Dave's CPSC 121 Tutorial Notes – Week Four

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Formula Sheet Tips

- **Proof Methods**

Universal Modus Ponens:

$$\begin{array}{l} \forall x, P(x) \to Q(x) \\ P(a) \\ \hline \therefore\ Q(a) \end{array}$$

Universal Modus Tollens:

$$\begin{array}{l} \forall x, P(x) \to Q(x) \\ \sim Q(a) \\ \hline \therefore \sim P(a) \end{array}$$

Direct Proof (Existential):

$$\begin{array}{l} a \in D \\ P(a) \\ \hline \therefore\ \exists x \in D, P(x) \end{array}$$

Direct Proof (Counterexample):

$$\begin{array}{l} a \in D \\ \sim P(a) \\ \hline \therefore \sim (\forall x \in D, P(x)) \end{array}$$

Direct Proof (Exhaustive):

$$\begin{array}{l} D = \{a, b, c\} \\ P(a) \\ P(b) \\ P(c) \\ \hline \therefore\ \forall x \in D, P(x) \end{array}$$

Direct Proof (Generalization):

$$\begin{array}{l} P(x) \text{ for arbitrary } x \in D \\ \hline \therefore\ \forall x \in D, P(x) \end{array}$$

Direct Proof (by cases):

$$\begin{array}{l} D = \{x | (x \in E) \vee (x \in F)\} \\ \forall x \in E, P(x) \\ \forall x \in F, P(x) \\ \hline \therefore\ \forall x \in D, P(x) \end{array}$$

Indirect Proof (Contradiction):

$$\begin{array}{l} \text{assume} \sim P(x) \\ \hline \therefore\ a \\ \therefore \sim a \\ \hline \therefore\ P(x) \end{array}$$

Indirect Proof (Contraposition):

$$\begin{array}{l} \forall x \in D, \sim Q(x) \to \sim P(x) \\ \hline \therefore\ \forall x \in D, P(x) \to Q(x) \end{array}$$

- **Mathematical Definitions:**

x is even	$\Leftrightarrow$	$\exists k \in \mathbb{Z}, x = 2k$
x is odd	$\Leftrightarrow$	$\exists k \in \mathbb{Z}, x = 2k + 1$
$x \in \mathbb{Q}$	$\Leftrightarrow$	$\exists a, b \in \mathbb{Z}, (x = \frac{a}{b}) \wedge (b \neq 0)$
x is prime	$\Leftrightarrow$	$\forall k, m \in \mathbb{Z}, (x = mn) \to [(m = x) \vee (m = 1)]$
$3 \mid x$	$\Leftrightarrow$	$\exists k \in \mathbb{Z}, x = 3k$
$3 \nmid x$	$\Leftrightarrow$	$\exists k \in \mathbb{Z}, (x = 3k + 1) \vee (x = 3k + 2)$

- **Counterexample Tips:**

$\mathbb{Z}$	Integers:	Check $-1, 0, 1$
$\mathbb{Q}$	Rational Numbers:	Check above plus $\frac{a}{a}, \frac{-a}{a}, \frac{a}{0}$
$\mathbb{R}$	Real Numbers:	Check above plus $(0 < x < 1)$
$\forall x \exists y$	Multiple Vars:	Check above plus $(y = x), (y = -x), (y = \frac{1}{x})$

Sample Problems

1. **Simple Oddity**

   Prove that if x is odd then $(x+4)$ is odd.

   Mathematically:

   $O(x) \Leftrightarrow x$ is odd
   $\forall x \in \mathbb{Z}, O(x) \rightarrow O(x+4)$

   or, recall that (x is odd) $\Leftrightarrow$ $(\exists k \in \mathbb{Z}, x = 2k+1)$
   so it can also be written as:

   $\forall x \in \mathbb{Z}, (\exists k \in \mathbb{Z}, x = 2k+1) \rightarrow (\exists m \in \mathbb{Z}, x+4 = 2m+1)$

   Direct Proof (Generalization):

   Let x be an arbitrary odd integer.
   By definition, $\exists k \in \mathbb{Z}, x = 2k+1$.
   $x + 4$
   $= (2k+1) + 4$
   $= 2k + 4 + 1$
   $= 2(k+2) + 1$
   $= 2m + 1$ where $m = (k+2)$
   Since $k \in \mathbb{Z}$, $(k+2) \in \mathbb{Z}$, $\therefore\ m \in \mathbb{Z}$
   Since $(x+4)$ can be written as $(2m+1)$ where $m \in \mathbb{Z}$
   $\therefore\ (x+4)$ is odd

2. **Another Oddity**

   Prove that $(x^2 - 3x + 5)$ is odd.

   Mathematically:

   $\forall x \in \mathbb{Z}, \exists k \in \mathbb{Z}, x^2 - 3k + 5 = 2k + 1$

   Direct Proof (by cases):

   By definition, if $x \in \mathbb{Z}$ then x is either odd or even.

   Case One: (x is even)

   By definition, $\exists k \in \mathbb{Z}, x = 2k$.
   $x^2 - 3x + 5$
   $= (2k)^2 - 3(2k) + 5$
   $= 4k^2 - 6k + 4 + 1$
   $= 2(2k^2 - 3k + 2) + 1$
   $= 2m + 1$ where $m = (2k^2 - 3k + 2)$
   Since $k \in \mathbb{Z}$, $(2k^2 - 3k + 2) \in \mathbb{Z}, \therefore\ m \in \mathbb{Z}$
   Since $(x^2 - 3x + 5)$ can be written as $(2m + 1)$ where $m \in \mathbb{Z}$
   $\therefore\ (x^2 - 3x + 5)$ is odd when x is even.

   Case Two: (x is odd)

   By definition, $\exists k \in \mathbb{Z}, x = 2k + 1$.
   $x^2 - 3x + 5$
   $= (2k + 1)^2 - 3(2k + 1) + 5$
   $= (4k^2 + 4k + 1) - (6k + 3) + 5$
   $= 4k^2 - 2k + 3$
   $= 2(2k^2 - k + 1) + 1$
   $= 2m + 1$ where $m = (2k^2 - k + 1)$
   Since $k \in \mathbb{Z}$, $(2k^2 - k + 1) \in \mathbb{Z}, \therefore\ m \in \mathbb{Z}$
   Since $(x^2 - 3x + 5)$ can be written as $(2m + 1)$ where $m \in \mathbb{Z}$
   $\therefore\ (x^2 - 3x + 5)$ is odd when x is odd.

   Since [$(x^2 - 3x + 5)$ is odd when x is even] and [$(x^2 - 3x + 5)$ is odd when x is odd]
   $\therefore\ (x^2 - 3x + 5)$ is odd for all $x \in \mathbb{Z}$

3. **A Third Oddity**

   Show that if x is odd then x^2 has the form $8m + 1$ for some integer m.

   Mathematically:

   $\forall x \in \mathbb{Z}, (\exists k \in \mathbb{Z}, x = 2k + 1) \rightarrow (\exists m \in \mathbb{Z}, x^2 = 8m + 1)$

   This problem is solved in the Epp Textbook: Theorem 3.4.3 on page 162

   The key is to identify that odd numbers can be written as either $(4q + 1)$ or $(4q + 3)$

4. **The Smallest Rational Number**

   Prove there is no smallest positive rational number.

   Mathematically:

   $\forall x \in \mathbb{Q}^+ \exists y \in \mathbb{Q}^+, y < x$

   Direct Proof (Generalization):

   Let x be an arbitrary positive rational number $(x \in \mathbb{Q}^+)$

   By definition, $\exists a, b \in \mathbb{Z}^+ (b \neq 0), x = \frac{a}{b}$
   $x = \frac{a}{b}$
   Choose
   $y = \frac{x}{2} = \frac{a}{2b}$

   Part One: Demonstrate $y \in \mathbb{Q}^+$ (Trivial, but important):
   Since $b \in \mathbb{Z}^+ \therefore\ 2b \in \mathbb{Z}^+$
   Since $b \neq 0 \therefore\ 2b \neq 0$
   Because $(a), (2b) \in \mathbb{Z}^+$ and $2b \neq 0$,
   $\frac{a}{2b}$ is rational and $y \in \mathbb{Q}^+$

   Part Two: Demonstrate $y < x$
   $y = \frac{x}{2}$
   $\therefore\ 2y = x$
   $\therefore\ y = x - y$
   $\therefore\ 0 < x - y$ (because $y \in \mathbb{Q}^+$ and so $(0 < y)$)
   $\therefore\ y < x$

   So for any $(x \in \mathbb{Q}^+)$ there exists $y = \mathbb{Q}^+$ where $y = \frac{x}{2}$ and $y < x$.