Dave's CPSC 121 Tutorial Notes – Week Eleven

Cheat Sheet Tips

- **Strong Mathematical Induction:**

$$\begin{array}{l} P(a) \\ P(a+1) \\ \ldots P(b) \\ [P(a) \wedge P(a+1) \wedge P(a+2) \wedge \ldots \wedge P(k-1)] \rightarrow P(k) \\ \hline \therefore\ \forall n \in \mathbb{Z}^+ (n \geq a)[P(n)] \end{array}$$

Sample Problems

1. **A simple strong induction**

   Prove for all positive integers the last digit of $5n$ is either a 0 or a 5.

   Notation: % is the mod function: so $x\%10 =$ the last digit of x.

   Step One: State $P(n)$
   $P(n)$: $(5n\%10 = 0) \vee (5n\%10 = 5)$

   Step Two: Show $P(1)$ and $P(2)$ are true
   $P(1)$: $(5\%10 = 0) \vee (5\%10 = 5)$
   $(5\%10) = 5 \therefore\ P(1) \equiv (F \vee T) \equiv T$

   $P(2)$: $(10\%10 = 0) \vee (10\%10 = 5)$
   $(5\%10) = 0 \therefore\ P(2) \equiv (T \vee F) \equiv T$

   Step Three: State *I.H.*, show $P(k-2) \rightarrow P(k)$ is true
   Inductive Hypothesis (I.H.): $P(k-2) \equiv T$

   $5(k-2)\%10$
   $= (5k-10)\%10$
   $= 5k\%10 - 10\%10$ [$(a+b)\%c = a\%c + b\%c$]
   $= 5k\%10$ [since 10 % 10 = 0]

   Since $5k\%10 = 5(k-2)\%10$
   and $(5(k-2)\%10 = 0) \vee (5(k-2)\%10 = 5)$ [by *I.H.*]
   $\therefore\ P(k) \equiv T$

   Step Four: Conclusion

   Since $P(1) \wedge P(2) \wedge [P(k-2) \rightarrow P(k)]$
   $\therefore\ \forall n \in \mathbb{Z}^{+}[P(n)]$
   by the principle of strong mathematical induction.

2. **One with an unspecified base case**

   For which amounts of change can you make if you have six 13¢ coins and an infinite number of 7¢ coins?

   Step Zero: Determine the range of integers
   With just 7¢ coins, you can only make change for values x where $x\%7 = 0$.

   With just one 13¢ coin $(13\%7 = 6)$ you can make change for values x where $x \geq 13$ and $x\%7 = 6$.

   $26\%7 = 5$
   $39\%7 = 4$
   $52\%7 = 3$
   $65\%7 = 2$
   and
   $78\%7 = 1$

   So clearly any x where $x \geq 78$ can be made with up to six 13¢ coins and an infinite number of 7¢ coins. However, 71¢ is the *last* highest value that cannot be reached.

   Step One: State $P(n)$
   $P(n)$: You can make change for n¢ with up to six 13¢ coins and any number of 7¢ coints.

   Step Two: Show Base Cases are true
   $P(72) : 72 = 5 * 13 + 1 * 7 \therefore\ P(72) \equiv T$
   $P(73) : 73 = 4 * 13 + 3 * 7 \therefore\ P(73) \equiv T$
   $P(74) : 74 = 3 * 13 + 5 * 7 \therefore\ P(74) \equiv T$
   $P(75) : 75 = 2 * 13 + 7 * 7 \therefore\ P(75) \equiv T$
   $P(76) : 76 = 1 * 13 + 9 * 7 \therefore\ P(76) \equiv T$
   $P(77) : 77 = 0 * 13 + 11 * 7 \therefore\ P(77) \equiv T$
   $P(78) : 78 = 6 * 13 + 0 * 7 \therefore\ P(78) \equiv T$

   Step Three: State *I.H.*, show $P(k-7) \rightarrow P(k)$ is true
   Inductive Hypothesis (I.H.): $P(k-7) \equiv T$

   Because $P(k-7) \equiv T$, We can make change for $(k-7)$¢.
   To make change for k¢ , we only require one additional 7¢ coin.

   Step Four: Conclusion
   Since $P(72) \wedge P(73) \wedge P(74) \wedge P(75) \wedge P(76) \wedge P(77) \wedge P(78) \wedge [P(k-7) \rightarrow P(k)]$
   $\therefore\ \forall n \in \mathbb{Z}^{+}(n \geq 72)[P(n)]$
   by the principle of mathematical induction.