On {123,124,134}-free Hypergraphs

P. C. Li* and G. H. J. van Rees†
Department of Computer Science
University of Manitoba
Winnipeg, Manitoba
Canada R3T 2N2

D. R. Stinson‡
David R. Cheriton School of Computer Science
University of Waterloo
Waterloo, Ontario
Canada N2L 3G1

R. Wei§
Department of Computer Science
Lakehead University
Thunder Bay, Ontario
Canada P7B 5E1

Abstract

Let H be the 3-hypergraph having edges {123,124,134} and points {1,2,3,4}. A 3-hypergraph is *H-free* if it does not contain three edges isomorphic to H. The integer $\mathrm{ex}(n, H)$ denotes the maximum number of edges in any H-free hypergraph on n points. In this paper, it is shown that de Caen's upper bound, $n^2(n-1)/18$, cannot be met for $n > 6$. Then the exact values for $\mathrm{ex}(n, H)$ for $n = 9$, 10, 11 and 12 are determined. Finally, an improvement to $\mathrm{ex}(13, H)$ is given, which allows us to improve the upper bounds for $\mathrm{ex}(n, H)$ for $n = 14, \ldots, 24$. Using these numbers, Mubayi's asymptotic upper bound is improved to $1/3 - 1.89820 \times 10^{-5}$.

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1 Introduction

A 3-*hypergraph* is a set system in which every edge (or block or triple) has size three and in which there do not exist repeated blocks. Let $H(4,3)$, or H for brevity, be the 3-hypergraph having edges {123,124,134} and points {1,2,3,4}. A 3-hypergraph is *H-free* if it does not contain three blocks isomorphic to H. The integer ex(n, H) denotes the maximum number of blocks in any H-free hypergraph on n points. Some results on ex(n, H) can be found in [2, 3].

We define

$$\pi(H(4,3)) = \lim_{n\to\infty} \frac{\mathrm{ex}(n,H)}{\binom{n}{3}}.$$

For applications of $H(4,3)$ to computer science see [4, 9]. In 1983, de Caen [1] showed that ex$(n, H) \leq n^2(n-1)/18$. De Caen's bound implies that $\pi(H(4,3)) \leq 1/3$. Some efforts were made to improve de Caen's bound in both the asymptotic and non-asymptotic cases. In the asymptotic case, Matthias [6] proved that $\pi(H(4,3)) \leq 1/3 - 10^{-10}$. Then Mubayi [8] improved this result to $\pi(H(4,3)) \leq 1/3 - 0.45305 \times 10^{-5}$. Just recently, Talbot [10] has improved this to $\pi(H(4,3)) \leq .32975$. For the non-asymptotic case, Deng et al. [2] proved that, for some instances where $n^2(n-1)/18$ is not an integer, the bound could be improved. They also found ex(n, H) for $n = 3, 4, 5, 6, 7$ and 8.

In Section 2, we show that for $n > 6$, ex$(n, H) < n^2(n-1)/18$. In Section 3, we describe a computer search to find the exact value for ex(n, H) for $n = 9$. We then determine that ex(n, H) for $n = 10, 11, 12$. Finally, in Section 4, we show that ex$(13, H) \leq 103$, improving the currently best known upper bound. We use this result to improve the upper bounds on ex(n, H) for $n = 14, \ldots, 24$. In Section 5, we slightly modify Mubayi's method, use the upper bounds of ex(n, H) from Section 4 and use the exact number of blocks in an H-free hypergraph to improve Mubayi's bound to $\pi(H(4,3)) \leq 1/3 - 1.89820 \times 10^{-5}$. Note, however, that Talbot's recent bound is stronger than our bound.

2 If de Caen's Bound is an Integer

If de Caen's bound is an integer, then every pair must occur equally often in the triples of the 3-hypergraph (see Theorem 5.14 in [2]). Then the 3-hypergraph is a BIBD$(n, 3, (n/3))$, i.e., every element occurs $n(n-1)/6$ times. We examine this situation in the next theorem. Originally we had a long tedious proof from first principles, but we instead present a short proof supplied by an anonymous referee.

Theorem 2.1. *Suppose that $n > 6$ is an integer. If $n \equiv 0, 1, 3, 6 \pmod 9$, then ex(n, H)$\leq n^2(n-1)/18-1$. If $n \equiv 2, 4, 5, 7, 8 \pmod 9$, then ex(n, H)$< n^2(n-1)/18$.*

Proof. Suppose such a system exists with $n^2(n-1)/18$ triples. If n is not divisible by 3, then the number of pairs cannot all be equal and the bound cannot be met. If n is divisible by 3, then the system is a simple (no repeated blocks) $(n = 3\lambda, 3, \lambda)$-BIBD, where each pair xy occurs λ times in the system. Now if every four points span 0 or 2 triples, then, by Frankl and Füredi ([3]), the number of triples in such a hypergraph is strictly less than $n^2(n-1)/18$ for $n > 6$. These numbers are explicitly computed in Theorem 3.4. So there must be a set of four points, say $1, 2, 3, 4$, spanning precisely one edge, say the triple $\{1, 2, 3\}$. This means that 4 does not occur with the pairs 12, 13 or 23 in any triple. Further, the elements that occur with the pairs 12, 13 and 23 cannot be repeated, as otherwise we have H. So one of them has size at most $(n-4)/3 + 1 < \lambda$ and our assumption that ex$(n, H) = n^2(n-1)/18$ is false. ∎

3 Computing $\mathrm{ex}(9, \{123, 124, 134\})$

The first n where the exact value of ex(n, H) is unknown is $n = 9$. It has been previously shown that $32 \leq \mathrm{ex}(9, \{123, 124, 134\}) \leq 33$. In this section, we describe how we showed that $\mathrm{ex}(9, \{123, 124, 134\}) = 32$.

In order to determine the value of $\mathrm{ex}(9, \{123, 124, 134\})$, we will use an orderly backtracking algorithm which will try to find such a system with 33 triples. We begin by placing the lexicographical ordering on the $\binom{9}{3} = 84$ triples given by Algorithm 2.7 of [5]. The *rank* of a triple is its position in the ordering. At each stage of the algorithm, the next triple, T, chosen must have rank larger than the rank of the triple chosen in the previous stage. A choice set for each stage can now be defined. Suppose $\mathcal{C}_i$ is the choice set for stage i, and a $T \in C_i$ is selected to be in the partial solution as the i^{th} block of the partial solution. Then, C_{i+1} is the set of all triples in $\mathcal{C}_i$ whose rank is larger than that of T having the property that adding any one of these triples to the partial solution (as the $(i+1)^{st}$ block) will keep it H-free. C_1 is initialized to contain all of the 84 triples. It is important that the triples in the choice set be ordered (and selected) by increasing rank.

In order to speed up the search, we will use an isomorphic rejection technique. For each partial solution with b triples, we form the $b \times 9$ incidence matrix B where row i of B is the characteristic vector of the i^{th} triple. For example, suppose the triples $\{1, 2, 3\}$, $\{1, 2, 4\}$ and $\{1, 3, 5\}$ are selected in this order in the partial solution. Then the corresponding incidence matrix is

1	1	1	0	0	0	0	0	0
1	1	0	1	0	0	0	0	0
1	0	1	0	1	0	0	0	0

The algorithm will reject a partial solution if its corresponding incidence matrix is not in canonical form. We define the canonical form of an $b \times 9$ matrix B in the following manner. First, define $b(B)$ to be the binary string obtained by concatenating the rows of B, so that the $B[1][1]$ is the most significant bit of the string. The order of the natural numbers can them be used to define an ordering on all the $b \times 9$ incidence matrices. We say that the matrix B is in *canonical form* if for each $B^{'}$ obtained by permuting the rows and columns of B, we have $b(B) > b(B^{'})$. A backtracking algorithm from [7] can be used to determine if an incidence matrix is canonical or not. We have included the pseudocode, as given by Algorithm 3.1, for completeness. Here, π is a permutation of the columns, and M_i is the i^{th} row of M. Before attempting to add a triple to the partial solution (which is selected from the choice set), the search algorithm will determine if adding the triple leads an incidence matrix in canonical form. If it does, we add the triple to the partial solution and continue to the next stage. If not, then it will not add the triple, and try to add the next triple in the choice set.

Notice that if an incidence matrix B is in canonical form, the triple corresponding to row i has rank smaller than the rank of the triple corresponding to row j, where $i < j$. For if not, we can swap the two rows and get a matrix $B^{'}$ with $b(B^{'}) < b(B)$. In addition, if B is canonical, then it is easy to see that the matrix B^*, formed using the all but the last row of B, is also canonical. This implies that the way we define our choice set and how we pick the next candidate triple to add to the partial solution is consistent with the canonical test. We now state the pseudocode for the search algorithm. Algorithm 3.2 is called with $\mathcal{C}_1$ containing all 84 triples and *level* set to 1 and *stoplevel* set to 34.

Using this algorithm, we were able to show that ex(9,H)$\neq$ 33 and that there are 6 non-isomorphic H-free hypergraphs on 9 points and 32 blocks. These are listed in the appendix. This computation was done on a Pentium III, 700 MHz machine with 1 GB of RAM in about 134676 seconds. Note that the isomorphism test was done for only up to the ninth level. We found this to be a good place to stop performing the canonical test.

To use this result we need the following from [2]:

Theorem 3.1. *If there exists an H-free hypergraph on n points with b blocks and $x \in X$ occurs in b_x blocks, then there exists an H-free hypergraph on $n-1$ points with at least $b - b_x$ blocks.*

Algorithm 3.1: *isCanon*(M)

```
comment  M is a n by v matrix.
comment  Returns TRUE if it is canonical
I ← {1, 2, ..., n, n+1}
k ← 0
last ← 0
repeat
  j ← min{i ∈ I : i > last}
  if j < n+1
    then π[k+1] ← j, I ← I \ {j}, k ← k+1
    else { if k ≥ 1
             then I ← I ∪ π[k], last ← π[k], k ← k−1
             else return ( true )
  B ← matrix with rows M_π[1], ..., M_π[k]
  B* ←B, after it is sorted by descending order of its columns
  comment  Compare k^th row of B* and M
  if b(B*_k) > b(M_k)
    then return ( false )
  if b(B*_k) < b(M_k)
    then I ← I ∪ π[k], last ← π[k], k ← k−1
until true
```

Of course, any H-free hypergraph on n points with b blocks must have a point of frequency at most $\lfloor 3b/n \rfloor$. So we can state the following:

Corollary 3.2. *If there exists an H-free hypergraph on n points with b blocks, then there exists an H-free hypergraph on $n-1$ points with at least $b - \lfloor 3b/n \rfloor$ blocks.*

Using Theorem 3.1, Corollary 3.2 and $\mathrm{ex}(9, \{123, 124, 134\}) = 32$, we can get upper bounds for $\mathrm{ex}(n, H)$, $n = 10, 11, 12$. To get the best lower bounds for $n = 10, 11, 12$, we use the following constructions from [3].

Construction 3.3. *Let $|V| = n$ and partition V into six parts V_i where $|V_i| \geq \lfloor n/6 \rfloor$. Define a 3-hypergraph on n points with blocks $\{(v_{i_1}, v_{i_2}, v_{i_3}) : 1 \leq i_1 \leq i_2 \leq i_3 \leq 6, v_{i_j} \in V_{i_j}, (i_1, i_2, i_3) \in S(6)\}$ where $S(6)$ is the unique, up to isomorphism H-free hypergraph on 6 points and 10 blocks.*

The constructed hypergraph has the property that any set of 4 points spans either 0 or 2 blocks. It was shown in [3] that no hypergraph satisfying

Algorithm 3.2: *Backtrack*(*level*)

```
global C_i, blocks, stoplevel
local M, i, result
if level ≥ stoplevel
  then return ( true )
for i ← 1 to |C_level|
     ⎧ comment  Check if there are enough blocks in choice set
     ⎪ if (|C_level| − i) < (stoplevel − level)
     ⎪   then go to 10
     ⎪ blocks[level] ← C_level[i]
     ⎪ comment  Perform canonical test
  do ⎨ M ← ConstructIncidenceMatrix(blocks, level)
     ⎪ if isCanon(M)
     ⎪           ⎧ comment  Compute choice set C_level+1
     ⎪           ⎪ ComputeChoiceSet(level + 1)
     ⎪   then    ⎨ result ← Bactrack(level + 1)
     ⎪           ⎪ if result = true
     ⎩           ⎩   then return ( true )
10 :
return ( false )
```

this property has more blocks than the one constructed above. In the next sections, knowing the actual number of blocks in this construction will be useful so we list these in the following theorem. Since we know the actual blocks of $S(6)$, the proof is relatively straight forward.

Theorem 3.4. *Let* $|V| = 6s + r$*, where* $0 \leq r < 6$*. Then the number of blocks in Construction 3.3 is* $10s^3, 10s^3 + 5s^2, 10s^3 + 10s^2 + 2s, 10s^3 + 15s^2 + 6s + 1, 10s^3 + 20s^2 + 12s + 2, 10s^3 + 25s^2 + 20s + 5$ *for* $r = 0, 1, 2, 3, 4, 5$ *respectively.*

When $|V|$ is large enough and any 4 points are allowed to span 0,1 or 2 triples, Construction 3.3 can be improved by adding more blocks.

Construction 3.5. *When* $|V_i| \geq 3$*,* $\sum_{i=1}^{6} ex(|V_i|, H)$ *blocks of the form* (i, j, k) *where* $i, j, k \in V_i$ *can be added to the blocks of Construction 3.3 to form an* H*-free hypergraph.*

Applying Corollary 3.2 and Construction 3.3, it is easy to see show that $44 \leq \text{ex}(10, H) \leq 45$, $60 \leq \text{ex}(11, H) \leq 61$, and $80 \leq \text{ex}(12, H) \leq 81$.

Now, if we can show that $ex(10,H) = 44$, then Corollary 3.2 implies that ex(11,H) = 60 and ex(12,H) = 80.

So we now discuss how we go about showing that $ex(10,H) \neq 45$. We begin by supposing that $ex(10,H) = 45$ and try to look for such a design. The average frequency of the elements in such a design is $45 \times 3/10 = 13.5$. In such a hypergraph, no element can appear in less than 13 of the 45 blocks as ex(9,H) = 32. That is, each element must have frequency at least 13 and so one element must have frequency exactly 13. Hence to look for a design on 45 blocks, it is enough to enumerate all the non-isomorphic designs on 9 points with 32 blocks (these six designs are listed in the appendix) and try to extend each one to 45 blocks, with each of the 13 blocks being added containing a "new" element 10, which does not occur in the starting 32 blocks. Note that once we fix the first 32 blocks, isomorphism rejection is not applied in the search algorithm. With only 13 blocks to extend, it was easy for our search program to show such a design on 45 blocks does not exist. We can now state the following theorem.

Theorem 3.6. *$ex(10,H) = 44$, $ex(11,H) = 60$ and $ex(12,H) = 80$.*

4 ex(13,H)

Using Corollary 3.2, we know that ex(13,H) ≤ 104. If ex(13,H) = 104 then it must be a simple (13,3,4) BIBD. But we will show that this cannot be the case in the following theorem.

Theorem 4.1. *$101 \leq \mathrm{ex}(13,H) \leq 103$.*

Proof. If one applies Construction 3.5 for 13 points one gets 101 blocks, including the one block consisting of elements from the same part. So ex(13,H) ≥ 101.

Now let us assume that there is an H-free hypergraph on 13 points with 104 blocks. By Corollary 3.2, no element can have frequency 23 so every element has frequency 24. Note that the fact that ex(10,H) = 60 implies that no pair of elements can have frequency 5. So every pair of elements have frequency 4. This shows that the hypergraph is a (13,3,4) BIBD. Using a computer search, we found that there are only 11 non-isomorphic ways that the blocks, containing a particular element, could occur. We will call the element that occurs in the 24 blocks, 13. We will list the blocks vertically without putting in element 13. We will represent 10 by a, 11 by b, 12 by c and 13 by d.

Consider the 4 blocks containing the pair dx where x is one of 1,2,$\ldots$, c. The 4 elements that occur with dx will be called *linked(x) elements*. Let y be a linked(x) element. The 3 other elements that occur in the triples containing dy are called *extra elements* of the linked(x) element. As in the proof of Theorem 2.1, these elements will prove useful. We have three claims that we need to prove.

The first claim is that extra elements coming from 2 linked(x) elements can not intersect in 3 elements. If they did, then we have triples dxy_1, dxy_2, dxy_3, dxy_4, dy_1m, dy_1n, dy_1p, dy_2m, dy_2n, dy_2p, then there are 3 more triples containing the pair xy_1. Because these triples are H-free, those third elements in those triples can not be from $d, x, y_1, y_2, y_3, y_4, , m, n, p$. So the third elements must be 3 elements from a 4-element set. But this is also true for the third elements in the triple containing xy_2. The 4-element sets are equal. So we must have triples xy_1r, xy_2s, xy_2r, xy_2s. But now we do not have enough elements to be third elements in the triples containing y_1y_2. So this situation can not happen. This eliminates the first 4 cases.

The second claim is that extra elements from 3 linked(x) elements, say y_1, y_2, y_3, can not all contain two elements, say m and n. If they did, then again we would be forced to have a situation where we have triples $xy_ir, xy_is, xy_jr, xy_js$, where i and j come from 1,2 or 3. So this can not happen. This eliminates 2 cases

The third claim is that if extra elements from 2 linked(x) elements, say y_1, y_2, both contain two elements, say m and n, then some triples will be forced into the design. To see this consider the third elements that occur with triples containing the pair xy_i where i is from 1 or 2. For each y_i, there is a choice of 4 elements with 3 of the choices being the same for each y_i and one choice being different. Let the different choices be u for y_1 and w for y_2. To ensure that we do not get the previous situations, we must have triple xy_1u and xy_2w. This is useful in eliminating 4 cases.

That leaves only one possibility, up to isomorphism for any element. Using this fact and a computer search, this last case was also ruled out. The details of the proof and the 11 cases are listed in the appendix.

Since all cases lead to a contradiction then ex(13,H) ≤ 103. ∎

Iteratively using Corollary 3.2, with this new upper bound for ex(13,H), new upper bounds on ex(n, H) for larger n, can be obtained. Also, using Construction 3.3 and Construction 3.5, lower bounds for ex(n, H) can also be given. We record these facts in Table 1.

The upper bounds from $n = 18$ to $n = 24$ will be used in the next section to improve the asymptotic upper bound.

5 Improving Mubayi's Bound

In 2003, Mubayi [8] showed that $\pi(H(4, 3)) \leq 1/3 - .45305 \times 10^{-5}$. We review Mubayi's method and give an improvement. Even though Talbot [10] has the best current upper bound of $\pi(H(4, 3)) < .32975$, we think our techniques are short and interesting enough to be included.

Table 1: Lower and Upper Bounds for $\mathrm{ex}(n,H)$, $n = 4, 5, \ldots, 24$

n	$\mathrm{ex}(n,H)$	n	$\mathrm{ex}(n,H)$	n	$\mathrm{ex}(n,H)$
4	2	5	5	6	10
7	15	8	22	9	32
10	44	11	60	12	80
13	101–103	14	126–131	15	156–163
16	190–200	17	230–242	18	276–290
19	322–344	20	374–404	21	433–471
22	498–545	23	571–626	24	652–715

Suppose that $(X, \mathcal{B})$ is a H-free hypergraph in which $|X| = n$ and $|\mathcal{B}| = \alpha\binom{n}{3}$. A 4-set spans a triple if all 3 of the points of the triple are in the 4-set. Let q_1 denote the number of 4-subsets of X that span exactly 1 triple of $\mathcal{B}$. Mubayi proved the following by simple counting:

$$q_1 \leq \binom{n}{3}(\alpha n - 3\alpha^2(n-2)). \tag{1}$$

Let $m \leq n$ and suppose that there are $\delta\binom{n}{m}$ m-subsets of X that span more than $10(m/6)^3 = 5m^3/108$ triples in $\mathcal{B}$. By a result of Frankl and Füredi [3], any such m-subset contains at least one 4-subset that spans exactly one triple in $\mathcal{B}$. Therefore the following holds:

$$q_1 \geq \frac{\delta\binom{n}{m}}{\binom{n-4}{m-4}} = \frac{\delta\binom{n}{4}}{\binom{m}{4}}. \tag{2}$$

Combining (1) and (2) and simplifying, we get the following:

$$\delta \leq \frac{4\binom{m}{4}(\alpha n - 3\alpha^2(n-2))}{n-3}. \tag{3}$$

The following equation results from counting pairs of the form (B, Y), where $B \in \mathcal{B}$, $B \subseteq Y \subseteq X$, $|Y| = m$, and using the fact that $\mathrm{ex}(m, H) \leq m^2(m-1)/18$:

$$\alpha\binom{n}{3}\binom{n-3}{m-3} \leq \delta\binom{n}{m}\frac{m^2(m-1)}{18} + (1-\delta)\binom{n}{m}\frac{5m^3}{108}. \tag{4}$$

Rearranging and simplifying (4), we get:

$$\delta \geq \frac{18\alpha(m-1)(m-2)}{m(m-6)} - \frac{5m}{m-6}. \tag{5}$$

Combining (3) and (5) we get:

$$\frac{18\alpha(m-1)(m-2)}{m(m-6)} - \frac{5m}{m-6} \leq \frac{4\binom{m}{4}(\alpha n - 3\alpha^2(n-2))}{n-3}. \qquad (6)$$

Then, letting $n \to \infty$, we see that $\pi(H(4,3)) \leq r$, where r is the largest root of the quadratic equation

$$\frac{18x(m-1)(m-2)}{m(m-6)} - \frac{5m}{m-6} = 4\binom{m}{4}(x - 3x^2). \qquad (7)$$

Mubayi's bound is obtained by taking $m = 18$. However, since we have better bounds for $\mathrm{ex}(n, H)$ than de Caen's bound and since we know the actual number of triples in Construction 3.3, we can obtain a better bound, as shown in the following theorem.

Theorem 5.1. $\pi(H(4,3)) \leq 1/3 - 1.89820 \times 10^{-5}$.

Proof. Rather than using de Caen's bound for $\mathrm{ex}(n, H)$, we use the best bounds known for $n = 18, \ldots, 24$ that are in Table 1. Also, rather than using $5m^3/108$ for the number of blocks in Construction 3.3, we use the numbers in Theorem 3.4. The best value to use is $m = 20$. Doing this in equations (4), (5), (6) we get $\pi(H(4,3)) \leq 1/3 - 1.89820 \times 10^{-5}$. □

6 Conclusion

We showed that for $n > 6$, $\mathrm{ex}(n, H) < n^2(n-1)/18$. We then determined $\mathrm{ex}(n, H)$ for $n = 9, 10, 11, 12$. Finally, we lowered the upper bound on $\mathrm{ex}(11, H)$ to 103. We used this last result to get upper bounds on $\mathrm{ex}(n, H)$ for $14 \leq n \leq 24$. These results on the small hypergraphs allowed us to prove that $\pi(H(4,3)) \leq 1/3 - 1.89820 \times 10^{-5}$.

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7 Appendix

The 11 non-isomorphic cases of triples containing d is listed below with a brief indication of how they were ruled out.

Case 1:
1111222333444555778899aa
234567867869a69abcbcbcbc
The extra elements of linked(1) elements 2 and 3 intersect in 3 element which is a contradiction.

Case 2:
1111222333444555778899aa
234567867869a6bc91bcbcbc
The extra elements of linked(1) elements 2 and 3 intersect in 3 element which is a contradiction.

Case 3:
1111222333444555677889ab
234567867869a9acbac9cbbc
The extra elements of linked(1) elements 2 and 3 intersect in 3 element which is a contradiction.

Case 4:
1111222333444555678899ab

23456786787ab6abc99cabcc
The extra elements of linked(1) elements 2 and 3 intersect in 3 element which is a contradiction.

Case 5:
1111222333444555667788 9b
23456786ab7ab8ab9c9c9cac
The linked(1) elements 3,4 and 5 each contain extra elements a and b which is a contradiction.

Case 6:
11112223334445556677888b
23456786ab7ab9ab9c9c9bcc
The linked(1) elements 3,4 and 5 each contain extra elements a and b which is a contradiction.

Case 7:
1111222333444555667788 9ab
23456786ab7ab68c99c9abcc
The linked(1) elements 3 and 4 force the triple 137, the linked(3) elements 1 and 6 force the triple 139, and the linked(7) elements 2 and 9 force triple 179. This arrangement of triples is not H-free.

Case 8:
1111222333444555667788 9a
23456786799ac8ababac9cbc
The linked(7) elements 1 and 2 force the triple 279, the linked(9) elements 8 and b force the triple 29b, and the linked(b) elements 9 and c force the triple 79b. This arrangement of triples is not H-free.

Case 9:
1111222333444555667788 9a
23456786ab7ab89a9c9cbcbc
The linked(3) elements 1 and a force the triple 13c, the linked(4) elements 1 and a force the triple 14c, the linked(5) elements 1 and a force the triple 15c, the linked(6) elements 2 and c force triple 16c, and the linked(7) elements 2 and c force the triple 17c. The pair 1c then occurs 5 times which is a contradiction.

Case 10:
11112223334445556677888b
23456786ab7ab9ac9c9c9abc
The linked(1) elements 3 and 4 force the triple 13c, the linked(2) elements

6 and 7 force the triple 237, the linked(9) elements 6 and 7 force the triple 379, the linked(a) elements 3 and 4 force triple 37a, and the linked(b) elements 3 and 4 force the triple 37b. The pair 37 then occurs 5 times which is a contradiction.

Case 11:
11112223334445556677889b
23456786ab7ac89b9c9bacac
All blocks containing a particular element must look like this. Using this fact and a computer search, this case was also ruled out.

The 6 non-isomorphic H-free hypergraphs on 9 points and 32 blocks:

Hypergraph 1
11111111111122222222333333444444
22223344567834556678555666555666
34567878999999787899789789789789

Hypergraph 2
11111111111122222222333333334445557
22223344568345566844456665676678
34567879899987978958977896788989

Hypergraph 3
11111111111122222222333333444445567
22223344568345566844455755676678
34567879899987978968967989788999

Hypergraph 4
1111111111112222222223333333445567
22233445567334455674445567686878
34567687998897968985786989797999

Hypergraph 5
1111111111112222222223333333445567
22233445567334455674445567686888
34567687998897968985786989797999

Hypergraph 6
11111111112222222223333333344444566
22233445773345556744455565566878
34567689898976789978967986879989