathbb{R}^n$ that is $\frac{1}{8}$-far from convex and a constant $c > 0$ such that a set $X = \{x^{(1)}, \ldots, x^{(m)}\} \in K$ of $m = 2^{cn}$ points drawn uniformly and independently at random from $K$ satisfies*

$$\Pr_X \left[\mathrm{conv}(X) \not\subseteq K\right] = 2^{-\Omega(n)}.$$

Theorem 1.2 shows that any $\epsilon$-tester for convexity built on the convex hull test must have query complexity $2^{\Omega(n)}$. It also relates to a conjecture of Rademacher and Vempala [18]: they conjecture that when $K \subset \mathbb{R}^n$ is $\epsilon$-far from convex and $x, y, z \in K$ are drawn uniformly at random from $K$, then the intersection of $K$ with the two-dimensional subspace spanned by $x$, $y$, and $z$ is non-convex with probability at least $\Omega(\epsilon/n)$. Theorem 1.2 shows that it is impossible to strengthen the conjecture by replacing the subspace spanned by $\{x, y, z\}$ with the convex hull of these points, since in that case the resulting statement is false.

## 1.1 Proof overview

Theorems 1.1 and 1.2 are established constructively. The construction that achieves the bounds promised in the theorems is obtained by taking the union of two truncated cones, as pictured in Figure 1. The main technical component of the proof of the theorems lies in the task of showing that, contrary to what our low-dimensional intuition might suggest, the union of two truncated cones is far from being convex, even when the radius at the point of intersection of both cones is very close to the maximum radius of both truncated cones. The construction is defined precisely and is shown to be far from convex in Section 3.

The proof of Theorem 1.1 is completed in Section 4.1, where we show that with high probability the line segment joining two points drawn uniformly at random from the union of two truncated cones is contained within the body. We then build on this result in Section 4.2 to show that the convex hull of $m \geq 2$ points drawn uniformly from that body is also contained within the body with high probability.

## 1.2 Discussion

### Testing convexity efficiently.

Our results do not rule out the possibility that convexity of high-dimensional sets can be tested with a number of queries to random and membership oracles that is polynomial in

$n$ and $1/\epsilon$, but they do show that new algorithmic techniques that go beyond convex hull testing are required if such an efficient convexity tester exists. To determine which additional techniques might be useful in obtaining such an efficient convexity tester (or ruling out their existence), it might be instructive to point out that the body constructed in Section 3 is in fact very easy to distinguish from convex bodies. One way to do this is to notice that the union of truncated cones has poor *expansion*: if we take a random walk from a point within one of the two truncated cones, with high probability it will remain within the same truncated cone. By contrast, a random walk in a convex body quickly converges to a distribution that is close to uniform in the body. Is it possible to efficiently test if an unknown body is expanding or far from it? And is it also possible to efficiently distinguish convex sets from expanding sets that are far from convex? Affirmative answers to both of these questions—for any reasonable formalization of the expansion testing problem—would likely lead to a new efficient tester for convexity; the question of testing expansion of high-dimensional sets also appears to be worth studying for independent interest as well.

#### Testing convexity over the Gaussian distribution.

There is another formalization of the geometric convexity testing problem in which we measure the distance to convexity in terms of of the Gaussian distribution on $\mathbb{R}^n$. Chen, Freilich, Servedio, and Sun [10] studied *sample-based* testers for convexity in this model—testers that have access to the membership oracle but can only observe its responses to points drawn from the Gaussian distribution. They showed that all such sample-based testers for convexity have exponential sample complexity. Could the construction introduced in this paper be extended to show a similar bound for the query complexity of a wider class of testers in the same setting? Such results do not follow immediately from the current work since the argument showing that the union of truncated cones is far from convex does not hold in the Gaussian distribution setting.

#### Testing convex functions.

Another problem that has received a considerable amount of attention in the property testing literature, starting with the work of Parnas, Ron, and Rubinfeld [17], is that of testing the convexity of *functions* [8, 4, 9, 3]. There is a close connection between convexity of sets and convexity of functions. Namely, a function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if its epigraph is a convex set in $\mathbb{R}^{n+1}$. The definitions of distance to convexity, however, make the problems of testing convex functions and testing convex sets quite different in general. Nonetheless, as Berman, Raskhodnikova, and Yaroslavtsev [8] pointed out, the two problems are closely connected when we consider the testing of convex functions under the $\ell_1$ norm, and it would be interesting to see if the techniques or results introduced here could yield any progress on the problem of testing convex functions with a polynomial number of queries. (See [8, 22] for more details on this problem.)

## 2 Preliminaries

We use standard notions and results regarding high-dimensional convex sets. For general introductions to the topic and to algorithmic implications, see [1, 15, 16, 14, 21].

## 2.1 Convex bodies and slices

The *distance* between two bodies $A, B \in \mathbb{R}^n$ is defined to be

$$\mathrm{dist}(A, B) = \mathrm{Vol}(A \triangle B) = \mathrm{Vol}(A \setminus B) + \mathrm{Vol}(B \setminus A),$$

the measure of the symmetric difference of the two bodies. We will repeatedly use the following simple lower bound on the distance of two bodies.

▶ **Proposition 2.1.** *The distance between two bodies $A, B \subset \mathbb{R}^n$ is bounded below by*

$$\mathrm{dist}(A, B) \geq \max\left\{\mathrm{Vol}(A) - \mathrm{Vol}(B), \mathrm{Vol}(B) - \mathrm{Vol}(A)\right\}.$$

*Furthermore, equality holds whenever $A \subseteq B$ or $B \subseteq A$.*

**Proof.** The distance between $A$ and $B$ is bounded below by

$$\mathrm{dist}(A, B) = \mathrm{Vol}(A \setminus B) + \mathrm{Vol}(B \setminus A) \geq \max\{\mathrm{Vol}(A \setminus B), \mathrm{Vol}(B \setminus A)\}.$$

The lower bound then follows from the observation that $\mathrm{Vol}(A \setminus B) = \mathrm{Vol}(A) - \mathrm{Vol}(A \cap B) \geq \mathrm{Vol}(A) - \mathrm{Vol}(B)$ and, similarly, that $\mathrm{Vol}(B \setminus A) \geq \mathrm{Vol}(B) - \mathrm{Vol}(A)$. Finally, when $A \subseteq B$, then $\mathrm{Vol}(A \setminus B) = 0$ and $\mathrm{Vol}(B \setminus A) = \mathrm{Vol}(B) - \mathrm{Vol}(A)$, as $\mathrm{Vol}(A \cap B) = \mathrm{Vol}(A)$ so equality holds. Similarly, equality also holds when $B \subseteq A$. ◀

Much of our analysis in Section 3 is concerned with various slices of a high-dimensional body. To make this notion precise, for each $t \in \mathbb{R}$ we define

$$H_t = \{x \in \mathbb{R}^n : x_1 = t\}$$

to be the hyperplane of all points with first coordinate value $t$. The corresponding halfspaces are denoted by $H_{\leq t} = \{x \in \mathbb{R}^n : x_1 \leq t\}$ and $H_{\geq t} = \{x \in \mathbb{R}^n : x_1 \geq t\}$. The $t$-th *slice* of a body $A \subset \mathbb{R}^n$ is

$$A_t = A \cap H_t = \{x \in A : x_1 = t\}.$$

For $t_1 \leq t_2 \in \mathbb{R}$, we also define $A_{[t_1,t_2]} = A \cap H_{\geq t_1} \cap H_{\leq t_2}$ to be the set of points in $A$ with first coordinate between $t_1$ and $t_2$.

A fundamental property of the slices of a convex body is that the $(n-1)$-th root of their volumes is a concave function.

▶ **Brunn's Theorem.** *For any convex body $C \subset \mathbb{R}^n$, the function $t \mapsto \mathrm{Vol}_{n-1}(C_t)^{\frac{1}{n-1}}$ is concave on its support.*

## 2.2 High-dimensional balls and cones

We use $B_n(x, r) = \{y \in \mathbb{R}^n : \|y - x\| \leq r\}$ to denote the ball of radius $r$ around a point $x \in \mathbb{R}^n$. We use $B(r)$ as a shorthand for $B_n(o, r)$, and $B_{n-1}(r)$ for $B_{n-1}(o, r)$, where $o$ is the origin. Similarly, we use $S_n(x, r) = \{y \in \mathbb{R}^n : \|y - x\| = r\}$ to denote the sphere of radius $r$ around a point $x \in \mathbb{R}^n$. We will use the following standard approximation on the volume of the ball.

▶ **Proposition 2.2.** *The volume of ball $B(r) \subset \mathbb{R}^n$ with radius $r$ is*

$$\mathrm{Vol}(B(r)) = r^n \cdot \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)} = r^n \cdot \frac{1}{\sqrt{\pi n}} \left(\frac{2\pi e}{n}\right)^{\frac{n}{2}} \cdot (1 + O(n^{-1})),$$

*where $\Gamma$ is Euler's gamma function.*

We also use the following standard concentration inequality for high-dimensional balls. (See, e.g., [1].)

▶ **Proposition 2.3.** *Let $x \in \mathbb{R}^n$ be drawn uniformly at random from $B(r)$. Then*

$$\Pr\left[|x_1| \geq \tfrac{r}{100}\right] \leq 2^{-\Omega(n)}.$$

▶ **Definition 2.4.** *Let $H \subset \mathbb{R}^n$ be a hyperplane, $S \subset H$ be an $(n-1)$-dimensional convex body, and $x \in \mathbb{R}^n \setminus H$ be a point. The* cone *with $x$ as vertex and $S$ as base is the convex hull of $x$ with the body $S$;*

$$\mathrm{cone}(x, S) = \mathrm{conv}(x \cup S).$$

We use the following result on the volume of cones.

▶ **Proposition 2.5.** *Let $H \subset \mathbb{R}^n$ be a hyperplane, $S \subset H$ be an $(n-1)$-dimensional convex body, and $x \in \mathbb{R}^n \setminus H$ be a point at a distance $h = \min_{y \in H} \|x - y\|$ from the hyperplane $H$. Then the volume of the cone is*

$$\mathrm{Vol}(\mathrm{cone}(x, S)) = \frac{h}{n} \mathrm{Vol}_{n-1}(S).$$

▶ **Definition 2.6.** *A* truncated cone *is the convex hull of two balls $B_{n-1}((t_1, 0 \ldots, 0), r_1) \subset H_{t_1}$ and $B_{n-1}((t_2, 0 \ldots, 0), r_2) \subset H_{t_2}$, $\mathrm{conv}(B_{n-1}((t_1, 0 \ldots, 0), r_1) \cup B_{n-1}((t_2, 0 \ldots, 0), r_2))$.*

# 3 Union of truncated cones

Theorems 1.1 and 1.2 are both established by analyzing a construction obtained by taking the union of two truncated cones. We describe this construction in Section 3.1. The main technical component of the proofs is in Section 3.2, where we show that the union of truncated cones is far from convex.

## 3.1 Description of the union of truncated cones $D$

The body $D \subset \mathbb{R}^n$ that will show the non-robustness of the line and convex hull definition is defined as follows. First, let $d > 0$ be some distance parameter. This distance parameter does not affect the results in the following sections; the reader may fix $d = 1$ for simplicity.

Let $B_L$, $B_R$, and $B_O$ be three $(n-1)$-dimensional balls in the hyperplanes $H_{-d}$, $H_d$, and $H_0$, respectively. We define the balls $B_L$ and $B_R$ to have radius 1.1 each and be centered on the points $(-d, 0, 0, \ldots, 0)$ and $(d, 0, 0, \ldots, 0)$, respectively, while $B_O$ has radius 1 and is centered at the origin. Define the body $D$ to be the union of the truncated cones $\mathrm{conv}(B_L, B_O)$ and $\mathrm{conv}(B_O, B_R)$,

$$D = \mathrm{conv}(B_L, B_O) \cup \mathrm{conv}(B_O, B_R).$$

The definition of the body $D$ is illustrated in Figure 1.

One of the basic properties of the body $D$ that we will use in later sections is that a point drawn uniformly at random from $D$ will have a large value in its first coordinate with high probability.

▶ **Proposition 3.1.** *Let $x = (x_1, \ldots, x_n)$ be drawn uniformly at random from $D$. Then*

$$\Pr\left[|x_1| \leq \frac{d}{2}\right] = 2^{-\Omega(n)}.$$

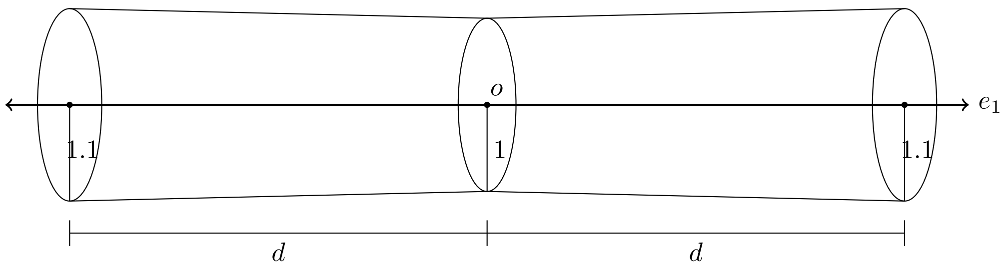


**Figure 1** The body $D$ obtained by taking the union of two truncated cones.

**Proof.** Using the formula in Proposition 2.5 for the volume of a cone, the volume of the body $D$ is bounded below by

$$\begin{aligned}\mathrm{Vol}(D) = 2\,\mathrm{Vol}(D_{[0,d]}) &= 2\Big(\frac{11d}{n}\,(1.1)^{n-1}\,\mathrm{Vol}_{n-1}\big(B_{n-1}(1)\big) \\ &\qquad - \frac{10d}{n}\,(1)^{n-1}\,\mathrm{Vol}_{n-1}(B_{n-1}(1))\Big) \\ &\geq \frac{2d}{n}\,(1.1)^{n-1}\,\mathrm{Vol}_{n-1}(B_{n-1}(1)).\end{aligned}$$

Similarly, the volume of the body $D_{[-\frac{d}{2},\frac{d}{2}]}$ is bounded above by

$$\begin{aligned}\mathrm{Vol}(D_{[-\frac{d}{2},\frac{d}{2}]}) = 2\,\mathrm{Vol}(D_{[0,\frac{d}{2}]}) &= 2\Big(\frac{10.5d}{n}\,(1.05)^{n-1}\,\mathrm{Vol}_{n-1}(B_{n-1}(1)) \\ &\qquad - \frac{10d}{n}\,(1)^{n-1}\,\mathrm{Vol}_{n-1}(B_{n-1}(1))\Big) \\ &\leq \frac{21d}{n}\,(1.05)^{n-1}\,\mathrm{Vol}_{n-1}(B_{n-1}(1)).\end{aligned}$$

Therefore, the probability that the absolute value of the first coordinate of the point is less than or equal to $\frac{d}{2}$ is bounded by

$$\Pr\left[|x_1| \leq \frac{d}{2}\right] = \frac{\mathrm{Vol}(D_{[-\frac{d}{2},\frac{d}{2}]})}{\mathrm{Vol}(D)} \leq \frac{\frac{21d}{n}\,(1.05)^{n-1}\,\mathrm{Vol}_{n-1}(B_{n-1}(1))}{\frac{2d}{n}\,(1.1)^{n-1}\,\mathrm{Vol}_{n-1}(B_{n-1}(1))} \leq \frac{1}{2^{\Omega(n)}}. \qquad \blacktriangleleft$$

## 3.2 $D$ is far from convex

In this section, we prove that the body $D$ is $\frac{1}{8}$-far from convex. We prove this in three steps. First, we show that the closest convex body to $D$ must be symmetric about the axis $e_1 = (1, 0, 0, \ldots, 0)$. Next, we prove that if the closest convex body is symmetric about $e_1$, then it has to be a truncated cone. Finally, we prove that every truncated cone is $\frac{1}{8}$-far from our body $D$.

### 3.2.1 A partial converse to Brunn's Theorem

The first step in the proof—showing that the closest convex body to $D$ must be symmetric—uses Brunn's Theorem as well as the following partial converse result.

▶ **Lemma 3.2.** *Let $K \subset \mathbb{R}^n$ be a body such that for each $t \in \mathbb{R}$ the slice $K_t$ is an $(n-1)$ dimensional ball and the function $t \mapsto \mathrm{Vol}_{n-1}(K_t)^{\frac{1}{n-1}}$ is concave on its support. Then $K$ is convex.*

**Proof.** Assume for the sake of contradiction that $K$ is a non-convex body that satisfies the conditions of the theorem. Then there exist points $x, y \in K$ and $0 \leq \lambda \leq 1$ such that the point $z = \lambda x + (1-\lambda)y \notin K$.

Let $r : \mathbb{R} \to \mathbb{R}$ be the function defined by setting $r(t)$ to be the radius of the $(n-1)$-dimensional ball with volume $\mathrm{Vol}_{n-1}(K_t)$. By applying the formula for volume of the ball from Proposition 2.2, we have that

$$r(t) = \frac{\Gamma(\frac{n-1}{2}+1)^{\frac{1}{n-1}}}{\sqrt{\pi}} \mathrm{Vol}_{n-1}(K_t)^{\frac{1}{n-1}}.$$

Since the function $t \mapsto \mathrm{Vol}_{n-1}(K_t)^{\frac{1}{n-1}}$ is concave, the function $r$ is concave as well.

The concavity of $r$ and the fact that $x, y \in K$ imply that

$$r(z_1) = r(\lambda x_1 + (1-\lambda)y_1) \geq \lambda r(x_1) + (1-\lambda)r(y_1) \geq \lambda \sqrt{\sum_{2 \leq i \leq n} x_i^2} + (1-\lambda)\sqrt{\sum_{2 \leq i \leq n} y_i^2}.$$

The fact that $z \notin K$ also implies that

$$r(z_1) < \sqrt{\sum_{2 \leq i \leq n} z_i^2} = \sqrt{\sum_{2 \leq i \leq n} (\lambda x_i + (1-\lambda)y_i)^2}.$$

But by the convexity of Euclidean norm and Jensen's inequality, $\sqrt{\sum_{2 \leq i \leq n}(\lambda x_i + (1-\lambda)y_i)^2}$ $\leq \lambda\sqrt{\sum_{2 \leq i \leq n} x_i^2} + (1-\lambda)\sqrt{\sum_{2 \leq i \leq n} y_i^2}$ so the last two inequalities yield the desired contradiction. ◀

### 3.2.2 Symmetry of the closest convex body

A body $K \subset \mathbb{R}^n$ is *symmetric* about $e_1 = (1, 0, 0, \ldots, 0)$ if for every point $x \in K$, all points $y \in \mathbb{R}^n$ that satisfy $x_1 = y_1$ and $\sum_{i=2}^n x_i^2 = \sum_{i=2}^n y_i^2$ are also in $K$. In other words, a body $K$ is symmetric about $e_1$ if it is invariant under rotations about the axis $e_1$. We use a standard symmetrization argument to prove that the closest convex body to $D$ is symmetric about $e_1$.

► **Lemma 3.3.** *There exists a closest convex body to $D$ that is symmetric about $e_1$.*

**Proof.** Fix $C$ to be any convex body which minimizes $\mathrm{dist}(D, C)$. $C$ should be contained between $H_{-d}$ and $H_d$, otherwise we can truncate $C$ and get a convex body closer to $D$. Let $C^s$ be the body where for every $t \in \mathbb{R}$, the slice $C_t^s$ of the body is an $(n-1)$-dimensional ball centered at $(t, 0, 0, \ldots, 0)$ and has volume $\mathrm{Vol}_{n-1}(C_t^s)$ equal to the volume of the slice $C_t$ of $C$. By this construction, $C^s$ is symmetric about $e_1$. To complete the proof, we need to show that it satisfies $\mathrm{dist}(D, C^s) \leq \mathrm{dist}(D, C)$ and that it is convex.

We first establish the inequality $\mathrm{dist}(D, C^s) \leq \mathrm{dist}(D, C)$. The distance between $D$ and $C$ is

$$\mathrm{dist}(D, C) = \int_{t=-d}^{d} \mathrm{dist}(D_t, C_t)\,\mathrm{d}t = \int_{t=-d}^{d} \mathrm{Vol}_{n-1}(D_t \triangle C_t)\,\mathrm{d}t.$$

By Proposition 2.1, for every $t \in \mathbb{R}$, the volume of the symmetric difference between the slices $D_t$ and $C_t$ is bounded below by

$$\mathrm{Vol}_{n-1}(D_t \triangle C_t) \geq \max\Big\{\mathrm{Vol}_{n-1}(C_t) - \mathrm{Vol}_{n-1}(D_t), \mathrm{Vol}_{n-1}(D_t) - \mathrm{Vol}_{n-1}(C_t)\Big\}.$$

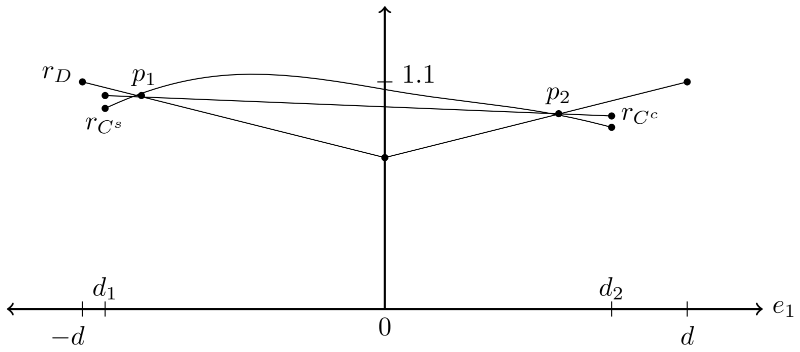


**Figure 2** Illustration of the radius functions $r_D$, $r_{C^s}$, and $r_{C^c}$ and of the points $p_1$ and $p_2$ in the proof of Lemma 3.5.

Since $\mathrm{Vol}_{n-1}(C_t) = \mathrm{Vol}_{n-1}(C_t^s)$, we then obtain

$$\mathrm{dist}(D, C) \geq \int_{t=-d}^{d} \max\left\{\mathrm{Vol}_{n-1}(C_t^s) - \mathrm{Vol}_{n-1}(D_t), \mathrm{Vol}_{n-1}(D_t) - \mathrm{Vol}_{n-1}(C_t^s)\right\} \mathrm{d}t.$$

Both $D_t$ and $C_t^s$ are balls with the same center, so one is a strict subset of the other and so we can apply the equality condition of Proposition 2.1 to obtain

$$\mathrm{dist}(D, C) \geq \int_{t=-d}^{d} \mathrm{Vol}_{n-1}(D_t \triangle C_t^s)\,\mathrm{d}t = \int_{t=-d}^{d} \mathrm{dist}(D_t, C_t^s)\,\mathrm{d}t = \mathrm{dist}(D, C^s),$$

as we wanted to show.

We now complete the proof of the lemma by showing that $C^s$ is convex. From Brunn's Theorem, the function $t \mapsto \mathrm{Vol}_{n-1}(C_t)^{\frac{1}{n-1}}$ is concave on its support. And from the construction we have that $\mathrm{Vol}_{n-1}(C_t^s)^{\frac{1}{n-1}} = \mathrm{Vol}_{n-1}(C_t)^{\frac{1}{n-1}}$. Hence, the function $t \mapsto \mathrm{Vol}_{n-1}(C_t^s)^{\frac{1}{n-1}}$ is also concave on its support and by Lemma 3.2, the body $C^s$ is convex. ◀

### 3.2.3 The closest convex body is a truncated cone

We now show that the closest symmetric convex body to $D$ is a truncated cone. The proof of this claim uses the following standard result about the separation of convex and concave functions.

▶ **Lemma 3.4.** *Fix any $d_1 \leq d_2 \in \mathbb{R}$. Let $f : [d_1, d_2] \to \mathbb{R}$ be a convex function and $g : [d_1, d_2] \to \mathbb{R}$ be a concave function such that $\forall t \in [d_1, d_2]$, $f(t) \geq g(t)$. Then there exists an affine function $h : [d_1, d_2] \to \mathbb{R}$ such that $g(t) \leq h(t) \leq f(t)$ for all $t \in [d_1, d_2]$.*

**Proof.** The proof follows from the fact that any two convex sets have a separating hyperplane. Let $S_1 = \{(t, x) : t \in [d_1, d_2], x \geq f(t)\}$ and $S_2 = \{(t, x) : t \in [d_1, d_2], x \leq g(t)\}$. The sets $S_1$ and $S_2$ are convex and their separating hyperplane corresponds to the function $h$. ◀

▶ **Lemma 3.5.** *The closest convex body to $D$ that is symmetric about $e_1$ is a truncated cone.*

**Proof.** Let $C^s$ be a convex body that is symmetric about $e_1$ and minimizes $\mathrm{dist}(D, C^s)$. We will construct a truncated cone $C^c$ that is also symmetric about $e_1$ and satisfies $\mathrm{dist}(D, C^c) \leq \mathrm{dist}(C^s, D)$.

Define the functions $r_D, r_{C^s} : \mathbb{R} \to \mathbb{R}$ where $r_D(t)$ and $r_{C^s}(t)$ are the radii of the $(n-1)$-dimensional balls $D_t$ and $C_t^s$, respectively. Let $d_1$ be the infimum of $t$ for which $r_{C^s}(t) > 0$

and $d_2$ be the supremum of $t$ for which $r_{C^s}(t) > 0$. Note that $-d \le d_1 < d_2 \le d$ since the body $C^s$ is contained between the hyperplanes $H_{-d}, H_d$.

We define an affine function $r_{C^c} : [d_1, d_2] \to \mathbb{R}$. We further define $C^c$ to be the body whose slices $C^c_t$ are $(n-1)$-dimensional balls of radius $r_{C^c}(t)$, for $t \in [d_1, d_2]$. Clearly $C^c$ is a truncated cone. We define $r_{C^c}$ differently for different cases mentioned below. In Case 1 we define it directly and in Cases $2, 3$ we define two values $a_1, a_2$. The affine function $r_{C^c} : [d_1, d_2] \to \mathbb{R}$ corresponding to the values $a_1, a_2$ is defined by the line joining the points $p_1 = (a_1, r_D(a_1)), p_2 = (a_2, r_D(a_2))$. See Figure 2 for an illustration of this construction.

- **Case 1:** $\forall t \in (d_1, d_2)$**,** $r_{C^s}(t) \le r_D(t)$
  Since $r_D$ is convex and $r_{C^s}$ is concave, from Lemma 3.4, there exists an affine function $r_{C^c} : [d_1, d_2] \to \mathbb{R}$ such that $r_D(t) \ge r_{C^c}(t) \ge r_{C^s}(t)$ for all $t \in [d_1, d_2]$.
- **Case 2:** $\forall t \in (d_1, d_2)$**,** $r_{C^s}(t) > r_D(t)$
  Let $a_1 = d_1$, $a_2 = d_2$.
- **Case 3:** $\exists t_1, t_2 \in [d_1, d_2]$ such that $r_{C^s}(t_1) \le r_D(t_1)$ and $r_{C^s}(t_2) > r_D(t_2)$
  This case be further divided into three sub-cases.
  - **Case 3a:** $r_{C^s}(d_1) \le r_D(d_1)$ and $r_{C^s}(d_2) \le r_D(d_2)$
    In this case since $r_{C^s}$ is concave and $r_D$ is convex the curves have exactly two points of intersection. Let $a_1, a_2$ be the values of $t$ where the curves intersect.
  - **Case 3b:** $r_{C^s}(d_1) \le r_D(d_1)$ and $r_{C^s}(d_2) > r_D(d_2)$
    In this case since $r_{C^s}$ is concave and $r_D$ is convex the curves have exactly one point of intersection. Let $a_1$ be the value of $t$ where the curves intersect and let $a_2 = d_2$.
  - **Case 3c:** $r_{C^s}(d_1) > r_D(d_1)$ and $r_{C^s}(d_2) \le r_D(d_2)$
    In this case since $r_{C^s}$ is concave and $r_D$ is convex the curves have exactly one point of intersection. Let $a_2$ be the value of $t$ where the curves intersect and let $a_1 = d_1$.

Since the function $r_{C^c}$ is affine, it is also concave and so by Lemma 3.2 the body $C^c$ is convex. To complete the proof, we need to show that $\mathrm{dist}(D, C^c) \le \mathrm{dist}(D, C^s)$ in all three cases.

By definition, the distance between $D$ and $C^s$ is

$$\mathrm{dist}(D, C^s) = \mathrm{Vol}(D \setminus C^s) + \mathrm{Vol}(C^s \setminus D) = \int_{-d}^{d} \mathrm{Vol}_{n-1}(D_t \setminus C^s_t) + \mathrm{Vol}_{n-1}(C^s_t \setminus D_t)\,\mathrm{d}t.$$

For Case 1, since $D, C^s, C^c$ are symmetric about $e_1$ and $r_D(t) \ge r_{C^c}(t) \ge r_{C^s}(t)$ for every $t \in [d_1, d_2]$,

$$\begin{aligned}
\mathrm{dist}(D, C^s) &= \int_{-d}^{d_1} \mathrm{Vol}_{n-1}(D_t)\,\mathrm{d}t + \int_{d_1}^{d_2} \mathrm{Vol}_{n-1}(D_t) \\
&\qquad\qquad - \mathrm{Vol}_{n-1}(C^s_t)\,\mathrm{d}t + \int_{d_2}^{d} \mathrm{Vol}_{n-1}(D_t)\,\mathrm{d}t \\
&\ge \int_{-d}^{d_1} \mathrm{Vol}_{n-1}(D_t)\,\mathrm{d}t + \int_{d_1}^{d_2} \mathrm{Vol}_{n-1}(D_t) \\
&\qquad\qquad - \mathrm{Vol}_{n-1}(C^c_t)\,\mathrm{d}t + \int_{d_2}^{d} \mathrm{Vol}_{n-1}(D_t)\,\mathrm{d}t \\
&= \mathrm{dist}(D, C^c).
\end{aligned}$$

For Cases 2 and 3, for $t \in (d_1, a_1) \cup (a_2, d_2)$, the ball $D_t$ contains the ball $C^s_t$. Conversely, for every $t \in [a_1, a_2]$, $C^s_t$ contains $D_t$. And for $t \in (-d, d_1) \cup (d_2, d)$ the ball $C^s_t$ has zero radius

Hence, the distance between $D$ and $C^s$ is

$$\begin{aligned}\mathrm{dist}(D, C^s) = &\int_{d_1}^{a_1} \mathrm{Vol}_{n-1}(D_t) - \mathrm{Vol}_{n-1}(C^s_t)\,\mathrm{d}t + \int_{a_1}^{a_2} \mathrm{Vol}_{n-1}(C^s_t) - \mathrm{Vol}_{n-1}(D_t)\,\mathrm{d}t \\ &+ \int_{a_2}^{d_2} \mathrm{Vol}_{n-1}(D_t) - \mathrm{Vol}_{n-1}(C^s_t)\,\mathrm{d}t + \int_{-d}^{d_1} \mathrm{Vol}_{n-1}(D_t)\,\mathrm{d}t \\ &+ \int_{d_2}^{d} \mathrm{Vol}_{n-1}(D_t)\,\mathrm{d}t.\end{aligned}$$

For every $t \in (d_1, a_1) \cup (a_2, d_2)$, we have that $r_D(t) \geq r_{C^c}(t) \geq r_{C^s}(t)$. And for every $t \in (a_1, a_2)$, we have the reverse inequalities $r_D(t) \leq r_{C^c}(t) \leq r_{C^s}(t)$. Therefore,

$$\begin{aligned}\mathrm{dist}(D, C^s) \geq &\int_{d_1}^{a_1} \mathrm{Vol}_{n-1}(D_t) - \mathrm{Vol}_{n-1}(C^c_t)\,\mathrm{d}t + \int_{a_1}^{a_2} \mathrm{Vol}_{n-1}(C^c_t) - \mathrm{Vol}_{n-1}(D_t)\,\mathrm{d}t \\ &+ \int_{a_2}^{d_2} \mathrm{Vol}_{n-1}(D_t) - \mathrm{Vol}_{n-1}(C^c_t)\,\mathrm{d}t + \int_{-d}^{d_1} \mathrm{Vol}_{n-1}(D_t)\,\mathrm{d}t \\ &+ \int_{d_2}^{d} \mathrm{Vol}_{n-1}(D_t)\,\mathrm{d}t \\ = &\ \mathrm{Vol}(D \setminus C^c) + \mathrm{Vol}(C^c \setminus D) = \mathrm{dist}(D, C^c).\end{aligned}$$ ◀

### 3.2.4 Every truncated cone is far from $D$

As the last step in the proof that $D$ is far from convex, we show that it is far from every truncated cone.

▶ **Lemma 3.6.** *Every truncated cone is $\frac{1}{8}$-far from $D$.*

**Proof.** Let $C^c$ be a truncated cone. Without loss of generality let the truncated cone have larger radius towards the left side. We consider the two cases where $\mathrm{Vol}(C^c_{[0,d]}) \leq \frac{1}{2}\mathrm{Vol}(D_{[0,d]})$ and where $\mathrm{Vol}(C^c_{[0,d]}) > \frac{1}{2}\mathrm{Vol}(D_{[0,d]})$ separately.

**Case 1:** $\mathrm{Vol}(C^c_{[0,d]}) \leq \frac{1}{2}\mathrm{Vol}(D_{[0,d]})$.

In this case, Proposition 2.1 and the case condition yield

$$\mathrm{Vol}(D \triangle C^c) \geq \mathrm{Vol}(D_{[0,d]} \triangle C^c_{[0,d]}) \geq \mathrm{Vol}(D_{[0,d]}) - \mathrm{Vol}(C^c_{[0,d]}) \geq \frac{1}{2}\mathrm{Vol}(D_{[0,d]}) = \frac{1}{4}\mathrm{Vol}(D).$$

**Case 2:** $\mathrm{Vol}(C^c_{[0,d]}) \geq \frac{1}{2}\mathrm{Vol}(D_{[0,d]})$

In this case, if $C^c_{[-d,-\frac{d}{2}]} = \emptyset$, then from Proposition 3.1

$$\mathrm{Vol}(D \triangle C^c) \geq \mathrm{Vol}(D_{[-d,-\frac{d}{2}]} \triangle C^c_{[-d,-\frac{d}{2}]}) = \mathrm{Vol}(D_{[-d,-\frac{d}{2}]}) \geq \frac{1}{4}\mathrm{Vol}(D).$$

If $C^c_{[-d,-\frac{d}{2}]} \neq \emptyset$, then using the fact that $C^c$ is a truncated cone with larger radius on the left side we get

$$\mathrm{Vol}(C^c_{[-\frac{d}{2},\frac{d}{2}]}) \geq \mathrm{Vol}(C^c_{[0,d]}) \geq \frac{1}{2}\mathrm{Vol}(D_{[0,d]}) \geq \frac{1}{4}\mathrm{Vol}(D).$$

Then Proposition 2.1 implies that

$$\begin{aligned}\mathrm{Vol}(D\triangle C^c) &\geq \mathrm{Vol}(D_{[-\frac{d}{2},\frac{d}{2}]}\triangle C^c_{[-\frac{d}{2},\frac{d}{2}]}) \geq \mathrm{Vol}(C^c_{[-\frac{d}{2},\frac{d}{2}]}) - \mathrm{Vol}(D_{[-\frac{d}{2},\frac{d}{2}]})\\ &\geq \frac{1}{4}\mathrm{Vol}(D) - \mathrm{Vol}(D_{[-\frac{d}{2},\frac{d}{2}]}).\end{aligned}$$

From Proposition 3.1, we also have that $\mathrm{Vol}(D_{[-\frac{d}{2},\frac{d}{2}]})$ is exponentially smaller than $\mathrm{Vol}(D)$. Hence,

$$\mathrm{Vol}(D\triangle C_C) \geq \frac{1}{4}\mathrm{Vol}(D) - o\big(\mathrm{Vol}(D)\big) \geq \frac{1}{8}\mathrm{Vol}(D). \qquad \blacktriangleleft$$

Putting our last three lemmas together completes the proof of the main result from this section.

▶ **Theorem 3.7.** *The body $D$ is $\frac{1}{8}$-far from convex.*

▶ Remark 1. In fact, the above argument shows that $D$ is $(\frac{1}{4} - o(1))$-far from convex. With more careful calculations, it is possible to show that $D$ is $(\frac{1}{2} - o(1))$-far from convex. This result is tight, since the body $D$ is $(\frac{1}{2} - o(1))$-close to the convex body obtained by deleting the right half of $D$ and extending the truncated cone in the left half to $d$.

# 4 Proofs of Theorems 1.1 and 1.2

We complete the proofs of Theorems 1.1 and 1.2 in this section. The proof of Theorem 1.1 is completed in Section 4.1, where we show that a line segment connecting two points drawn at random from the body is contained within the body with high probability. In Section 4.2, we generalize this result to show that the convex hull of a set of points picked uniformly at random lies inside the body with high probability.

## 4.1 Non-robustness of the line characterization

We are now ready to prove Theorem 1.1 by showing that when two points $x$ and $y$ are drawn uniformly at random from $D$, then with high probability the line segment $\overline{xy}$ that connects $x$ to $y$ is completely contained within $D$.

▶ **Lemma 4.1.** *When $x, y \in D$ are drawn uniformly at random from $D$, then the line segment $\overline{xy}$ that joins $x$ and $y$ satisfies*

$$\Pr\left[\overline{xy} \not\subseteq D\right] = 2^{-\Omega(n)}.$$

**Proof.** Let $x = (\alpha, x_2, \ldots, x_n)$ and $y$ be drawn independently and uniformly at random from $D$. By the symmetry of $D$ with respect to reflection on the axis $e_1$, we can assume without loss of generality that $\alpha \leq 0$. Furthermore, since $D$ is symmetric with respect to rotations around $e_1$, we can also assume that $x_2 \geq 0$ and rest of the $x_i = 0$. Hence, without loss of generality let $x = (\alpha, x_2, 0, 0, \ldots, 0)$ and let $y = (\beta, y_2, \ldots, y_n)$.

If $\beta \leq 0$, then both $x$ and $y$ lie in the same half of $D$, and that half is a convex set so the line segment $\overline{xy}$ is contained in $D$.

Consider the case now where $\beta > 0$. By Proposition 3.1, with probability $1 - 2^{-\Omega(n)}$ we have $\alpha \leq -d/2$ and $\beta \geq d/2$. Furthermore, for any given $\beta$ since $(y_2, \ldots, y_n)$ is uniformly distributed over an $(n-1)$-dimensional ball of radius at most 1.1, from Proposition 2.3 we have that $\Pr[|y_2| \leq \frac{1}{10}] = 1 - 2^{-\Omega(n)}$. In the rest of the proof, assume that all three inequalities $\alpha \le -\frac{d}{2}$, $\beta \ge \frac{d}{2}$, and $|y_2| \le \frac{1}{10}$ hold. We will show that in this case, the line passes through the center slice $D_0$ and, therefore, the line segment $\overline{xy}$ is contained in $D$, thus completing the proof of the theorem.

Consider the point $z = \frac{|\alpha|}{|\alpha|+|\beta|}(\beta, y_2, \dots, y_n) + \frac{|\beta|}{|\alpha|+|\beta|}(\alpha, x_2, 0, 0, \dots, 0)$. The point $z$ lies in the hyperplane $H_0$. We want to show that it is contained in the slice $D_0$ or, equivalently, that $\|z\|^2 \le 1$. By definition,

$$\begin{aligned}\|z\|^2 &= \left(\frac{1}{|\alpha|+|\beta|}\right)^2 (|\alpha|y_2 + |\beta|x_2)^2 + \left(\frac{|\alpha|}{|\alpha|+|\beta|}\right)^2 \sum_{i=3}^{n-1} y_i^2 \\ &= \left(\frac{|\beta|x_2}{|\alpha|+|\beta|}\right)^2 + \left(\frac{1}{|\alpha|+|\beta|}\right)^2 2|\alpha||\beta|x_2y_2 + \left(\frac{|\alpha|}{|\alpha|+|\beta|}\right)^2 \sum_{i=2}^{n} y_i^2.\end{aligned}$$

Since $x$ and $y$ are in $D$, then $\sum_{i=2}^n y_i^2 \le (1.1)^2$ and $x_2 \le 1.1$. And we have that $y_2 \le 0.1$. Substituting these bounds into the above expression, we obtain

$$\begin{aligned}\|z\|^2 &\le \left(\frac{1.1|\beta|}{|\alpha|+|\beta|}\right)^2 + \left(\frac{1}{|\alpha|+|\beta|}\right)^2 .22|\alpha||\beta| + \left(\frac{1.1|\alpha|}{|\alpha|+|\beta|}\right)^2 \\ &= (1.1)^2 - 2.2|\alpha||\beta| \left(\frac{1}{|\alpha|+|\beta|}\right)^2.\end{aligned}$$

Defining $\delta = |\alpha|/|\beta|$, the above equation simplifies to $\|z\|^2 \le 1.21 - 2.2\frac{\delta}{(1+\delta)^2}$. Since $|\alpha|$ and $|\beta|$ are both in the range $[\frac{d}{2}, d]$, then $\delta \in [\frac{1}{2}, 2]$. The minimum value of the function $\frac{\delta}{(1+\delta)^2}$ in the interval $[\frac{1}{2}, 2]$ is $\frac{2}{9}$, so $\|z\|^2 \le 1.21 - 2.2 \cdot \frac{2}{9} \le 1$. ◀

Theorem 1.1 follows immediately from Theorem 3.7 and Lemma 4.1.

## 4.2 Non-robustness of the convex hull characterization

In this section, we complete the proof of Theorem 1.2 by combining Lemma 4.1 with the following structural result about the body $D$.

▶ **Lemma 4.2.** *For any finite set $X \subset D$, if the line connecting any two points $x, y \in X$ satisfies $\overline{xy} \subseteq D$, then*

$$\mathrm{conv}(X) \subseteq D.$$

**Proof.** We prove the claim by induction on the number of points in $X$. The base case where $|X| = 2$ is trivially true. For the base case where $|X| = 3$, let $X = \{x, y, z\}$ be any set that satisfies $\overline{xy}, \overline{xz}, \overline{yz} \subseteq D$. We can assume without loss of generality that $x, y \in D_{\ge 0}$. If $z \in D_{\ge 0}$ as well, then $\mathrm{conv}(X) \subseteq D$ since $D_{\ge 0}$ is a convex set. Let us now consider the case where $z \in D_{<0}$. Note that a line joining two points $a \in D_{\le 0}, b \in D_{\ge 0}$ is contained in the body if and only if $\overline{ab} \cap D_0 \ne \emptyset$. From this observation, we get that $\overline{xz} \cap D_0 \ne \emptyset$ and $\overline{yz} \cap D_0 \ne \emptyset$. Define $x' = \overline{xz} \cap D_0$ and $y' = \overline{yz} \cap D_0$. Let $w$ be any point on the line $\overline{xy}$ and define $w' = \overline{zw} \cap H_0$. Since $w \in \overline{xy}$, we have that $w' \in \overline{x'y'}$. Since $x', y' \in D_0$ and $D_0$ is convex, we must also have that $w' \in D_0$, and so $\overline{zw} \subseteq D$. Since every point in the convex hull of $X$ is on the line $\overline{zw}$ for some $w \in \overline{xy}$, this means that $\mathrm{conv}(X) \subseteq D$.

For the induction step, we assume that the claim is true for all sets with at most $k$ points for some fixed $k \ge 2$. Fix any set $X \subseteq D$ with $k+1$ elements such that every line $\overline{xy}$ connecting $x, y \in X$ is contained in $D$. We want to show that $\mathrm{conv}(X) \subseteq D$.

Let $x \in X$ be an element for which there exists $y \in X$ that satisfies $x_1 y_1 \geq 0$, i.e $x, y$ are in the same half of $D$. Such an element $x$ is guaranteed to exist since $|X| \geq 3$. Without loss of generality, assume $x \in D_{\leq 0}$. Define $X_k = X \setminus \{x\}$. By the induction hypothesis, we must have that $\text{conv}(X_k) \subseteq D$. Furthermore, if every $x' \in \text{conv}(X_k)$ satisfies $\overline{x'x} \subseteq D$, then $\text{conv}(X) = \text{conv}(x \cup X_k) \subseteq D$. To complete the proof, let us now assume that there exists $x' \in \text{conv}(X_k)$ for which $\overline{x'x} \not\subseteq D$ and show that this leads to a contradiction.

Define $X_1 = \{y : y \in X_k \cap D_{\leq 0}\}$ and $X_2 = X_k \setminus X_1$. By our choice of $x$, $|X_1| \geq 1$ and so $|X_2| \leq k-1$. And since $x' \in \text{conv}(X_k) = \text{conv}(X_1 \cup X_2)$, there exist two points $x'' \in \text{conv}(X_1)$ and $x''' \in \text{conv}(X_2)$ such that $x' \in \overline{x''x'''}$. We have $\overline{xx''} \subseteq D$ as $x, x'' \in D_{\leq 0}$ and $D_{\leq 0}$ is convex. And $\overline{xx'''} \subseteq D$ because $\text{conv}(\{x\} \cup X_2) \subseteq D$ from the induction hypothesis. Finally, since $x'', x''' \in \text{conv}(X_k)$ we also have that $\overline{x''x'''} \subseteq D$. Hence, the three points $x, x'', x'''$ satisfy $\overline{xx''} \subseteq D$, $\overline{xx'''} \subseteq D$, and $\overline{x''x'''} \subseteq D$. Therefore, from the induction hypothesis on the set $\{x, x'', x'''\}$, $\text{conv}(x, x'', x''') \subseteq D$. This implies $\overline{xx'} \subseteq D$, which is a contradiction. Therefore, $\text{conv}(X) = \text{conv}(\{x\} \cup X_K) \subseteq D$. ◀

There exists a small constant $c > 0$ such that if we pick $m = 2^{cn}$ points, $X$, uniformly at random then the probability that $\forall x, y \in X$, $\overline{xy} \subset D$ is greater than $1 - \frac{1}{2^{\Omega(n)}}$. We get this by applying a union bound on Lemma 4.1. This combined with Lemma 4.2 completes the proof of Theorem 1.2.

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