Dave's CPSC 121 Tutorial Notes – Week Eight

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Formula Sheet Tips

- **Set Proofs:**

  To prove $A \subseteq B$,
  Let x be an arbitrary element of A and show that $x \in B$

  To prove $A = B$
  prove $A \subseteq B$ and $B \subseteq A$,
  $A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$

- **Cartesian Products**

  $(a, b) \neq (b, a) \Leftrightarrow a \neq b$
  $(a, b) = (c, d) \Leftrightarrow (a = c) \wedge (b = d)$

  $A \times B = \{(a, b) | (a \in A) \wedge (b \in B)\}$
  $(a, b) \in A \times B \Leftrightarrow (a \in A) \wedge (b \in B)$
  $|A \times B \times C| = |A| \cdot |B| \cdot |C|$

- **Functions**

  $F : X \mapsto Y \Leftrightarrow \forall x \in X, \exists y \in Y, F(x) = y$
  $F : X \mapsto Y \Leftrightarrow F \subseteq X \times Y$
  $(x, y) \in F \Leftrightarrow F(x) = y$

  $F : X \mapsto Y$ is one-to-one (injective) $\Leftrightarrow \forall a, b \in X, (F(a) = F(b)) \rightarrow (a = b)$
  $F : X \mapsto Y$ is onto (surjective) $\Leftrightarrow \forall y \in Y, \exists x \in X, F(x) = y$
  $F : X \mapsto Y$ is bijective: $\Leftrightarrow$ (F is one-to-one) $\wedge$ (F is onto) $\Leftrightarrow \exists F^{-1} : Y \mapsto X$

  $(G \circ F)(x) = G(F(x))$

Sample Problems

1. **Another Set Proof**

   Prove that for all sets A, B, C that $(A - B) \cap (C - B) = (A \cap C) - B$

   Approach One (Identities):

   LHS = $(A - B) \cap (C - B)$
   $= (A \cap \overline{B}) \cap (C \cap \overline{B})$
   $= A \cap \overline{B} \cap C \cap \overline{B}$
   $= A \cap \overline{B} \cap C$
   $= (A \cap C) \cap \overline{B}$
   $= (A \cap C) - B$
   $= RHS$

   Approach Two (Elements):

   Step One: Prove $X \subseteq Y$

(1)	let x be any arbitrary element of $(A - B) \cap (C - B)$	[Premise]
(2)	$\therefore\ x \in (A - B)$	[from (1), def'n of $\cap$]
(3)	$\therefore\ x \in A$	[from (2), def'n of (-)]
(4)	$\therefore\ x \notin B$	[from (2), def'n of (-)]
(5)	$\therefore\ x \in (C - B)$	[from (1), def'n of $\cap$]
(6)	$\therefore\ x \in C$	[from (5), def'n of (-)]
(7)	$\therefore\ x \in (A \cap C)$	[from (3)+(6), def'n of $\cap$]
(8)	$\therefore\ x \in (A \cap C) - B$	[from (4)+(7), def'n of (-)]
(9)	$\therefore\ (A - B) \cap (C - B) \subseteq (A \cap C) - B$	[from (1)+(8), def'n of $\subseteq$]

   Step Two: Prove $X \supseteq Y$

(1)	let x be any arbitrary element of $(A \cap C) - B$	[Premise]
(2)	$\therefore\ x \in (A \cap C)$	[from (1), def'n of (-)]
(3)	$\therefore\ x \in A$	[from (2), def'n of $\cap$]
(4)	$\therefore\ x \in C$	[from (2), def'n of $\cap$]
(5)	$\therefore\ x \notin B$	[from (1), def'n of (-)]
(6)	$\therefore\ x \in (A - B)$	[from (3)+(5), def'n of (-)]
(7)	$\therefore\ x \in (C - B)$	[from (4)+(5), def'n of (-)]
(8)	$\therefore\ x \in (A - B) \cap (C - B)$	[from (6)+(7), def'n of $\cap$]
(9)	$\therefore\ (A \cap C) - B \subseteq (A - B) \cap (C - B)$	[from (1)+(8), def'n of $\subseteq$]

   Step Three: $(X \subseteq Y) \wedge (X \supseteq Y) \Leftrightarrow (X = Y)$

   Because$[(A - B) \cap (C - B) \subseteq (A \cap C) - B] \wedge [(A - B) \cap (C - B) \supseteq (A \cap C) - B]$
   $\therefore\ (A - B) \cap (C - B) = (A \cap C) - B$

2. **Yet Another Set Proof**

I just wanted to give you a bonus example, so let's do the previous example with a union so you can see one with cases:

Prove that for all sets A, B, C that $(A - B) \cup (C - B) \subseteq (A \cup C) - B$

(1)	let x be any arbitrary element of $(A - B) \cup (C - B)$	[Premise]
	Case One: $x \in (A - B)$	
(2a)	$\therefore\ x \in (A - B)$	[from (1)]
(3a)	$\therefore\ x \in A$	[from (2a), def'n of (-)]
(4a)	$\therefore\ x \notin B$	[from (2a), def'n of (-)]
(5a)	$\therefore\ x \in (A \cup C)$	[from (3a), def'n of $\cup$]
(6a)	$\therefore\ x \in (A \cup C) - B$	[from (4a)+(5a), def'n of (-)]
	Case Two: $x \in (C - B)$	
(2b)	$\therefore\ x \in (C - B)$	[from (1)]
(3b)	$\therefore\ x \in C$	[from (2b), def'n of (-)]
(4b)	$\therefore\ x \notin B$	[from (2b), def'n of (-)]
(5b)	$\therefore\ x \in (A \cup C)$	[from (3b), def'n of $\cup$]
(6b)	$\therefore\ x \in (A \cup C) - B$	[from (3b)+(5b), def'n of (-)]
(7)	$\therefore\ (A - B) \cup (C - B) \subseteq (A \cap C) - B$	[from (6a)+(6b), def'n of $\subseteq$, $\cup$]

3. **Fun with Functions**

The Boolean NOT function:
$N : \mathbb{B} \mapsto \mathbb{B}$
$N = \{(0,1),(1,0)\} = \{(a,b) \in \mathbb{B} \times \mathbb{B} | (b \equiv \sim a)\}$

The Boolean AND function:
$A : \mathbb{B} \times \mathbb{B} \mapsto \mathbb{B}$
$A = \{((0,0),0),((0,1),0),((1,0),0),((1,1),1)\} = \{((a,b),c) \in (\mathbb{B} \times \mathbb{B}) \times \mathbb{B} | (c \equiv (a \wedge b))\}$

The Boolean NAND function:
$N \circ A$

4. **Proofs with Functions: Part 1a**

prove that $F : \mathbb{R}^+ \mapsto \mathbb{R}^+, F(x) = (x+1)/x$ is one-to-one.

By definition,
$F : X \mapsto Y$ is one-to-one (injective) $\Leftrightarrow \forall a, b \in X, (F(a) = F(b)) \rightarrow (a = b)$

so we need to show
$\forall a, b \in \mathbb{R}^+, (a+1)/a = (b+1)/b \rightarrow a = b$

Without Loss of Generality, let a and b be any arbitrary elements of $\mathbb{R}^+$ and assume
$(a+1)/a = (b+1)/b$
$\therefore\ b(a+1)/ab = a(b+1)/ab$
$\therefore\ ba + b = ab + a$
$\therefore\ a = b$

since $(a+1)/a = (b+1)/b \rightarrow a = b$, therefore $F(x) = (x+1)/x$ is one-to-one.

5. **Proofs with Functions: Part 1b**

Is $F : \mathbb{R}^+ \mapsto \mathbb{R}^+, F(x) = x + 1/x$ onto?

No: We can find a counterexample.

By definition,
$F : X \mapsto Y$ is onto (surjective) $\Leftrightarrow \forall y \in Y, \exists x \in X, F(x) = y$

so, for our function to be true,
$\forall y \in \mathbb{R}^+, \exists x \in \mathbb{R}^+, (x+1)/x = y$

Let $y = 1/2$,

$\therefore\ (x+1)/x = 1/2$
$\therefore\ x = -2$

$\therefore\ \forall y in \mathbb{R}^+, \nexists x \in \mathbb{R}^+, (x+1)/x = y$

$\therefore\ F(x) = (x+1)/x$ is not onto.

6. **Proofs with Functions: Part 2**

prove that $F : \mathbb{R} \mapsto \mathbb{R}, F(x) = (x+1)/4$ is bijective.

By definition,
$F : X \mapsto Y$ is one-to-one (injective) $\Leftrightarrow \forall a, b \in X, (F(a) = F(b)) \rightarrow (a = b)$

so we need to show
$\forall a, b \in \mathbb{R}, (a+1)/4 = (b+1)/4 \rightarrow a = b$

Without Loss of Generality, let a and b be any arbitrary elements of $\mathbb{R}$ and assume
$(a+1)/4 = (b+1)/4$
$\therefore\ a/4 + 1/4 = b/4 + 1/4$
$\therefore\ a = b$

since $(a+1)/4 = (b+1)/4 \rightarrow a = b$, therefore $F(x) = (x+1)/4$ is one-to-one.

By definition,
$F : X \mapsto Y$ is onto (surjective) $\Leftrightarrow \forall y \in Y, \exists x \in X, F(x) = y$

so, for our function to be true,
$\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, (x+1)/4 = y$

Let y be any arbitrary element in $\mathbb{R}$
and select $x = 4y - 1$

$x \in \mathbb{R}$
and $(x+1)/4 = y$

$\therefore\ F(x) = (x+1)/4$ is onto.

Since $F(x)$ is both one-to-one and onto, $F(x)$ is bijective.