Dave's CPSC 121 Tutorial Notes – Week Ten

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Formula Sheet Tips

Cheat Sheet Tips

- **Mathematical Induction:**

$$\frac{P(1) \quad P(k) \rightarrow P(k+1)}{\therefore\ \forall n \in \mathbb{Z}^+[P(n)]} \qquad \frac{P(1) \quad P(k-1) \rightarrow P(k)}{\therefore\ \forall n \in \mathbb{Z}^+[P(n)]} \qquad \frac{P(a) \quad P(k) \rightarrow P(k+1)}{\therefore\ \forall n \in \mathbb{Z}(n \geq a)[P(n)]}$$

- **4 Steps for a Proof by Mathematical Induction:**

  1. Clearly define your predicate P(n)
  2. Show P(1) is true
  3. State your *Inductive Hypothesis:* P(k) is true, and then show $P(k) \rightarrow P(k+1)$
  4. State that because
     $P(1) \land [P(k) \rightarrow P(k+1)]$
     $\therefore\ \forall n \in \mathbb{Z}^+[P(n)]$
     by the principle of mathematical induction.

Sample Problems

1. **The Classic: The sum of first n Integers**

   Prove for all integers $n \geq 1$ that $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

   Step One: State $P(n)$
   $P(n) : 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$

   Step Two: Show $P(1)$ is true
   $P(1) : 1 = \frac{1(1+1)}{2}$
   Clearly, LHS = RHS, $\therefore\ P(1) \equiv T$

   Step Three: State *I.H.*, show $P(k) \rightarrow P(k+1)$ is true
   Inductive Hypothesis (I.H.): $P(k) \equiv T$

   $\therefore\ 1 + 2 + 3 + \ldots + k = \frac{k(k+1)}{2}$ [by the *I.H.*]
   $\therefore\ 1 + 2 + 3 + \ldots + k + (k+1) = \frac{k(k+1)}{2} + (k+1)$ [add $(k+1)$ to both sides]
   $\therefore\ 1 + 2 + 3 + \ldots + k + k + 1 = \frac{k(k+1)+2(k+1)}{2}$
   $\therefore\ 1 + 2 + 3 + \ldots + k + k + 1 = \frac{(k+1)(k+2)}{2}$
   $\therefore\ P(k+1) \equiv T$

   Step Four: Conclusion

   Since $P(1) \wedge [P(k) \rightarrow P(k+1)]$
   $\therefore\ \forall n \in \mathbb{Z}^{+}[P(n)]$
   by the principle of mathematical induction.

2. **One with Inequalities**

Prove for all integers $n > 1$ that $\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}$

Step One: State $P(n)$

$P(n) : \sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}$

Step Two: Show $P(2)$ is true

$P(2) : \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2}$
LHS = 1 + 0.707 = 1.707
RHS = 1.414
LHS $>$ RHS,
$\therefore\ P(2) \equiv T$

Step Three: State *I.H.*, show $P(k) \rightarrow P(k+1)$ is true

Inductive Hypothesis (I.H.): $P(k) \equiv T$

$$\therefore\ \sum_{i=1}^{k} \frac{1}{\sqrt{i}} > \sqrt{k} \qquad \text{[by the } \textit{I.H.}\text{]}$$

$$\therefore\ \sum_{i=1}^{k} \frac{1}{\sqrt{i}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}} \qquad \text{[add } \tfrac{1}{\sqrt{k+1}} \text{ to both sides]}$$

$$\therefore\ \sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} > \sqrt{k} + \frac{1}{\sqrt{k+1}} \qquad \text{[change } \textstyle\sum \text{ limit]}$$

$$\therefore\ \sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} > \frac{\sqrt{k}\sqrt{k+1}+1}{\sqrt{k+1}}$$

$$\therefore\ \sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} > \frac{\sqrt{k}\sqrt{k}+1}{\sqrt{k+1}} \qquad \text{[because } k < k+1\text{]}$$

$$\therefore\ \sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} > \frac{k+1}{\sqrt{k+1}}$$

$$\therefore\ \sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} > \sqrt{k+1}$$

$\therefore\ P(k+1) \equiv T$

Step Four: Conclusion

Since $P(2) \wedge [P(k) \rightarrow P(k+1)]$
$\therefore\ \forall n \in \mathbb{Z}^{+}(n \geq 2)[P(n)]$
by the principle of mathematical induction.

3. **One with Divisibility**

   Prove for all integers greater than or equal to 1 that $n(n^2+5)$ is divisible by 6.

   Step One: State $P(n)$
   $P(n) : 6|[(n)(n^2+5)]$

   Step Two: Show $P(1)$ is true
   $P(1) : 6|[(2)(2^2+5)]$
   $(2)(2^2+5) = 2(9) = 18 = 3(6)$
   $6|[3(6)]$
   $\therefore\ P(1) \equiv T$

   Step Three: State *I.H.*, show $P(k) \rightarrow P(k+1)$ is true
   Inductive Hypothesis (I.H.): $P(k) \equiv T$

   $(k+1)((k+1)^2+5)$
   $= (k+1)(k^2+2k+1+5)$
   $= k^3+3k^2+8k+6$
   $= (k^3+5k)+(3k^2+3k+6)$
   $= (k)(k^2+5)+(3k^2+3k+6)$
   $= (k)(k^2+5)+3(k^2+k+2)$
   $= (k)(k^2+5)+3(k(k+1)+2)$

   Aside: we know that $2|[k(k+1)]$, because either k or $k+1$ is even.

   since $2|[(k)(k+1)]$ and clearly $2|2$ then $2|[k(k+1)+2]$

   since $2|[3(k(k+1)+2)]$ and clearly $3|[3(k(k+1)+2)]$ then $6|[3(k(k+1)+2)]$

   From the *I.H.* we know $6|[(k)(k^2+5)]$

   Because $6|[(k)(k^2+5)]$ AND $6|[3(k(k+1)+2)]$
   $\therefore\ 6|[(k)(k^2+5)+3(k(k+1)+2)]$
   $\therefore\ 6|[(k+1)((k+1)^2+5)]$
   $\therefore\ P(k+1)$

   Step Four: Conclusion

   Since $P(1) \wedge [P(k) \rightarrow P(k+1)]$
   $\therefore\ \forall n \in \mathbb{Z}^+[P(n)]$
   by the principle of mathematical induction.