Dave's CPSC 121 Tutorial Notes – Week Seven

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Formula Sheet Tips

- **Set Identities – Use the Logical Equivalences**

  All of the logical equivalence laws apply

(Logic) Distributive:	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
(Set) Distributive:	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

- **Additional Set Identities**

Definition of $-$:	$A - B = A \cap \overline{B}$
Definition of $\oplus$:	$A \oplus B = (A - B) \cup (B - A)$
Negation of $\mathbb{U}$:	$\overline{\mathbb{U}} = \varnothing$

- **Definitions**

  $A \subseteq B \Leftrightarrow \forall x \in \mathbb{U}, (x \in A) \rightarrow (x \in B)$
  $A \nsubseteq B \Leftrightarrow \exists x \in \mathbb{U}, (x \in A) \wedge (x \notin B)$
  $A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$

  $A \cup B = \{x \in \mathbb{U} | (x \in A) \vee (x \in B)\}$
  $A \cap B = \{x \in \mathbb{U} | (x \in A) \wedge (x \in B)\}$
  $\overline{A} = \{x \in \mathbb{U} | x \notin A\}$

  $A \subseteq B \Leftrightarrow A \in \mathcal{P}(B)$

  $|\mathcal{P}(A)| = 2^{|A|}$

Sample Problems

1. **Tricky Set Stuff**

   The following are some tricky operations on sets. Convince yourself the following are all true.

$A = \{1, 2\}$
$B = \{1, 2, 3\}$
$C = \{3\}$

$\therefore\ 3 \in B$
$\therefore\ 3 \notin A$
$\therefore\ 3 \neq C$
$\therefore\ C \notin B$

$\therefore\ A \subset B$
$\therefore\ A = B - C$
$\therefore\ C = B - A$
$\therefore\ B = A \cup C$
$\therefore\ A \cap C = \varnothing$
$\therefore\ B = A \oplus C$
$\therefore\ C = A \oplus B$
$\therefore\ C = \overline{A} \cap B$

$D = \{A \cup C\}$
$E = \{A, C\}$
$F = \{A, B, C\}$
$G = \{\varnothing\}$

$\therefore\ D \neq B$
$\therefore\ D = \{B\}$
$\therefore\ E \neq B$
$\therefore\ E \neq D$
$\therefore\ E = F - D$
$\therefore\ E \neq F - B$
$\therefore\ F = D \cup E$

$\therefore\ G \neq \varnothing$
$\therefore\ G - \varnothing \neq \varnothing$
$\therefore\ G \cap \varnothing = \varnothing$

$H = A \cup B \cup C \cup D \cup E \cup F \cup G = \{\varnothing, 1, 2, 3, \{1, 2\}, \{1, 2, 3\}, \{3\}\}$

$|A| = |E| = 2$
$|B| = |F| = 3$
$|C| = |D| = |G| = 1$
$|H| = 7$
$|\{H\}| = 1$

$\mathcal{P}(C) \neq \{\varnothing, 3\}$
$\mathcal{P}(C) = \{\varnothing, \{3\}\}$
$\mathcal{P}(C) \neq G \cup C$
$\mathcal{P}(C) = G \cup \{C\}$

$\mathcal{P}(\varnothing) = G$
$\mathcal{P}(G) = \{\varnothing, \{\varnothing\}\}$
$\mathcal{P}(\mathcal{P}(G)) = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}$

2. **A Counterexample Proof**

   Prove $A - (B - C) = (A - B) - C$

   This is actually *false* and can be shown to be false with a counterexample.

   $A = \{1, 2\}$
   $B = \{2, 3\}$
   $C = \{1, 3\}$

   LHS =
   = $\{1, 2\} - (\{2, 3\} - \{1, 3\})$
   = $\{1, 2\} - (\{2\})$
   = $\{1\}$

   RHS =
   = $(\{1, 2\} - \{2, 3\}) - \{1, 3\})$
   = $(\{1\}) - \{1, 3\})$
   = $\varnothing$
   $\neq LHS$

3. **An Indirect Proof**

   Prove $A \subseteq B \rightarrow \overline{B} \subseteq \overline{A}$

   For questions like this, it is best to deal with the two cases:

   Case One: $A = B$

   clearly $A \subseteq A \rightarrow \overline{A} \subseteq \overline{A}$
   or $T \rightarrow T$

   Case Two: $A \neq B$

   Indirect Proof (Contradiction):

   Assume that the statement is false:

   $\sim(A \subseteq B \rightarrow \overline{B} \subseteq \overline{A})$
   $\therefore\ (A \subseteq B) \wedge \sim(\overline{B} \subseteq \overline{A})$
   $\therefore\ (A \subseteq B) \wedge (\overline{B} \not\subseteq \overline{A})$

   $\therefore\ (A \subseteq B)$
   $\therefore\ (\overline{B} \not\subseteq \overline{A})$

   $(\overline{B} \not\subseteq \overline{A}) \Leftrightarrow \exists x \in \mathbb{U}, (x \in \overline{B}) \wedge (x \notin \overline{A})$

   Let x be such an arbitrary element of $\mathbb{U}$

   $\therefore\ x \in A$
   $\therefore\ x \notin B$

   However,
   $(A \subseteq B) \Leftrightarrow \forall x \in \mathbb{U}, (x \in A) \rightarrow (x \in B)$

   $x \in A$
   $\therefore\ x \in B$

   We have reached a contradiction $(x \in B) \wedge (x \notin B)$

   $\therefore$ Assumption was false
   $\therefore$ Original statement was true

   $\therefore\ A \subseteq B \rightarrow \overline{B} \subseteq \overline{A}$