Dave's CPSC 121 Tutorial Notes – Week Three

Last Updated: 2008.02.01 (v2.0)

Formula Sheet Tips

- **Domains:**

  $\mathbb{N}$ Natural Numbers: $\{0, 1, 2, \ldots\}$
  $\mathbb{Z}$ Integers $\{\ldots, -2, -1, 0, 1, +2, \ldots\}$
  $\mathbb{Q}$ Rational Numbers $\{\ldots, \frac{-1}{3}, 0, \frac{1}{2}, 1, \ldots\}$
  $\mathbb{R}$ Real Numbers $\{\ldots, \frac{-1}{2}, 0, \sqrt{2}, \pi, \ldots\}$

- **Quantifiers:**

  $\forall x \in \mathbb{U}[P(x)]$ $P(x)$ is true for *all* (every) x in $\mathbb{U}$
  $\exists x \in \mathbb{U}[P(x)]$ $P(x)$ is true for *at least one* x in $\mathbb{U}$

- **Equivalent Domain Representation:**

  if $P(x)$ is a predicate that defines a *subset* of another domain to be a new domain D:
  $D = \{x \in \mathbb{U} | P(x)\}$
  then
  $\forall x \in D[Q(x)] \equiv \forall x \in \mathbb{U}[P(x) \rightarrow Q(x)]$
  $\exists x \in D[Q(x)] \equiv \exists x \in \mathbb{U}[P(x) \wedge Q(x)]$

- **Negation:**

  $\sim[\forall x \in D[P(x)]] \equiv \exists x \in D[\sim[P(x)]]$
  $\sim[\exists x \in D[P(x)]] \equiv \forall x \in D[\sim[P(x)]]$

Sample Problems

1. **Domain Selection & Defining Predicates**

   You want to put some thought into how you define your domains and your predicates, as there may be more than one approach. Depending on your goal, some may be easier to work with than others.

   Example: Write *All brown puppies are cute* using predicate logic.

   Approach One:
   $U =$ { set of all brown puppies }
   $C(x)$: x is cute

   $\forall x \in U[C(x)]$

   Approach Two:
   $U =$ { set of all puppies }
   $B(x)$: x is brown
   $C(x)$: x is cute

   $\forall x \in U[B(x) \rightarrow C(x)]$

   Approach Three:
   $U =$ { set of all dogs }
   $B(x)$: x is brown
   $C(x)$: x is cute
   $Y(x)$: x is young

   $\forall x \in U[(B(x) \wedge Y(x)) \rightarrow C(x)]$

   Approach Four:
   $U =$ { set of all things }
   $B(x)$: x is brown
   $C(x)$: x is cute
   $Y(x)$: x is young
   $D(x)$: x is a dog

   $\forall x \in U[(D(x) \wedge B(x) \wedge Y(x)) \rightarrow C(x)]$

   All of these are valid representations of the original statement. While approach four may seem the most complicated, it may be the most flexible and powerful if you also have to also represent statements like *you can't teach an old dog new tricks* with $\forall x \in U[(D(x) \wedge \sim Y(x)) \rightarrow \sim T(x)]$.

   The lesson to be learned here is that you should take extra care defining your domains and your predicates, and be aware that alternate domain forms can exist which can make your life easier or harder.

2. **Negation**

   The negation of the statement: *All puppies are cute* is **not** *All puppies are not cute*, but rather *There exists at least one puppy that is not cute*.

   $\sim[\forall x \in U[C(x)]] \equiv \exists x \in U[\sim C(x)]$

   Note the difference using **Approach Two** from above:

   $U = \{$ set of all puppies $\}$
   $B(x)$: x is brown
   $C(x)$: x is cute

   $\sim[\forall x \in U[B(x) \rightarrow C(x)]]$
   $\equiv\sim[\forall x \in U[\sim B(x) \vee C(x)]]$
   $\equiv \exists x \in U[\sim[\sim B(x) \vee C(x)]]$
   $\equiv \exists x \in U[B(x) \wedge \sim C(x)]$

   $\sim[\forall x \in U[B(x) \rightarrow C(x)]] \equiv \exists x \in U[B(x) \wedge \sim C(x)]$

3. **Multiple Variable Predicates**

   Something to be aware of when you're working with multi-variable predicates is that the variables can be *un-ordered*:

   $N(x, y)$: x and y are both in Dave's tutorial

   N(Alice,Bob) $\equiv$ N(Bob,Alice)

   Or, they can be *ordered*:

   $L(x, y)$: x loves y

   L(Dave, Natalie Portman) is true
   L(Natalie Portman, Dave) is *(likely)* false

4. **All You Need is Love**

   When dealing with different quantifiers, the order can be *very important*

   $L(x, y)$: x loves y

   There are *six* different ways of quantifying $L(x, y)$ and they are all unique. As an exercise, make sure you understand all of these differences:

   1. $\forall x \forall y[L(x, y)]$ Everybody loves every other single person *Utopia*
   2. $\forall x \exists y[L(x, y)]$ Everybody loves at least one other person *Everybody loves some body*
   3. $\forall y \exists x[L(x, y)]$ Everybody has somebody love them *Everyone has a mother*
   4. $\exists x \forall y[L(x, y)]$ There exists someone who loves everybody *Barney loves everybody*
   5. $\exists y \forall x[L(x, y)]$ There exists someone that everybody loves *Everybody loves Raymond*
   6. $\exists x \exists y[L(x, y)]$ There exists someone who loves somebody *Love exists*

   Note that if both sequential quantifiers are the same, the order does not matter:
   $\exists x \exists y \equiv \exists y \exists x$

5. **Negating Love**

   For each of the above statements about love, we will look at the negation:

   1. $\sim[\forall x \forall y[L(x, y)]] \equiv \exists x \exists y[\sim L(x, y)]]$ At least one person doesn't love someone
   2. $\sim[\forall x \exists y[L(x, y)]] \equiv \exists x \forall y[\sim L(x, y)]]$ There exists someone who does not love anybody
   3. $\sim[\forall y \exists x[L(x, y)]] \equiv \exists y \forall x[\sim L(x, y)]]$ There exists someone that nobody loves
   4. $\sim[\exists x \forall y[L(x, y)]] \equiv \forall x \exists y[\sim L(x, y)]]$ Everyone has at least one person they don't love
   5. $\sim[\exists y \forall x[L(x, y)]] \equiv \forall y \exists x[\sim L(x, y)]]$ Everyone has at least one person who doesn't love them
   6. $\sim[\exists x \exists y[L(x, y)]] \equiv \forall x \forall y[\sim L(x, y)]]$ There is no love