Dave's CPSC 121 Tutorial Notes – Week Five

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Formula Sheet Tips

- **Floors and Ceilings**

  $\lfloor x \rfloor$ = largest integer $\leq x$
  $\lceil x \rceil$ = smallest integer $\geq x$
  $\lfloor 8.1 \rfloor = 8$
  $\lceil 8.1 \rceil = 9$
  $\lceil 8 \rceil = \lfloor 8 \rfloor = 8$
  $\lfloor -8.1 \rfloor = -9$

- **Converting Real Numbers to Integers**

  If $x \in \mathbb{R}, n \in \mathbb{Z}$ you CANNOT say Let $n = x$
  Instead, choose one of the following:

Let $n = \lceil x \rceil$	Let $n = \lfloor x \rfloor$
$\therefore\ n \geq x$	$\therefore\ n \leq x$

- **Dealing with Inequalities**

$x \leq y$	$x \leq y$
$\therefore\ x \leq y + b$ (if $b \geq 0$)	$\therefore\ x - b \leq y$ (if $b \geq 0$)

Sample Problems

1. **Back to The Oddity**

   Last day, we looked at the statement:
   If x is odd then $(x + 4)$ is odd.
   We can represent this as:

   $\forall x \in \mathbb{Z}, (\exists k \in \mathbb{Z}, x = 2k + 1) \rightarrow (\exists m \in \mathbb{Z}, x + 4 = 2m + 1)$

   Is that the same as saying:

   $\forall x \in \mathbb{Z}, \exists k \in \mathbb{Z}, \exists m \in \mathbb{Z}, (x = 2k + 1) \rightarrow (x + 4 = 2m + 1)$????

   Even though both statements are true, they are NOT saying the same thing: The second is vacuously true, because if we select say $k = x$, then the statement becomes:
   $(F) \rightarrow (x + 4 = 2m + 1)$ (which is always true)

   This becomes more obvious when we consider:
   If x is odd then $(x + 5)$ is odd.

   which is clearly false:

   $F \equiv \forall x \in \mathbb{Z}, (\exists k \in \mathbb{Z}, x = 2k + 1) \rightarrow (\exists m \in \mathbb{Z}, x + 5 = 2m + 1)$
   But,
   $T \equiv \forall x \in \mathbb{Z}, \exists k \in \mathbb{Z}, \exists m \in \mathbb{Z}, (x = 2k + 1) \rightarrow (x + 5 = 2m + 1)$
   because of the same vacuously true argument we used before.

   For better insight into this, recall that:
   $p \rightarrow q \equiv \sim p \vee q$

   so if we re-write the original statement in *or* form,
   $\forall x \in \mathbb{Z}, (\sim \exists k \in \mathbb{Z}, x = 2k + 1) \vee (\exists m \in \mathbb{Z}, x + 4 = 2m + 1)$

   we can see that the $\sim \exists$ cannot be brought to the left. However, we can re-write as:

   $\forall x \in \mathbb{Z}, (\sim \exists k \in \mathbb{Z}, x = 2k + 1) \vee (\exists m \in \mathbb{Z}, x + 4 = 2m + 1)$
   $\equiv \forall x \in \mathbb{Z}, (\forall k \in \mathbb{Z}, \sim (x = 2k + 1)) \vee (\exists m \in \mathbb{Z}, x + 4 = 2m + 1)$
   $\equiv \forall x \in \mathbb{Z}, \forall k \in \mathbb{Z}, \exists m \in \mathbb{Z}, \sim (x = 2k + 1) \vee (x + 4 = 2m + 1)$
   $\equiv \forall x \in \mathbb{Z}, \forall k \in \mathbb{Z}, \exists m \in \mathbb{Z}, (x = 2k + 1) \rightarrow (x + 4 = 2m + 1)$

   which is valid.

2. $\forall\exists$ **vs.** $\exists\forall$

As we have seen before, there is a big difference between $\exists x \forall y$ and $\forall y \exists x$ and there is a big difference between proving $\exists x \forall y, P(x, y)$ and $\forall y \exists x, P(x, y)$.

Whenever you have a proof in this form ($\forall y \exists x$) it can help to think of it as a *game*, where player 1 thinks of a y, and then player 2 has to think of an x that makes the predicate true. When given a problem like this, play the game with yourself a few times to see how it works. To prove the statement true, you have to show that player 2 can *always* win and come up with a solution **(Direct proof by generalization)**. To prove the statement false, you have to show that player 1 can *stump* player 2 and come up with a value where player 2 has no solution **(Direct proof by contradiction)**.

Example i) $\forall y \in \mathbb{Z}^+, \exists x \in \mathbb{Z}^+, y > (x - 3)^3$

In this case, the statement is true. Player 2 can just choose $x = 3$ every time to win the game, regardless of what y is.

Example ii) $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, x < y$

In this case, the statement is true: For any value y, we can choose $x = (y - 1)$

What if we reverse the order of the quantifiers?

For proofs in this form ($\exists x \forall y$) you may have to be more clever. Note that you *are* allowed to pick a single value of x and show that it's true. Depending on the problem it may be tricky to find that value of x or it may be hard to show that no x can exist.

Example i) $\exists x \in \mathbb{Z}^+, \forall y \in \mathbb{Z}^+, y > (x - 3)^3$

In this example again, the statement is also true. We can set $x = 3$ and show that $(y > 0)$ for all $y \in \mathbb{Z}^+$.

Example ii) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x < y$

In this case, the statement is **false**. Here it helps to show that the domain for y must include whatever value was chosen for x: No matter what is selected for x, it is false when $y = x$.

Just because $\forall y \exists x, P(x, y)$ is true, it does not mean $\exists x \forall y, P(x, y)$ is true (and vice-versa).

3. **A $\forall\exists$ Example**

Prove that no matter how we choose some positive integer c, there will be a positive real number x for which $c > 1000x$

Mathematically:
$\forall c \in \mathbb{Z}^+, \exists x \in \mathbb{R}^+, c > 1000x$

Direct Proof (Generalization):

For any arbitrary $c \in \mathbb{Z}^+$, choose $x = \frac{c}{2000}$.

Because $c \in \mathbb{Z}^+$, $x \in \mathbb{R}^+$.

Note: It's important at this stage to start with *what you know* $(x = \frac{c}{2000})$ and then get to what you are trying to prove $(c > 1000x)$: *Not the other way around (A common mistake)!*

$x = \frac{c}{2000}$

$\therefore\ x = \frac{1}{2}\frac{c}{1000}$

$\therefore\ x < \frac{c}{1000}$ (This is true because $c \in \mathbb{Z}^+ \therefore\ c > 0$).

$\therefore\ c > 1000x$

$\therefore\ \forall c \in \mathbb{Z}^+, \exists x \in \mathbb{R}^+, c > 1000x$

4. **An Indirect Proof by Contradiction I**

For any integer n, prove that $n^2 - 2$ is not divisible by 4.

Note that there are several equivalent ways of writing this mathematically:
a) $\forall n \in \mathbb{Z}, 4 \nmid (n^2 - 2)$
b) $\forall n \in \mathbb{Z}, \sim(4 \mid (n^2 - 2))$
c) $\forall n \in \mathbb{Z}, \sim\exists m \in \mathbb{Z}, 4m = n^2 - 2$
d) $\forall n \in \mathbb{Z}, \nexists m \in \mathbb{Z}, 4m = n^2 - 2$
e) $\forall n \in \mathbb{Z}, \forall m \in \mathbb{Z}, 4m \neq n^2 - 2$

Indirect Proof (Contradiction):

Assume that the statement is false:
There exists an integer n where $n^2 - 2$ is divisible by 4. Using form c) from above,

$\sim\forall n \in \mathbb{Z}, \sim\exists m \in \mathbb{Z}, 4m = n^2 - 2$
$\equiv \exists n \in \mathbb{Z}, \exists m \in \mathbb{Z}, 4m = n^2 - 2$

Let n,m be two integers where $4m = n^2 - 2$

because n is either odd or even, n can be written in the form $n = 2k + b$ where k is an integer and b is either 0 or 1. $\forall n \in \mathbb{Z}, \exists k \in \mathbb{Z}, \exists b \in \{0, 1\}, n = 2k + b$

$4m = n^2 - 2$
$4m = (2k + b)^2 - 2$
$4m = 4k^2 + 4bk + b^2 - 2$
$4m - 4k^2 - 4bk = b^2 - 2$
$4(m - k^2 - bk) = b^2 - 2$
$4p = b^2 - 2$, where $p = (m - k^2 + bk)$
$p = \frac{b^2-2}{4}$

but we know that b is either 0 or 1, so
$p = -\frac{1}{2}$ or $p = -\frac{1}{4}$
$\therefore$ p is not an integer

because m,k and b are all integers and $p = (m - k^2 + bk)$
$\therefore$ p is an integer

We have reached a contradiction. Therefore our original assumption was false.
$\therefore$ For any integer n, $n^2 - 2$ is not divisible by 4.

5. **An Indirect Proof by Contradiction II**

Show that for any positive integers a and b if $a + b = 11$ then only one of a or b is greater than 5.

Mathematically:
$\forall a \in \mathbb{Z}^+, \forall b \in \mathbb{Z}^+, (a + b = 11) \rightarrow, (a > 5) \oplus (b > 5)$

Indirect Proof (Contradiction)

Sometimes it can help to abstract the problem:
$\sim(p \rightarrow (q \oplus r))$
$\equiv \sim(\sim p \vee (q \oplus r))$
$\equiv (\sim\sim p \wedge \sim(q \oplus r))$
$\equiv (p \wedge \sim(q \oplus r))$
$\equiv (p \wedge (q \leftrightarrow r))$

or in our case:
$\sim\forall a \in \mathbb{Z}^+, \forall b \in \mathbb{Z}^+, (a + b = 11) \rightarrow [(a > 5) \oplus (b > 5)]$
$\equiv \exists a \in \mathbb{Z}^+, \exists b \in \mathbb{Z}^+, (a + b = 11) \wedge [(a > 5) \leftrightarrow (b > 5)]$

In English: There exists positive integers a and b where $a + b = 11$ and either a and b are both greater than 5 or a and b are both less than or equal to 5.

Let a and b be positive integers where $a + b = 11$.

Case One: a and b are both greater than 5
$(a > 5) \wedge (b > 5)$
$\therefore\ (a \geq 6) \wedge (b \geq 6)$
$\therefore\ a + b \geq 12$
$\therefore\ a + b \neq 11$ which is a contradiction

Case Two: a and b are both less than or equal to 5
$(a \leq 5) \wedge (b \leq 5)$
$\therefore\ a + b \leq 10$
$\therefore\ a + b \neq 11$ which is a contradiction

We have reached a contradiction in all cases. Therefore our original assumption was false.
$\therefore$ for any positive integers a and b if $a + b = 11$ then only one of a or b is greater than 5.

6. **An Indirect Proof by Contraposition**

Prove that if $x^3 + 2x + 1$ is even then x is odd.

Mathematically (several equivalent variants):
$\forall x \in \mathbb{Z}, (x^3 + 2x + 1 \text{ is even }) \rightarrow (x \text{ is odd })$
$\forall x \in \mathbb{Z}, E(x^3 + 2x + 1) \rightarrow \sim E(x)$
$\forall x \in \mathbb{Z}, (\exists m \in \mathbb{Z}, 2m = x^3 + 2x + 1) \rightarrow (\exists k \in \mathbb{Z}, 2k + 1 = x)$

Indirect Proof (Contraposition):

We must show that if x is even then $x^3 + 2x + 1$ is odd.

$\forall x \in \mathbb{Z}, \sim(x \text{ is odd }) \rightarrow \sim(x^3 + 2x + 1 \text{ is even })$
$\equiv \forall x \in \mathbb{Z}, (x \text{ is even}) \rightarrow (x^3 + 2x + 1 \text{ is odd })$

Let x be any arbitrary even integer where $x = 2k$ and k is an integer.

$x = 2k$
$\therefore\ x^3 + 2x + 1 = (2k)^3 + 2(2k) + 1$
$\therefore\ x^3 + 2x + 1 = 8k^3 + 4k + 1$
$\therefore\ x^3 + 2x + 1 = 2(4k^3 + 2k) + 1$
$\therefore\ x^3 + 2x + 1 = 2m + 1$, where $m = (4k^3 + 2k)$
$\therefore\ x^3 + 2x + 1$ is odd

Since $\equiv \forall x \in \mathbb{Z}, (x \text{ is even}) \rightarrow (x^3 + 2x + 1 \text{ is odd })$
$\therefore\ \forall x \in \mathbb{Z}, (x^3 + 2x + 1 \text{ is even }) \rightarrow (x \text{ is odd })$