ON THE EXISTENCE OF INCOMPLETE DESIGNS OF BLOCK SIZE FOUR HAVING ONE HOLE

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Abstract. The obvious necessary conditions for the existence of a $(v, 4, 1)$ balanced incomplete block design (BIBD) containing as a subdesign a $(w, 4, 1)$-BIBD are $v \equiv 1$ or 4 modulo 12, $w \equiv 1$ or 4 modulo 12, and $v \geq 3w + 1$. More generally, we can consider the existence of pairwise balanced designs on v points, having blocks of size 4, except for one block of size w. Such a design can exist only if $v \geq 3w + 1$; and $v \equiv 1$ or 4 modulo 12 and $w \equiv 1$ or 4 modulo 12, or $v \equiv 7$ or 10 modulo 12 and $w \equiv 7$ or 10 modulo 12. We show that these conditions are sufficient for the existence of such a design.

1. Introduction.

A *pairwise balanced design* (or, PBD) is a pair $(X, \mathcal{A})$, such that X is a set of elements (called *points*) and $\mathcal{A}$ is a set of subsets of X (called *blocks*), such that every unordered pair of points is contained in a unique block of $\mathcal{A}$. If v is a positive integer and K is a set of positive integers, then we say that $(X, \mathcal{A})$ is a $(v; K)$-PBD if $|X| = v$, and $|A| \in K$ for every $A \in \mathcal{A}$. The integer v is called the *order* of the PBD.

Using this notation, we can define $(v, 4, 1)$-BIBD to be a $(v; \{4\})$-PBD. It is of course well-known that a $(v, 4, 1)$-BIBD exists if and only if $v \equiv 1$ or 4 modulo 12.

Let $(X, \mathcal{A})$ be a PBD. If a set of points $Y \subseteq X$ has the property that, for any $A \in \mathcal{A}$, either $|Y \cap A| \leq 1$ or $A \subseteq Y$, then we say that Y is a *subdesign* or *flat* of the PBD. The *order* of the subdesign is $|Y|$. The subdesign Y is *proper* if $Y \neq X$. If Y is a subdesign, then we can delete all blocks $A \subseteq Y$ and replace them by a single block, Y, and the result is a PBD. Also, any block or point of a PBD is itself a subdesign.

The problem of constructing $(v, 4, 1)$-BIBDs containing subdesigns was first studied by Brouwer and Lenz ([7] and [8]) and more recently by Wei and Zhu ([28] and [29]). The obvious necessary conditions for the existence of a $(v, 4, 1)$-BIBD containing a $(w, 4, 1)$-BIBD as a proper subdesign are $v \geq 3w + 1$, $v \equiv 1$ or 4 modulo 12, and $w \equiv 1$ or 4 modulo 12. An almost complete solution to the problem has recently been given by Wei and Zhu. They have proved the following in [28] and [29].

Theorem 1.1. *Suppose* $v \equiv 1$ *or* 4 *modulo 12*, $w \equiv 1$ *or* 4 *modulo 12*, $v > w$, *and* $w \geq 88$, $w \not\equiv 133$. *Then there exists a* $(v, 4, 1)$*-BIBD containing a* $(w, 4, 1)$*-BIBD as a subdesign if and only if* $v \geq 3w + 1$.

In this paper, we shall prove an analogous result for the remaining small values of w not covered by Theorem 1.1, thus completing the spectrum.

If we allow a subdesign of a PBD to be missing (i.e., a hole), we have an incomplete PBD, as follows. An *incomplete* PBD (or IPBD) is a triple $(X, Y, \mathcal{A})$, where X is a set of points, $Y \subseteq X$, and $\mathcal{A}$ is a set of blocks which satisfies the following properties:

1) for any $A \in \mathcal{A}$, $|A \cap Y| \leq 1$, and
2) any two points x, y, not both in Y, occur in a unique block.

Hence, Y is the hole. Note that $(X, Y, \mathcal{A})$ is an IPBD if and only if $(X, \mathcal{A} \cup \{Y\})$ is a PBD. We say that $(X, Y, \mathcal{A})$ is a $(v, w; K)$-IPBD if $|X| = v$, $|Y| = w$, and $|A| \in K$ for every $A \in \mathcal{A}$.

There is a $(v, w; \{4\})$-IPBD whenever the hypotheses of Theorem 1.1 are satisfied. However, existence of a $(v, w; \{4\})$-IPBD does not require that $v \equiv 1$ or 4 modulo 12 and $w \equiv 1$ or 4 modulo 12. The necessary conditions (when $v > w$) are easily seen to be as follows:

1) $v \geq 3w + 1$, and
2) $v \equiv 1$ or 4 modulo 12 and $w \equiv 1$ or 4 modulo 12;
   or $v \equiv 7$ or 10 modulo 12 and $w \equiv 7$ or 10 modulo 12.

An ordered pair (v, w), where $v > w$, which satisfies 1) and 2), is said to be *admissible*.

The existence of $(v, 7; \{4\})$-IPBDs was studied by Brouwer in [4]. He proved the following result.

Theorem 1.2. *For all* $v \equiv 7$ *or* 10 *modulo 12*, $v \geq 22$, *there is a* $(v, 7, \{4\})$*-IPBD.*

A similar result concerning $(v, 10; \{4\})$-IPBDs was proved by Bermond and Bond in [2].

Theorem 1.3. *For all* $v \equiv 7$ *or* 10 *modulo 12*, $v \geq 31$, *there is a* $(v, 10, \{4\})$*-IPBD.*

In this paper, we study the existence of $(v, w; \{4\})$-IPBDs for $w \equiv 7$ or 10 modulo 12, $w \geq 19$. We show that there exists a $(v, w; \{4\})$-IPBD for all admissible ordered pairs (v, w).

Let us also observe that a $(v, w; \{4\})$-IPBD is equivalent to another type of design introduced in [11]. A *parallel class* is a set of blocks that form a partition of the point set. A *partially resolvable partition* PRP 2-$(p, s, v; m)$ can be defined to be a $(v; \{p, s\})$-PBD in which the blocks of size p can be partitioned into m parallel classes. It is not difficult to see that a PRP 2-$(s-1, s, v; m)$ is equivalent to a $(v+m, m; \{s\})$-IPBD. Thus, our results completely determine the spectrum of PRP 2-$(3,4,v;m)$.

Finally, let us remark that the existence of $(v, w; \{3\})$-IPBDs has been determined in [9] and [11].

2. Definitions and results concerning related designs.

We need to define several types of designs. First, we define a useful generalization of a PBD called a group-divisible design. A *group-divisible design* (or GDD), is a triple $(X, \mathcal{G}, \mathcal{A})$, which satisfies the following properties:

1) $\mathcal{G}$ is a partition of X into subsets called *groups*,
2) $\mathcal{A}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point, and
3) every pair of points from distinct groups occurs in a unique block.

The *group-type* (or *type*) of a GDD $(X,\mathcal{G},\mathcal{A})$ is the multiset $\{|G| : G \in \mathcal{G}\}$. We usually use an "exponential" notation to describe group-types: a group-type $1^i 2^j 3^k \ldots$ denotes i occurrences of 1, j occurrences of 2, etc. As with PBDs, we will say that a GDD is a K-GDD if $|A| \in K$ for every $A \in \mathcal{A}$.

We will also use the notation GD$[K, M; x]$ to denote a GDD$(X, \mathcal{G}, \mathcal{A})$ where $|X| = x$, $|G| \in M$ for every $G \in \mathcal{G}$, and $|A| \in K$ for every $A \in \mathcal{A}$.

As our first observation, we note that a $(v, w; \{4\})$-IPBD is equivalent to a $\{4\}$-GDD of type $3^{(v-w)/3}(w-1)^1$.

In this paper, we shall make extensive use of $\{4\}$-GDDs. The following result has been proved by Brouwer, Hanani and Schrijver [6] concerning $\{4\}$-GDDs where every group has the same size.

Theorem 2.1. *Suppose* $u > 1$. *Then, there exists a* $\{4\}$*-GDD of type* t^u *if and only if* $u \geq 4$, $t(u-1) \equiv 0$ *modulo 3,* $t^2u(u-1) \equiv 0$ *modulo 4, and* $(t, u) \neq (2,4)$ *or* $(6,4)$.

A PBD or GDD is *resolvable* if the block set can be partitioned into parallel classes. It is not difficult to see that if a $\{k\}$-GDD is resolvable, then all groups have the same size. We also observe that we can add a "point at infinity" to any parallel class in a design. Hence, it follows that a resolvable $\{k\}$-GDD of type t^u is equivalent to a $\{k+1\}$-GDD of type $t^u r^1$, where $r = t(u-1)/(k-1)$. We call this process *completing* the resolvable design.

The existence of resolvable $\{3\}$-GDDs has recently been studied in several papers. The following summarizes the known results.

Theorem 2.2. *([18], [1]) Suppose g and* u *are positive integers such that* $u \geq 3$, $tu \equiv 0$ *modulo 3,* $t(u-1)$ *is even, and* $(t, u) \neq (2,3), (2,6)$, *or* $(6,3)$. *Then, there exists a resolvable* $\{3\}$*-GDD of type* t^u, *except possibly when* $t \equiv 2$ *or* 10 *modulo 12 and* $u = 6$.

Of course, this result contains as a special case the result that a resolvable $(v,3,1)$-BIBD (i.e., a Kirkman triple system of order v, or KTS(v)) exists if and only if $v \equiv 3$ modulo 6 ([17]). As well, we note that the designs having group-size two are known as nearly Kirkman triple systems.

The authors have also studied the existence of Kirkman triple systems which contain subdesigns which also are Kirkman triple systems (where we require that the parallel classes of the subdesign are induced from the larger design). The following is proved in [20].

Theorem 2.3. *Suppose $v \equiv w \equiv 3$ modulo 6 and $v \geq 3w$. Then there is a Kirkman triple system of order v which contain as a subdesign a Kirkman triple system of order w.*

The existence of resolvable $(v,4,1)$-BIBDs was determined by Hanani, Ray-Chaudhuri and Wilson [10]. They proved the following.

Theorem 2.4. *There exists a resolvable $(v,4,1)$-BIBD if and only if $v \equiv 4$ modulo 12.*

Now, we define the idea of a GDD with a hole. Informally, an *incomplete* GDD, or IGDD, is a GDD from which a sub-GDD is missing (this is the "hole"). We give a formal definition. An IGDD is a quadruple $(X,Y,\mathcal{G},\mathcal{A})$ which satisfies the following properties:

1) X is a set of *points*, and $Y \subseteq X$,
2) $\mathcal{G}$ is a partition of X into *groups*,
3) $\mathcal{A}$ is a set of *blocks*, each of which intersects each group in at most one point,
4) no block contains two members of Y, and
5) every pair of points $\{x,y\}$ from distinct groups, such that at least one of x, y is in $X \backslash Y$, occurs in a unique block of $\mathcal{A}$.

We say that an IGDD $(X,Y,\mathcal{G},\mathcal{A})$ is a K-IGDD if $|A| \in K$ for every block $A \in \mathcal{A}$. The *type* of the IGDD is defined to be the multiset of ordered pairs $\{(|G|, |G \cap Y|) : G \in \mathcal{G}\}$. As with GDDs, we shall use an exponential notation to describe types. Note that if $Y = \phi$, then the IGDD is a GDD.

We have already defined the idea of a PBD having a hole. We also employ a more general type of incomplete PBD. We are interested in the situation when we have two subdesigns, of given sizes, which intersect in a third subdesign of a given size. However, as usual, the subdesigns need not be present, that is, we allow holes. We will refer to these designs as $\diamondsuit$-IPBDs, in order to suggest the structure of the holes. We give a formal definition. A *$\diamondsuit$-IPBD* is a quadruple $(X,Y_1,Y_2,\mathcal{A})$, where $Y_1 \subseteq X$, $Y_2 \subseteq X$, and $\mathcal{A}$ is a set of blocks such that every pair of points $\{x,y\}$ occurs in a unique block, unless $\{x,y\} \subseteq Y_1$ or $\{x,y\} \subseteq Y_2$, in which case the pair occurs in no block. We say that the $\diamondsuit$-IPBD is a $(v; w_1, w_2; w_3; K)$-$\Diamond$-IPBD if $|X| = v$, $|Y_1| = w_1$, $|Y_2| = w_2$, $|Y_1 \cap Y_2| = w_3$, and $|A| \in K$ for every $A \in \mathcal{A}$.

We also utilize (incomplete) transversal designs, which we now define. A *transversal design* TD(k, n) is a $\{k\}$-GDD of type n^k. It is well-known that a TD(k, n) is equivalent to $k - 2$ mutually orthogonal Latin squares (MOLS) of order n. We also define a TD(k, n) - TD(k, m) (an *incomplete* transversal design) to be a $\{k\}$-IGDD of group-type $(n, m)^k$.

We now record the known results concerning TDs with 4, 5, or 6 groups (see [3], [22], [26], and [27]).

Theorem 2.5. *There exists a TD$(4, n)$ if and only if $n \neq 2$ or 6. There exists a TD$(5, n)$ if $n \geq 4$, $n \neq 6, 10$. There exists a TD$(6, n)$ if $n \geq 5$, $n \neq 6$, $10, 14, 18, 22, 26, 28, 30, 34, 38, 42, 44$, or 52.*

We shall make extensive use of incomplete TDs with four groups. The existence of these designs was completely determined by Heinrich and Zhu in [12].

Theorem 2.6. *For all positive integers v and w such that $v \geq 3w$, $(v, w) \neq (6, 1)$, there is a TD$(4, v)$ - TD$(4, w)$.*

Finally, we record the existence of several small GDDs which we shall use in recursive constructions.

Theorem 2.7. *There exist $\{4\}$-GDDs of the following types: $6^4 3^1, 6^1 3^5, 6^2 3^4$, $6^4 9^1, 6^5 9^1, 6^6 3^1, 6^5 12^1, 6^4 12^2, 9^4 6^1, 9^5 6^1, 9^2 3^6, 18^2 3^{12}$, and $6^5 12^1 15^1$. Also, there exists a resolvable $\{4\}$-GDD of type 3^8.*

Proof: The designs of types $6^4 3^1$, $9^4 6^1$, $9^5 6^1$, and $9^2 3^6$ are constructed in [19]. The designs of types $6^6 3^1$ and $6^2 3^4$ are constructed in [21]. The design of type $6^5 9^1$ is constructed in [18]. The designs of types $6^4 12^2$, $18^2 3^{12}$, and $6^5 12^1 15^1$ are constructed in the Appendix. A resolvable $\{4\}$-GDD of type 3^8 is constructed in [13]. The remaining three designs are obtained by completing resolvable $\{3\}$-GDDs. ∎

Theorem 2.8. *There exists a $\{5\}$-IGDD of type $(16, 4)^6$.*

Proof: Start with a $\{5\}$-GDD of type 4^6 (obtained from an affine plane of order 5) and give every point weights $(4, 1)$. Apply the Fundamental Construction (defined in Section 3), using $\{5\}$-IGDDs of type $(4, 1)^5$, which are just TD$(5, 4)$ - TD$(5, 1)$. ∎

Theorem 2.9. *For $k = 5$ and 6, and for all $0 \leq i \leq k$, there exists a $\{4\}$-IGDD of type $(9, 3)^i (6, 0)^{k-i}$.*

Proof: These designs are constructed in [18], [19], and [21]. ∎

3. General constructions for designs containing subdesigns.

It will be necessary for us to build families of IGDDs. Our basic construction for IGDDs is the "Fundamental IGDD Construction" (see [16] and [19]).

Construction 3.1 Fundamental IGDD Construction: Suppose $(X, Y, \mathcal{G}, \mathcal{A})$ is an IGDD, and let t, s: $X \rightarrow \mathbf{Z}^+ \cup \{0\}$ be functions such that $t(x) \leq s(x)$, for every $x \in X$. For every block $A \in \mathcal{A}$, suppose that we have a K-IGDD of type $\{(s(x), t(x)): x \in A\}$. Suppose also that we have a K-IGDD of type $\{(\sum_{x \in G \cap Y} s(x), \sum_{x \in G \cap Y} t(x)): G \in \mathcal{G}\}$. Then there exists a K-IGDD of type $\{(\sum_{x \in G} s(x), \sum_{x \in G} t(x)): G \in \mathcal{G}\}$.

As an immediate corollary, we obtain Wilson's Fundamental GDD construction (see [30]).

Construction 3.2 Fundamental GDD Construction: Suppose $(X, \mathcal{G}, \mathcal{A})$ is a GDD, and let s: $X \rightarrow \mathbf{Z}^+ \cup \{0\}$ be a function. For every block $A \in \mathcal{A}$, suppose that we have a K-GDD of type $\{s(x): x \in A\}$. Then there exists a K-GDD of type $\{\sum_{x \in G} s(x): G \in \mathcal{G}\}$.

We will refer to both Constructions 3.1 and 3.2 by the abbreviation FC. It will be clear from the context which applies.

We defined $\diamondsuit$-IPBDs in Section 2. Our main application of these designs involves using them to fill in the groups of IGDDs. The next construction was presented in [25].

Construction 3.3 Filling in groups: Let K be a set of positive integers, and let $b \geq a \geq 0$. Suppose that the following designs exist:

1) a K-IGDD of type $\{(t_1, u_1), (t_2, u_2), \ldots, (t_n, u_n)\}$;
2) a $(t_i + b; u_i + a, b; a; K)$-$\diamondsuit$-IPBD, for $1 \leq i \leq n - 1$; and
3) a $(t_n + b, u_n + a; K)$-IPBD.

Then there exists a $(t + b, u + a; K)$-IPBD, where $t = \sum t_i$ and $u = \sum u_i$.

As a simpler form of filling in groups, we have the following corollary (see, for example, [16]).

Construction 3.4 Filling in groups: Let K be a set of positive integers, and let $a \geq 0$. Suppose that the following designs exist:

1) a K-IGDD of type $\{(t_1, u_1), (t_2, u_2), \ldots, (t_n, u_n)\}$; and
2) a $(t_i + a; u_i + a; K)$-IPBD, for $1 \leq i \leq n$.

Then there exists a $(t + a, u + a; K)$-IPBD, where $t = \sum t_i$ and $u = \sum u_i$.

Finally, we mention the special case when we start with a GDD (see, for example, [30]).

Construction 3.5 Filling in groups: Let K be a set of positive integers, and let $a \geq 0$. Suppose that the following designs exist:

1) a K-GDD of type $\{t_1, t_2, \ldots, t_n\}$; and
2) a $(t_i + a; a; K)$-IPBD, for $1 \leq i \leq n - 1$.

Then there exists a $(t + a, t_n + a; K)$-IPBD, where $t = \sum t_i$.

We refer to any of Constructions 3.3, 3.4, or 3.5 as Filling in Groups. Again, it will be clear from the context which one applies. We will say that the new points have been *adjoined* to the groups of the IGDD or GDD.

We now mention several constructions which are known as the product constructions. The most general form is the following, first stated in [25].

Construction 3.6: (Generalized singular indirect product, or GSIP) Suppose there exists a $TD(4, r-b)$ - $TD(4, s-a)$, an $(r; s, b; a; \{4\})$-◇-IPBD, and a $(b, a; \{4\})$-IPBD. Then there is a $(4(r-b)+b, 4(s-a)+a; \{4\})$-IPBD.

The following special case of GSIP was first stated in [16].

Construction 3.7: (Singular indirect product, or SIP) Suppose there exists a $TD(4, r-a)$ - $TD(4, s-a)$, and an $(r, s; \{4\})$-IPBD. Then there is a $(4(r-a)+a, 4(s-a)+a; \{4\})$-IPBD.

A further specialization of SIP is as follows.

Construction 3.8: (Singular direct product, or SDP) Suppose there is a $TD(4, r-s)$ and an $(r, s; \{4\})$-IPBD. Then there is a $(4(r-s)+s, s; \{4\})$-IPBD and a $(4(r-s)+s, r; \{4\})$-IPBD.

Now, we present several specific constructions for designs with block-size four, by applying the recursive constructions described above. All GDDs required as ingredients have been shown to exist in Section 2.

Lemma 3.9. *Suppose there is a $(v, w; \{k: k \equiv 0$ or 1 modulo $4\})$-IPBD. Then there exists a $(3v+1, 3w+1; \{4\})$-IPBD.*

Proof: Give every point weight 3, and apply FC. We require $\{4\}$-GDDs of types 3^k, for all relevant $k \equiv 0$ or 1 modulo 4. Then adjoin $a = 1$ new point, filling in $(4; \{4\})$-PBDs. ∎

Lemma 3.10. *Suppose there is a $TD(5, m)$, and $0 \leq u \leq m$. Then there is a $\{4\}$-GDD of type $(3m)^4(3u)^1$.*

Proof: Give points in four groups of the TD weight 3, give the points in the fifth group weights 0 or 3. Apply FC, filling in $\{4\}$-GDDs of types 3^4 or 3^5. ∎

Lemma 3.11. *Suppose there is a $TD(6, m)$, and $m \leq u \leq 2m$. Then there is a $\{4\}$-GDD of type $(3m)^4(6m)^1(3u)^1$.*

Proof: Give points in four groups of the TD weight 3, give the points in the fifth group weights 3 or 6, and give the points in the sixth group weight 6. Apply FC, filling in $\{4\}$-GDDs of types $3^4 6^2$ or $3^5 6^1$. ∎

Lemma 3.12. *Suppose there is a $TD(6, m)$, and $0 \leq u \leq m$. Then there is a $\{4\}$-GDD of type $(3m)^5(6u)^1$.*

Proof: Give points in five groups of the TD weight 3, and give the points in the sixth group weights 0 or 6. Apply FC, filling in $\{4\}$-GDDs of types 3^5 or $3^5 6^1$. ∎

Lemma 3.13. *Suppose there is a* $TD(6, m)$, *and* $m \le u' \le 2m$, *and* $m \le u \le 2m$. *Then there is a* $\{4\}$*-GDD of type* $(6m)^4 (6u)^1 (6u')^1$.

Proof: Give points in four groups of the TD weight 6, and give the points in the fifth and sixth groups weights 6 or 12. Apply FC, filling in $\{4\}$-GDDs of types $6^6, 6^5 12^1$, or $6^4 12^2$. ∎

4. Designs with large holes.

In this section, we give several constructions for $(v, w; \{4\})$-IPBDs when $3w < v < 4w$. These constructions are generalizations and/or modifications of constructions used by Brouwer and Lenz, and by Wei and Zhu.

Lemma 4.1. *Suppose there is a resolvable* $\{4\}$*-GDD of type* t^u, *and* $0 \le s \le t(u-1)/3$. *Let* $b \ge a \ge 0$. *Suppose there is a* $(3t+b; t+a, b; a; \{4\})$*-◇-IPBD, and a* $(3s+b, a; \{4\})$*-IPBD. Then there is a* $(3tu+3s+b, tu+a; \{4\})$*-IPBD.*

Proof: Adjoin infinite points to s of the parallel classes of the GDD. This produces a $\{4,5\}$-GDD of type $t^u s^1$ in which every block of size 5 hits the group of size s. Assign weights $(3,1)$ to every point of the original GDD, and assign weights $(3,0)$ to the s infinite points. Apply FC, using $\{4\}$-IGDDs of types $(3,1)^4 (3,0)^1$ and $(3,1)^4$. (These arise by deleting a block from $\{4\}$-GDDs of types 3^5 and 3^4, respectively.) This yields a $\{4\}$-IGDD of type $(3t,t)^u (3s)^1$. Then, we fill in the groups of this IGDD with ◇-IPBDs. This gives us the desired IPBD. ∎

Lemma 4.2. *(Brouwer and Lenz [7], Wei and Zhu [28]). Suppose* $w \equiv 4$ *modulo 12,* $v \equiv 1$ *or* 4 *modulo 12, and* $3w+1 \le v \le 4w-3$. *Then there is a* $(v, w; \{4\})$*-IPBD and a* $(v, v-3w; \{4\})$*-IPBD.*

Proof: Set $t = 4, u = w/4, s = (v-3w-1)/3, b = 1$, and $a = 0$. Apply Lemma 4.1. There is a resolvable $\{4\}$-GDD of type 4^u for all $u \equiv 1$ modulo 3 (Theorem 2.4). We fill in $(13; 4, 1; 0; \{4\})$-◇-IPBDs, which are just $(13, 4; \{4\})$-IPBDs; and a $(3s+1, 0; \{4\})$-IPBD, which is just a $(v-3w; \{4\})$-PBD. ∎

Lemma 4.3. *For all* $t \equiv 0$ *modulo 3, there is a* $(3t+4; t+1, 4; 1; \{4\})$*-◇-IPBD.*

Proof: This design is constructed by adjoining $t+1$ points at infinity to the parallel classes of a resolvable $(2t+3, 3, 1)$-BIBD (Theorem 2.2). A block of size four is then deleted, giving the desired ◇-IPBD. ∎

Lemma 4.4. *Suppose* $w \equiv 1$ *modulo 12,* $w \not\equiv 13$ *or* 25, $v \equiv 1$ *or* 4 *modulo 12, and* $3w + 1 \le v \le (15w + 1)/4$. *Then there is a* $(v, w; \{4\})$*-IPBD.*

Proof: Set $u = 4, t = (w - 1)/4$, $s = (v - 3w - 1)/3$, $b = 4$, and $a = 1$. Apply Lemma 4.1. The necessary ◇-IPBDs are constructed in Lemma 4.3. ∎

Lemma 4.5. *For all* $t \equiv 0$ *modulo 3,* $t \ge 3$, *there is a* $(4t + 1; t + 1, t + 1; 1; \{4\})$*-◇-IPBD.*

Proof: The desired designs are equivalent to $\{4\}$-GDDs of type $t^2 3^{2t/3}$. For $t \equiv 0$ or 3 modulo 12, start with a TD$(4, t)$, and replace each of two groups by $\{4\}$-GDDs of type $3^{t/3}$. For $t = 6, 9$, and 18, the designs are given in Theorem 2.7. For $t \equiv 6$ or 9 modulo 12, $t \ge 21$, we proceed as follows. Start with a TD$(4, t)$ - TD$(4, 6)$. Replace each of two groups by $\{4\}$-GDDs of type $6^1 3^{(t-6)/3}$ (such a GDD is obtained by deleting a point from a $(t + 1, 7; \{4\})$-IPBD). Now, we have a $\{4\}$-IGDD of type $(t, 6)^2\ (3, 0)^{(2t-12)/3}\ (6, 6)^2$. Fill in the hole with a $\{4\}$-GDD of type $6^2 3^4$. This produces the required GDD. ∎

Remark: For $t \not\equiv 6, 9, 18, 21$, and 33, similar designs were constructed by Wei and Zhu in [29].

Lemma 4.6. *Suppose* $w \equiv 1$ *modulo 12,* $w \not\equiv 13$ *or* 25, $v \equiv 1$ *or* 4 *modulo 12, and* $(13w - 9)/4 \le v \le 4w - 3$. *Then there is a* $(v, w; \{4\})$*-IPBD.*

Proof: Set $u = 4, t = (w - 1)/4, s = (4v - 13w + 9)/12$, $b = t + 1$, and $a = 1$. Apply Lemma 4.1. The necessary ◇-IPBDs are constructed by Lemma 4.5. ∎

Combining Lemmata 4.4 and 4.6, we have

Lemma 4.7. *Suppose* $w \equiv 1$ *modulo 12,* $w \not\equiv 13$ *or* 25, $v \equiv 1$ *or* 4 *modulo 12, and* $3w + 1 \le v \le 4w - 3$. *Then there is a* $(v, w; \{4\})$*-IPBD.*

We derive two more corollaries to Lemma 4.1.

Lemma 4.8. *Suppose* $w \equiv 7$ *modulo 12,* $w \ge 67$, $v \equiv 7$ *or* 10 *modulo 12, and* $3w + 1 \le v \le (15w - 17)/4$. *Then there is a* $(v, w; \{4\})$*-IPBD.*

Proof: Set $u = 4, t = (w - 7)/4$, $s = (v - 3w - 1)/3$, $a = 7$, and $b = 22$. The required ◇-IPBDs can be constructed from a Kirkman triple system of order $(w + 23)/2$ which contains as a subdesign a Kirkman triple system of order 15 (Theorem 2.3). ∎

Lemma 4.9. *Suppose* $w \equiv 10$ *modulo 12,* $w \ge 94$, $v \equiv 7$ *or 10 modulo 12, and* $3w + 1 \le v \le (15w - 26)/4$. *Then there is a* $(v, w; \{4\})$*-IPBD.*

Proof: Set $u = 4, t = (w - 10)/4$, $s = (v - 3w - 1)/3$, $a = 10$, and $b = 31$. The required ◇-IPBDs can be constructed from a Kirkman triple system of order $(w + 32)/2$ which contains as a subdesign a Kirkman triple system of order 21 (Theorem 2.3). ∎

Finally, we note the following.

Lemma 4.10. *For all* $w \equiv 1$ *modulo 3, there exists a* $(3w+1, w; \{4\})$*-IPBD.*

Proof: Adjoin w infinite points to a resolvable $(2w+1, \{3\})$-PBD (which is just a Kirkman triple system of order $2w+1$). ∎

5. A general construction when $v - w \equiv 3$ or 9 modulo 12.

In this section, we prove an analogue of Lemma 3.2 in [21]. This is a general construction that applies when v and w are of opposite parity.

Lemma 5.1. *Suppose* $(X, \mathcal{G}, \mathcal{A})$ *is a* GD[$\{5,6\}, \{2,3,4,5,6,7,8,9,10,11,15,r^*\}; s$], *where* $r \geq 1$, *having more than one group. Let* O_g *denote the number of groups having odd size, and suppose* $3O_g \leq u \leq 3s$, $u \equiv 3O_g$ *modulo 6. Then there is a* GD[$4, \{3, u^*\}, 6s + u + 3$].

Proof: Let G' be the group of size r. Define $d: X \rightarrow \{0,3\}$ such that the following conditions are satisfied:

1) for every group G, $|\{x \in G: d(x) = 3\}| \equiv |G| \bmod 2$,
2) $\sum_{x \in X} d(x) = u$, and
3) $\sum_{x \in G'} d(x) = 0, 3$, or $3r$.

This can always be done, as follows. Suppose r is even (the argument is similar if r is odd). We can define d satisfying 1) so that $\sum_{x \in X \setminus G'} d(x)$ takes on any value $\equiv 3O_g$ modulo 6 between $3O_g$ and $3(s-r)$. Also, $\sum_{x \in G'} d(x)$ can take on the value 0 or $3r$. Hence, we can attain all desired values u provided $3O_g + 3r \leq 3(s-r) + 6$, or equivalently, $O_g + 2r \leq s + 2$. Now, no group has size 1 (since r is even), so we have $O_g \leq (s-r)/3$. Also, every block has size at least 5, so we have $s \geq 4r + 1$. These two inequalities imply that $O_g + 2r \leq (9s-5)/12 < s + 2$, as desired.

Next, define $w(x) = 6 + d(x)$ for every $x \in X$, and apply the Fundamental IGDD Construction (Construction 3.1). For each block $A \in \mathcal{A}$, we require an IGDD of type $(9,3)^i(6,0)^{|A|-i}$, for some i, $0 \leq i \leq |A|$. Since $|A| = 5$ or 6 for every $A \in \mathcal{A}$, the required IGDDs exist by Theorem 2.9. We now fill in the groups of this large IGDD with the $\{4\}$-GDDs listed in Table 1 (incorporating $a = 3$ new points). These GDDs have the form $3^u t^1$. When $t = 3$, the GDDs exist by Theorem 2.1. When $t = 6$ or 9, the GDDs are constructed by deleting points from $(v, 7; \{4\})$-IPBDs or from $(v, 10; \{4\})$-IPBDs, which exist by Theorem 1.2 and Theorem 1.3. When $t = (3u-3)/2$, the desired $\{4\}$-GDD is obtained by adjoining t infinite points to a Kirkman triple system of order $3u$, as indicated in Table 1. The remaining GDDs are constructed later in this paper or elsewhere; we give references in Table 1. ∎

Table 1

$\lvert G\rvert$	$\lvert\{x \in G: d(x) = 3\}\rvert$	{4}-GDD	$\lvert G\rvert$	$\lvert\{x \in G: d(x) = 3\}\rvert$	{4}-GDD
2	0	3^5	2	2	$3^5 6^1$ (KTS(15))
3	1	3^8	3	3	$3^7 9^1$ (KTS(21))
4	0	3^9	4	2	$3^9 6^1$
4	4	$3^9 12^1$ (KTS(27))	5	1	3^{12}
5	3	$3^{11} 9^1$	5	5	$3^{11} 15^1$ (KTS(33))
6	0	3^{13}	6	2	$3^{13} 6^1$
6	4	$3^{13} 12^1$ (Thm. 6.1)	6	6	$3^{13} 18^1$ (KTS(39))
7	1	3^{16}	7	3	$3^{15} 9^1$
7	5	$3^{15} 15^1$ (Thm. 6.1)	7	7	$3^{15} 21^1$ (KTS(45))
8	0	3^{17}	8	2	$3^{17} 6^1$
8	4	$3^{17} 12^1$ (Table 2)	8	6	$3^{17} 18^1$ [15]
8	8	$3^{17} 24^1$ (KTS(51))	9	1	3^{20}
9	3	$3^{19} 9^1$	9	5	$3^{19} 15^1$ (Table 2)
9	7	$3^{19} 21^1$ [14]	9	9	$3^{19} 27^1$ (KTS(57))
10	0	3^{21}	10	2	$3^{21} 6^1$
10	4	$3^{21} 12^1$ (Table 2)	10	6	$3^{21} 18^1$ [15]
10	8	$3^{21} 24^1$ (Thm. 6.1)	10	10	$3^{21} 30^1$ (KTS(63))
11	1	3^{24}	11	3	$3^{23} 9^1$
11	5	$3^{23} 15^1$ (Table 2)	11	7	$3^{23} 21^1$ [14]
11	9	$3^{23} 27^1$ (Lem. 4.2)	11	11	$3^{23} 33^1$ (KTS(69))
15	1	3^{32}	15	3	$3^{31} 9^1$
15	5	$3^{31} 15^1$ (Table 2)	15	7	$3^{31} 21^1$ [14]
15	9	$3^{31} 27^1$ (Thm. 6.1)	15	11	$3^{31} 33^1$ (Table 5)
15	13	$3^{31} 39^1$ (Lem. 4.2)	15	15	$3^{31} 45^1$ (KTS(93))
r	r	$3^{2r+1}(3r)^1$ (KTS(6r + 3))			
r (even)	0	3^{2r+1}			
r (odd)	1	3^{2r+2}			

Lemma 5.2. *Suppose* $(X, \mathcal{G}, \mathcal{A})$ *is a* $GD[\{5,6\}, \{2,3,4,5,6,7,8,9,10, 11,15,r^\star\}; s]$, *where* $r \geq 1$, *having more than one group. Let* O_g *denote the number of groups having odd size, and suppose* $4\,O_g + 1 \leq s$. *Finally, suppose that* $v - w = 6s + 3$, $3w + 1 \leq v \leq 9w + 4$, *and* (v, w) *is admissible. Then there is a* $(v, w, \{4\})$*-IPBD.*

Proof: Since $v \leq 9w + 4$, we have $v - w \leq 8w + 4$. But $v - w = 6s + 3 \geq 6(4\,O_g + 1) + 3 = 24\,O_g + 9$. Hence, $w \geq (24\,O_g + 9 - 4)/8 > 3\,O_g$. Furthermore, since (v, w) is admissible, it follows that $w - 1 \equiv 3\,O_g$ modulo 6. Hence, Lemma 5.1 can be applied. ∎

Theorem 5.3. *Suppose that* $v - w = 6s + 3$, $3w + 1 \leq v \leq 9w + 4$, *and* (v, w) *is admissible. If* $s \geq 24$, $s \neq 31, 32, 33, 34$, *then there is a* $(v, w, \{4\})$*-IPBD.*

Proof: It suffices to show that there exist GDDs which satisfy the hypotheses of Lemma 5.2 for the stated values of s. For $s \in \{24, 25, 28, 29, 30, 36, 40, 44, 45,$

52, 59, 60, 63, 64, 65} ∪ {n: 68 ≤ n ≤ 80} ∪ {n ≥ 88}, the GDDs constructed in [21, Theorem 3.4] satisfy the requirements. The remaining values of s are handled as follows.

$s = 26, 27$	adjoin a group of size 1 or 2 to an affine plane of order 5
$35 \leq s \leq 42$	delete $42 - s$ points from a group of a TD(6, 7)
$s = 43$	delete 5 points from a group of a TD(6, 8)
$45 \leq s \leq 54$	delete $54 - s$ points from a group of a TD(6, 9)
$55 \leq s \leq 66$	delete $66 - s$ points from a group of a TD(6, 11)
$s = 67$	adjoin a group of size two to a resolvable (65, 5, 1)-BIBD
$81 \leq s \leq 87$	delete $90 - s$ points from a group of a TD(6, 15).

It is easy to check that $4\,O_g + 1 \leq s$ in each case. ∎

We can do most of the cases corresponding to $s = 31$, as follows.

Lemma 5.4. *Suppose that $v - w = 189$, $28 \leq w \leq 94$, and (v, w) is admissible. Then there is a $(v, w, \{4\})$-IPBD.*

Proof: Apply Lemma 5.1 with $s = 31$, using a $\{5\}$-GDD of type $3^8 7^1$ (this GDD is constructed by completing the resolvable $\{4\}$-GDD of type 3^8 constructed in Theorem 2.7). Adjoin one new point to the groups of the resulting GDD. ∎

6. BIBDs with subdesigns: completing the spectrum.

The following result, proved by Wei and Zhu in [28], will be useful.

Theorem 6.1. *Suppose $w \equiv 1$ or 4 modulo 12, $v \equiv 1$ or 4 modulo 12, and $4w - 12 \leq v \leq 5w - 4$. Then there is a $(v, w; \{4\})$-IPBD.*

Recall that there always exists a $(3w + 1, w, \{4\})$-IPBD (Lemma 4.10). After applying Theorems 5.3 and 6.1, and Lemmata 4.2, 4.7 and 5.4, the only cases remaining when $v - w$ is odd and $v \leq 9w + 4$ are listed in Table 2, together with constructions in each case.

Table 2

w	v	Construction
13	64	Lemma 3.9, delete 4 points from a group of a TD(5, 5), $21 = 4 \cdot 5 + 1$
13	76	Lemma 3.9, (25, 4, 1)-BIBD
13	88	Lemma 3.9, adjoin a point at infinity to a resolvable (28, 4, 1)-BIBD.
13	100	Lemma 3.9, delete 7 points from a group of a TD(5, 8) ($33 = 4 \cdot 8 + 1$)
13	112	Lemma 3.9, delete 8 points from a group of a TD(5, 9) ($37 = 4 \cdot 9 + 1$)
16	85	Lemma 3.9, adjoin infinite points to the groups and to 3 parallel classes of a resolvable {4}-GDD of type 3^8, constructing a {4, 5}-GDD of type 4^7

Table 2 (continued)

16	97	Lemma 3.9, adjoin 4 infinite points to a resolvable (28, 4, 1)-BIBD
16	109	Lemma 3.9, delete 4 points from a group of a TD(5, 8) (36 = 4·8 + 4)
16	121	Lemma 3.9, TD(5, 8) (40 = 5·8)
16	133	Lemma 3.9, delete 1 point from a group of a TD(5, 9) (44 = 4·9 + 8)
16	145	Lemma 3.9, adjoin 8 infinite points to a resolvable (40, 4, 1)-BIBD
25	124	Lemma 3.9, delete 4 points from a *block* of a TD(5, 9) (41 = 4·8 + 9)
25	136	start with a {5}-GDD of type 4^6, and give every point weight 6, except for three points in one group, which get weight 3. Apply FC, using {4}-GDDs of types 6^5 and $6^4 3^1$. This yields a {4}-GDD of type $24^5 15^1$. Fill in the groups with a = 1 new point.
25	148	start with a {5}-GDD of type 4^6, and give every point weight 6, except for one point, which gets weight 9. Apply FC, using {4}-GDDs of types 6^5 and $6^4 9^1$. This yields a {4}-GDD of type $24^5 27^1$. Fill in the groups with a = 1 new point.
25	160	apply Lemma 3.12 with m = 9, u = 4, producing a {4}-GDD of type $27^5 24^1$. Fill in the groups with a = 1 new point.
25	220	Lemma 3.9, delete 8 points from a *block* of a TD(9, 9) (73 = 8·8 + 9)
28	145	delete 4 points from a block of a TD(5, 4), giving rise to a {4, 5}-GDD of type $3^4 4^1$. Then give every point weight 9, and apply FC, to get a {4}-GDD of type $27^4 36^1$. Fill in the groups with a = 1 new point.
28	157	there is a {4}-GDD of type $6^6 3^1$, which gives rise to a {4, 7}-IGDD of type $(3, 1)^7 (3, 0)^6$. Give every point weight 4 and apply FC, obtaining a {4}-IGDD of type $(12, 4)^7 (12, 0)^6$. Fill in the groups with one new point (a = 0, b = 1), to get a (157, 28; {4})-IPBD.
28	169	apply GSIP with the equations 169 = 4(43 − 1) + 1, 28 = 4(7 − 0) + 0.
28	229	Lemma 3.9, delete 5 points from a group of a TD(9, 9) (76 = 8·9 + 4)
37	232	if we delete points from the hole of a {5}-IGDD of type $(16, 4)^6$ (Theorem 2.8), we can construct a {4, 5}-IGDD of type $(16, 4)^1 (13, 1)^1 (12, 0)^4$, and hence a (77, 12; {4, 5, 12})-IPBD. Now apply Lemma 3.9.
37	244	if we delete points from the hole of a {5}-IGDD of type $(16, 4)^6$ (Theorem 2.8), we can construct a {4, 5}-IGDD of type $(16, 4)^2 (13, 1)^1 (12, 0)^3$, and hence an (81, 12; {4, 5, 9, 12})-IPBD. Now apply Lemma 3.9.
40	241	if we delete points from the hole of a {5}-IGDD of type $(16, 4)^6$ (Theorem 2.8), we can construct a {4, 5}-IGDD of type $(16, 4)^1 (13, 1)^4 (12, 0)^1$, and hence an (80, 13; {4, 5, 8, 12})-IPBD. Now apply Lemma 3.9.
49	244	adjoin one infinite point to an affine plane of order 4, giving rise to a {4, 5}-GDD of type $4^4 1^1$. Give all original points weight (5, 1) and give the new point weight (1, 0), and apply FC, using {4, 5}-IGDDs of types $(5, 1)^4$ and $(5, 1)^4 (1, 0)^1$. This produces a {4, 5}-IGDD of type $(20, 4)^4 (1, 0)^1$, from which we obtain an (81, 16; {4, 5})-IPBD. Now apply Lemma 3.9.
49	256	if we delete points from the hole of a {5}-IGDD of type $(16, 4)^6$ (Theorem 2.8), we can construct a {4, 5}-IGDD of type $(16, 4)^3 (13, 1)^1 (12, 0)^2$, and hence an (85, 16; {4, 5, 12})-IPBD. Now apply Lemma 3.9.

So, to this point, we have proved the following.

Theorem 6.2. *If* $v \equiv 1$ *or* 4 *modulo 12,* $w \equiv 1$ *or* 4 *modulo 12,* $v - w$ *is odd and* $3w + 1 \leq v \leq 9w + 4$*, then there exists a* $(v, w; \{4\})$*-IPBD.*

Next, we consider when $v - w$ is even. Brouwer and Lenz have proven several results concerning these cases in [7] and [8]. However, they do not deal with small values of w. Here, we give a complete proof for all values of w, using very similar techniques.

Lemma 6.3. *Suppose* $(X, \mathcal{G}, \mathcal{A})$ *is a GDD in which every block has size at least 4. Let* $a = 1$ *or* 4*. Then, for every* $G \in \mathcal{G}$*, there is a* $(12|X| + a, 12|G| + a, \{4\})$*-IPBD.*

Proof: Give every point weight 12 and apply FC, using $\{4\}$-GDDs of type 12^k, $k \geq 4$. Then adjoin a new points. ∎

Applying Lemma 6.3 to truncated TDs, we have the following.

Lemma 6.4.

i) *Suppose there is a TD*(k,t)*, where* $k \geq 4$*. Let* $a = 1$ *or* 4*. Then for all* s *such that* $4t \leq s \leq kt$*, there is a* $(12s + a, 12t + a, \{4\})$*-IPBD.*
ii) *Suppose there is a TD*(k,n)*, where* $k \geq 5$*. Let* $a = 1$ *or* 4 *and let* $0 \leq t \leq n$*. Then for all* s *such that* $4n + t \leq s \leq (k-1)n + t$*, there is a* $(12s + a, 12t + a, \{4\})$*-IPBD.*

Define $T_6 = \{n \geq 5\} \setminus \{6, 10, 14, 18, 22, 26, 28, 30, 34, 38, 42, 44, 52\}$. Then, for every $n \in T_6$, there exists a TD$(6, n)$ (Theorem 2.5). The following property can be easily verified.

Lemma 6.5. *If* $n \in T_6$ *and* $n \geq 7$*, then there exists an* $n_1 > n$ *such that* $n_1 \in T_6$ *and* $4n_1 \leq 5n + 1$*.*

Consequently, we obtain

Lemma 6.6. *Suppose* $t \geq 1$*,* $n \in T_6$ *and* $n \geq \max\{7, t\}$*. Then, for all* $s \geq 4n + t$*, and for* $a = 1, 4$*, there is a* $(12s + a, 12t + a, \{4\})$*-IPBD.*

Proof: This is an immediate consequence of Lemmata 6.4 and 6.5. ∎

Similarly, we have

Lemma 6.7. *Suppose* $t \in T_6$ *and* $t \geq 7$*. Then, for all* $s \geq 4t$*, and for* $a = 1, 4$*, there exists a* $(12s + a, 12t + a, \{4\})$*-IPBD.*

Some missing cases will be supplied by

Lemma 6.8. *Suppose there is a* $TD(5, t+1)$, *where* $t \geq 4$. *Let* $a = 1$ *or* 4, *and let* $r = 1, 2,$ *or* 3. *Then there exists a* $(12(5t+r)+a, 12t+a, \{4\}$*-IPBD).*

Proof: Delete $5 - r$ points from a block of a TD$(5, t+1)$. Take the blocks (and group) through one of the deleted points as groups of a new GDD. Apply Lemma 6.3. ∎

Now, we have

Lemma 6.9. *Suppose* $t \geq 7$ *and* $s \geq 4t$. *Let* $a = 1$ *or* 4. *Then there is a* $(12s+a, 12t+a, \{4\})$*-IPBD.*

Proof: If $t \in T_6$, $t \geq 7$, then Lemma 6.7 applies. If $t \notin T_6$, $t \geq 7$ proceed as follows. First, applying Lemma 6.6 with $n = t+1$ handles the cases $s \geq 5t+4$. Lemma 6.8 handles the cases $5t+1 \leq s \leq 5t+3$, and then, Theorem 6.1 handles the cases $4t \leq s \leq 5t$. ∎

At this point, we consider the cases corresponding to $t \leq 6$. Some constructions for these cases are done in Table 3. These include all cases corresponding to $t \geq 4$ and $s \geq 4t$, proving

Lemma 6.10. *Suppose* $4 \leq t \leq 6$ *and* $s \geq 4t$. *Let* $a = 1$ *or* 4. *Then there is a* $(12s+a, 12t+a, \{4\})$*-IPBD.*

Table 3

t	s	Construction
2	$22 \leq s \leq 27$	apply Lemma 6.4 ii) with k = 6 and n = 5
3	$23 \leq s \leq 28$	apply Lemma 6.4 ii) with k = 6 and n = 5
4	$s \geq 32$	apply Lemma 6.6
4	s = 30, 31	apply Lemma 6.3, adjoining 2 or 3 points to a resolvable (28, 4, 1)-BIBD
4	$24 \leq s \leq 29$	apply Lemma 6.4 ii) with k = 6 and n = 5
4	$20 \leq s \leq 24$	apply Lemma 6.3, deleting up to 4 points from a group of a {5}-GDD of type 4^6
4	$16 \leq s \leq 20$	Theorem 6.1
5	$s \geq 33$	apply Lemma 6.6
5	s = 31, 32	a resolvable TD(5, 7) gives rise to a {5, 7}-GDD of type 5^7. Delete 3 or 4 points from a group of this GDD and apply Lemma 6.3.
5	$20 \leq s \leq 30$	apply Lemma 6.4 i) with k = 6 and t = 5
6	$s \geq 34$	apply Lemma 6.6
6	$31 \leq s \leq 33$	Lemma 6.8
6	$24 \leq s \leq 30$	Theorem 6.1

Next, the cases corresponding to $t = 1$ are easy.

Lemma 6.11. *Suppose $s \geq 4$, and $a = 1$ or 4. Then there is a $(12s + a, 12 + a, \{4\})$-IPBD.*

Proof: Adjoin $a = 1$ or 4 new points to a $\{4\}$-GDD of type 12^s. ∎

When $t = 2$ and 3, observe that we have $(12s + a, 12t + a, \{4\})$-IPBDs for $s \geq 28 + t$, applying Lemma 6.6 with $n = 7$. The cases $20 + t \leq s \leq 25 + t$ were done in Table 3. So, it remains to do the following cases (when $s \geq 4t$):

$$t = 2, \quad 8 \leq s \leq 21, \quad s = 28, 29$$
$$t = 3, \quad 12 \leq s \leq 22, \quad s = 29, 30.$$

Four difficult cases are done in Lemmata 6.12 and 6.13, and the remaining cases are handled in Table 4. Most of the constructions are applications of the singular indirect product. Many of the others consist of adjoining $a = 1$ or 4 points to a suitable $\{4\}$-GDD.

Lemma 6.12. *There is a $(205, 37; \{4\})$-IPBD and a $(208, 40; \{4\})$-IPBD.*

Proof: Delete three points from a block of a TD(5,4), constructing a $\{4,5\}$-IGDD of type $(4,1)^2 3^3$. Give every point weight 12 and apply FC, obtaining a $\{4\}$-IGDD of type $(48, 12)^2 36^3$. To construct a $(205, 37; \{4\})$-IPBD, adjoin $a = 1$ infinite point, filling in $(49, 13; \{4\})$-IPBDs and $(37; \{4\})$-PBDs, and a $(25, \{4\})$-PBD. To construct a $(208, 40; \{4\})$-IPBD, adjoin $a = 4$ infinite points, filling in $(52, 16; \{4\})$-IPBDs, $(40, 4; \{4\})$-IPBDs, and a $(28, 4, \{4\})$-IPBD. ∎

Lemma 6.13. *There is a $(349, 37; \{4\})$-IPBD and a $(352, 40; \{4\})$-IPBD.*

Proof: Adjoin 2 infinite points to a resolvable $(28, 4, 1)$-BIBD, and then delete an old point, constructing a $\{4,5\}$-IGDD of type $(4,1)^2 3^7$. Proceed as in Lemma 6.12. ∎

Table 4

t	s	a	w	v	Construction
2	8	1	25	97	$\{4\}$-GDD of type 24^4
2	9	1	25	109	apply SIP, $109 = 4(31-5)+5$, $25 = 4(10-5)+5$
2	10	1	25	121	apply SIP, $121 = 4(31-1)+1$, $25 = 4(7-1)+1$
2	11	1	25	133	apply SIP, $133 = 4(34-1)+1$, $25 = 4(7-1)+1$
2	12	1	25	145	$\{4\}$-GDD of type 24^6
2	13	1	25	157	apply SIP, $157 = 4(43-5)+5$, $25 = 4(10-5)+5$
2	14	1	25	169	apply SIP, $169 = 4(43-1)+1$, $25 = 4(7-1)+1$
2	15	1	25	181	apply SIP, $181 = 4(46-1)+1$, $25 = 4(7-1)+1$
2	16	1	25	193	$\{4\}$-GDD of type 24^8
2	17	1	25	205	apply SIP, $205 = 4(55-5)+5$, $25 = 4(10-5)+5$

Table 4 (continued)

2	18	1	25	217	apply SIP, 217 = 4(55 – 1) + 1, 25 = 4(7 – 1) + 1
2	19	1	25	229	apply SIP, 229 = 4(58 – 1) + 1, 25 = 4(7 – 1) + 1
2	20	1	25	241	{4}-GDD of type 24^{10}
2	21	1	25	253	apply SIP, 253 = 4(67 – 5) + 5, 25 = 4(10 – 5) + 5
2	28	1	25	337	{4}-GDD of type 24^{14}
2	29	1	25	349	apply SIP, 349 = 4(91 – 5) + 5, 25 = 4(10 – 5) + 5
2	8	4	28	100	{4}-GDD of type 24^4
2	9	4	28	112	apply SIP, 112 = 4(31 – 4) + 4, 28 = 4(10 – 4) + 4
2	10	4	28	124	apply SIP, 124 = 4·31, 28 = 4·7
2	11	4	28	136	apply SIP, 136 = 4·34, 28 = 4·7
2	12	4	28	148	{4}-GDD of type 24^6
2	13	4	28	160	apply SIP, 160 = 4(43 – 4) + 4, 28 = 4(10 – 4) + 4
2	14	4	28	172	apply SIP, 172 = 4·43, 28 = 4·7
2	15	4	28	184	apply SIP, 184 = 4·46, 28 = 4·7
2	16	4	28	196	{4}-GDD of type 24^8
2	17	4	28	208	apply SIP, 208 = 4(55 – 4) + 4, 28 = 4(10 – 4) + 4
2	18	4	28	220	apply SIP, 220 = 4·55, 28 = 4·7
2	19	4	28	232	apply SIP, 232 = 4·58, 28 = 4·7
2	20	4	28	244	{4}-GDD of type 24^{10}
2	21	4	28	256	apply SIP, 256 = 4(67 – 4) + 4, 28 = 4(10 – 4) + 4
2	28	4	28	340	{4}-GDD of type 24^{14}
2	29	4	28	352	apply SIP, 352 = 4(91 – 4) + 4, 28 = 4(10 – 4) + 4
3	12	1	37	145	apply SIP, 145 = 4(40 – 5) + 5, 37 = 4(13 – 5) + 5
3	13	1	37	157	{4}-GDD of type $24^5 36^1$ (give every point in a {4}-GDD of type $6^5 9^1$ weight 4 and apply FC)
3	14	1	37	169	apply SIP, 169 = 4(43 – 1) + 1, 37 = 4(10 – 1) + 1
3	15	1	37	181	apply SIP, 181 = 4(46 – 1) + 1, 37 = 4(10 – 1) + 1
3	16	1	37	193	apply SIP, 193 = 4(52 – 5) + 5, 37 = 4(13 – 5) + 5
3	17	1	37	205	Lemma 6.12
3	18	1	37	217	apply SIP, 217 = 4(55 – 1) + 1, 37 = 4(10 – 1) + 1
3	19	1	37	229	apply SIP, 229 = 4(58 – 1) + 1, 37 = 4(10 – 1) + 1
3	20	1	37	241	apply SIP, 241 = 4(64 – 5) + 5, 37 = 4(13 – 5) + 5
3	21	1	37	253	{4}-GDD of type 36^7
3	22	1	37	265	apply SIP, 265 = 4(67 – 1) + 1, 37 = 4(10 – 1) + 1
3	29	1	37	349	Lemma 6.13
3	30	1	37	361	apply SIP, 361= 4(91 – 1) + 1, 37 = 4(10 – 1) + 1
3	12	4	40	148	apply SIP, 148 = 4(40 – 4) + 4, 40 = 4(13 – 4) + 4
3	13	4	40	160	{4}-GDD of type $24^5 36^1$ (give every point in a {4}-GDD of type $6^5 9^1$ weight 4 and apply FC)
3	14	4	40	172	apply SIP, 172 = 4·43, 40 = 4·10
3	15	4	40	184	apply SIP, 184 = 4·46, 40 = 4·10
3	16	4	40	196	apply SIP, 196 = 4(52 – 4) + 4, 40 = 4(13 – 4) + 4
3	17	4	40	208	Lemma 6.12
3	18	4	40	220	apply SIP, 220 = 4·55, 40 = 4·10
3	19	4	40	232	apply SIP, 232 = 4·58, 40 = 4·10
3	20	4	40	244	apply SIP, 244 = 4(64 – 4) + 4, 40 = 4(13 – 4) + 4
3	21	4	40	256	{4}-GDD of type 36^7
3	22	4	40	268	apply SIP, 268 = 4·67, 40 = 4·10
3	29	4	40	352	Lemma 6.13
3	30	4	40	364	apply SIP, 364 = 4·91, 40 = 4·10

So, we have the result for $t = 2$ and 3.

Lemma 6.14. *Suppose $t = 2$ or 3 and $s \geq 4t$. Let $a = 1$ or 4. Then there is a $(12s + a, 12t + a, \{4\})$-IPBD.*

We now can prove our main Theorem.

Theorem 6.15. *Suppose $v \equiv 1$ or 4 modulo 12 and $w \equiv 1$ or 4 modulo 12. Then there exists a $(v, 4, 1)$-BIBD containing a $(w, 4, 1)$-BIBD as a subdesign if and only if $v \geq 3w + 1$.*

Proof: First, suppose $v - w$ is even. If $v \geq 4w - 3$, then the design exists by Lemmata 6.9, 6.10, 6.11 and 6.14. If $3w + 4 \leq v < 4w - 3$, then the design exists by Lemmata 4.2 and 4.7, unless $w = 25$ and $v = 85$. This case is done by adjoining $a = 1$ point to a $\{4\}$-GDD of type $12^5 24^1$ (to construct this GDD, give every point in a $\{4\}$-GDD of type $3^5 6^1$ weight 4 and apply FC).

Next, suppose $v - w$ is odd. If $3w + 1 \leq v \leq 9w + 4$, apply Theorem 6.2. If $v > 9w + 4$, then there exists a $(3w + 1, w; \{4\})$-IPBD by Lemma 4.10, and a $(v, 3w + 1; \{4\})$-IPBD, since $v - (3w + 1)$ is even. Hence, the desired $(v, w; \{4\})$-IPBD exists. This completes the proof. ∎

7. IPBDs where $v - w \equiv 3$ or 9 modulo 12, $3w + 1 \leq v \leq 9w + 4$.

We have noted that there always exists a $(3w + 1, w; \{4\})$-IPBD (Lemma 4.10). After applying Theorem 5.3, the only cases remaining when $v - w$ is odd and $v \leq 9w + 4$ are listed in Table 5, together with constructions in most cases.

Table 5

w	v	Construction
19	70	Mills [15]
19	82	Mills [15]
19	94	using Construction 3.12 with m = 5, u = 2, build a {4}-GDD of type $15^5 12^1$. Adjoin a = 7 points, filling in (22, 7; {4})-IPBDs.
19	106	Mills [15]
19	118	delete 3 points from a group of a TD(5, 8) – TD(5, 1), constructing a {4, 5}-IGDD of type $(8, 1)^4(5, 0)^1$. Give every point weight 3 and apply FC, to build a {4}-IGDD of type $(24, 3)^4(15, 0)^1$. Adjoin a = 7 new points, filling in (31, 10; {4})-IPBDs and a (22, 7; {4})-IPBD.
19	130	delete 4 points from a group of a TD(5, 9) – TD(5, 1), constructing a {4, 5}-IGDD of type $(9, 1)^4(5, 0)^1$. Give every point weight 3 and apply FC, to build a {4}-IGDD of type $(27, 3)^4(15, 0)^1$. Adjoin a = 7 new points, filling in (34, 10; {4})-IPBDs and a (22, 7; {4})-IPBD.
19	142	delete a block from a {4}-GDD of type 9^5, constructing a {4}-IGDD of type $(9, 1)^4(9, 0)^1$. Give every point weight 3 and apply FC, to build a {4}-IGDD of type $(27, 3)^4(27, 0)^1$. Adjoin a = 7 new points, filling in (34, 10; {4})-IPBDs and a (34, 7; {4})-IPBD.
19	154	start with a {4}-GDD of type $9^5 6^1$. Give every point weight 3 and apply FC, producing a {4}-GDD of type $27^5 18^1$. Now, adjoin a = 1 new point, filling in (28, {4})-PBDs.

Table 5 (continued)

22	79	Mills [14]
22	91	Mills [14]
22	103	Mills [15]
22	115	Mills [14]
22	127	start with a TD(5, 8) – TD(5, 1), give every point weight 3 and apply FC, to build a {4}-IGDD of type $(24, 3)^5$. Adjoin a = 7 new points, filling in (31, 10; {4})-IPBDs.
22	139	delete 1 point from a group of a TD(5, 9) – TD(5, 1), constructing a {4, 5}-IGDD of type $(9, 1)^4(8, 1)^1$. Give every point weight 3 and apply FC, to build a {4}-IGDD of type $(27, 3)^4(24, 3)^1$. Adjoin a = 7 new points, filling in (34, 10; {4})-IPBDs and a (31, 10; {4})-IPBD.
22	151	a TD(6, 10) – TD(6, 2) is presented in [5]. Delete all points in the hole, to construct a a {5}-GDD of type 8^6. Then, delete a block from this design, constructing a {5}-IGDD of type $(8, 1)^5(8, 0)^1$. Give every point weight 3 and apply FC, to build a {4}-IGDD of type $(24, 3)^5(24, 0)^1$. Adjoin a = 7 new points, filling in (31, 10; {4})-IPBDs and a (31, 7; {4})-IPBD.
22	163	Mills [15]
31	106	start with a TD(5, 8) – TD(5, 2) (see [24]). Delete one point of the hole from each group, producing a {4, 5}-IGDD of type $(7, 1)^5$ in which every block of size 5 hits the hole. Give the 30 original points weight (3, 1) and give the 5 new points weight (3, 0) and then apply FC. In this way, we build a {4}-IGDD of type $(21, 6)^5$. Now, adjoin one new point, filling in (22, 7; {4})-IPBDs.
31	118	delete 3 points from a group of a TD(5, 8), constructing a {4, 5}-GDD of type $8^4 5^1$. Give every point weight 3 and apply FC, to build a {4}-GDD of type $24^4 15^1$. Adjoin a = 7 new points, filling in (31, 7; {4})-IPBDs and a (22, 7; {4})-IPBD.
31	130	a (43, 10; {4})-IPBD gives rise to a {4}-GDD of type $1^{33}10^1$. Give every point weight 3 and apply FC, obtaining a {4}-GDD of type $3^{33}30^1$. Fill in a = 1 new point.
31	142	start with a TD(5, 5) – TD(5, 1). Give every point weight 6, except for 3 points in the last group which get weight 3. Apply FC to build a {4}-IGDD of type $(30, 6)^4(21, 6)^1$. Adjoin a = 1 new point, filling in (31, 7; {4})-IPBDs and a (22, 7; {4})-IPBD.
31	154	start with a TD(5, 5) – TD(5, 1). Give every point weight 6 and apply FC, to build a {4}-IGDD of type $(30, 6)^5$. Adjoin b = 4 new points, incorporating a = 1 of them into the hole, filling in (34; 7, 4; 1; {4})-◊-IPBDs and a (34, 7; {4})-IPBD.
31	166	start with a TD(6, 5) – TD(6, 1), and delete 3 points from a group, constructing a {5, 6}-IGDD of type $(5, 1)^5(2, 0)^1$. Give every point weight 6 and apply FC, to build a {4}-IGDD of type $(30, 6)^5(12, 0)^1$. Adjoin b = 4 new points, incorporating a = 1 of them into the hole, filling in (34; 7, 4; 1; {4})-◊-IPBDs and a (16, {4})-PBD.
31	226	start with a TD(9, 9), and delete 8 points from a block, constructing an {8, 9}-GDD of type $8^8 9^1$. Give every point weight 3 and apply FC, building a {4}-GDD of type $24^8 27^1$. Adjoin a = 7 new points, filling in (31, 7; {4})-IPBDs and a (34, 7; {4})-IPBD.
31	238	start with a TD(6, 7). Delete a point, and let a group be the hole in a new {6, 7}-IGDD of type $(5, 1)^7(6, 0)^1$. Next, delete two points from the hole, constructing a {5, 6, 7}-IGDD of type $(5, 1)^5(4, 0)^2(6, 0)^1$.

Table 5 (continued)

		Give one point in the group of size 6 weight 9, and give all other points weight 6. Apply FC. This builds a {4}-IGDD of type $(30, 6)^5(24, 0)^2(39, 0)^1$. Adjoin a = 1 new point, filling in (31, 7; {4})-IPBDs, (25, {4})-PBDs, and a (40; {4})-PBD.
34	115	apply SDP with the equation $115 = 4(34 - 7) + 7$
34	127	delete 1 point from a group of a TD(5, 8), constructing a {4, 5}-GDD of type $8^4 7^1$. Give every point weight 3 and apply FC, to build a {4}-GDD of type $24^4 21^1$. Adjoin a = 10 new points, filling in (34, 10; {4})-IPBDs and a (31, 10; {4})-IPBD.
34	139	delete 1 point from a group of a TD(5, 9), constructing a {4, 5}-GDD of type $9^4 8^1$. Give every point weight 3 and apply FC, to build a {4}-GDD of type $27^4 24^1$. Adjoin a = 7 new points, filling in (34, 7; {4})-IPBDs and a (31, 7; {4})-IPBD.
34	151	start with a TD(5, 4), and delete 4 points from a block, constructing a {4, 5}-GDD of type $3^4 4^1$. Give every point weight 9 and apply FC, building a {4}-GDD of type $27^4 36^1$. Adjoin a = 7 new points, filling in (34, 7; {4})-IPBDs and a (43, 7; {4})-IPBD.
34	163	start with a TD(5, 5) – TD(5, 1). Give every point weight 6, except for 3 points in the last group which get weight 9. Apply FC to build a {4}-IGDD of type $(30, 6)^4(39, 6)^1$. Adjoin a = 4 new points, filling in (34, 10; {4})-IPBDs and a (43, 10; {4})-IPBD.
34	175	start with a TD(6, 5) – TD(6, 1). Delete two points in a group, constructing a {5, 6}-IGDD of type $(5, 1)^5(3, 0)^1$. Give every point weight 6, except for 1 point in the last group which gets weight 9. Apply FC to build a {4}-IGDD of type $(30, 6)^5(21, 0)^1$. Adjoin a = 4 new points, filling in (34, 10; {4})-IPBDs and a (25, 4; {4})-IPBD.
34	223	Lemma 5.4
34	235	start with a TD(9, 9), and delete 5 points from a block, constructing a {4, 8, 9}-GDD of type $8^5 9^4$. Give every point weight 3 and apply FC, building a {4}-GDD of type $24^5 27^4$. Adjoin a = 7 new points, filling in (34, 7; {4})-IPBDs and (31, 7; {4})-IPBDs.
43	142	Mills [15]
43	154	Mills [15]
43	166	start with a TD(5, 4), give every point weight 9, except for 3 points in the last group, one of which gets weight 6 and two of which get weight 0. Apply FC, to build a {4}-GDD of type $36^4 15^1$. Adjoin a = 7 new points, filling in (43, 7; {4})-IPBDs and a (22, 7; {4})-IPBD.
43	178	start with a TD(5, 4), give every point weight 9, except for 3 points in the last group, which get weight 6. Apply FC, to build a {4}-GDD of type $36^4 27^1$. Adjoin a = 7 new points, filling in (43, 7; {4})-IPBDs and a (34, 7; {4})-IPBD.
43	238	start with a TD(6, 13), and delete 6 points from one group, giving rise to a {5, 6}-GDD of type $13^5 7^1$. Give the points in the group of size 7 weight 6, and give all other points weight 3. Apply FC, constructing a {4}-GDD of type $39^5 42^1$. Adjoin a = 1 new point, filling in (40, {4})-PBDs.
43	250	start with a TD(5, 19) – TD(5, 3) (see [23]). Delete 14 points in a group, constructing a {4, 5}-IGDD of type $(19, 3)^4(5, 0)^1$. Give every

Table 5 (continued)

		point weight 3 and apply FC, building a {4}-IGDD of type $(57,9)^4(15,0)^1$. Adjoin a = 7 new points, filling in (64, 16; {4})-IPBDs and a (22, 7; {4})-IPBD.
46	151	start with a TD(6, 7), give every point in 5 groups weight 3, and give every point in the last group weight 6. Apply FC, to build a {4}-GDD of type $21^5 42^1$. Adjoin a = 4 new points, filling in (25, 4; {4})-IPBDs.
46	163	start with a TD(6, 7), give every point in 4 groups weight 3, give every point in the fifth group weight 6, give 4 points in the sixth group weight 6, and give the remaining 3 points in the sixth group weight 3. Apply FC. This builds a {4}-GDD of type $21^4 42^1 33^1$. Adjoin a = 4 new points, filling in (25, 4; {4})-IPBDs, and a (37, 4; {4})-IPBD.
46	175	start with a TD(5, 4), give every point weight 9, except for 3 points in the last group, two of which get weight 6, and one of which gets weight 0. Apply FC. This builds a {4}-GDD of type $36^4 21^1$. Adjoin a = 10 new points, filling in (46, 10; {4})-IPBDs and a (31, 10; {4})-IPBD.
46	187	start with a TD(5, 4), give every point weight 9, except for 1 point in the last group which gets weight 6. Apply FC, building a {4}-GDD of type $36^4 33^1$. Adjoin a = 10 new points, filling in (46, 10; {4})-IPBDs and a (43, 10; {4})-IPBD.
46	235	Lemma 5.4
46	247	start with a TD(6, 8). Delete a point, yielding a new {6, 8}-GDD of type $5^8 7^1$. Next, delete 7 points from B, one of the blocks of size 8, constructing a {5, 6, 8}-GDD of type $4^7 5^1 7^1$. Give all points weight 6, except for the point not deleted from B, which gets weight 9. Apply FC. This builds a {4}-GDD of type $24^7 33^1 42^1$. Adjoin a = 4 new points, filling in (28, 4; {4})-IPBDs and a (37, 4; {4})-IPBD.
55	178	apply Lemma 3.11 with m = 8, u = 9, to construct a {4}-GDD of type $24^4 48^1 27^1$. Adjoin a = 7 new points, filling in (31, 7; {4})-IPBDs and a (34, 7; {4})-IPBD.
55	190	apply Lemma 3.11 with m = 8, u = 13, to construct a {4}-GDD of type $24^4 48^1 39^1$. Adjoin a = 7 new points, filling in (31, 7; {4})-IPBDs and a (46, 7; {4})-IPBD.
55	250	start with a TD(5, 17), and delete 4 points from a block, constructing a {4, 5}-GDD of type $16^4 17^1$. Give every point weight 3 and apply FC, building a {4}-GDD of type $48^4 51^1$. Adjoin a = 7 new points, filling in (55, 7; {4})-IPBDs and a (58, 7; {4})-IPBD.
55	262	start with a TD(5, 20) – TD(5, 4) (this can be constructed by taking the direct product of a TD(5, 4) and TD(5, 5)). Delete 15 points in a group, constructing a {4, 5}-IGDD of type $(20, 4)^4(5, 0)^1$. Give every point weight 3 and apply FC, building a {4}-IGDD of type $(60, 12)^4(15, 0)^1$. Adjoin a = 7 new points, filling in (67, 19; {4})-IPBDs (Table 7) and a (22, 7; {4})-IPBD.
58	187	apply Lemma 3.11 with m = 8, u = 11, to construct a {4}-GDD of type $24^4 48^1 33^1$. Adjoin a = 10 new points, filling in (34, 10; {4})-IPBDs and a (43, 10; {4})-IPBD.
58	199	apply Lemma 3.11 with m = 8, u = 15, to construct a {4}-GDD of type $24^4 48^1 45^1$. Adjoin a = 10 new points, filling in (34, 10; {4})-IPBDs and a (55, 10; {4})-IPBD.

Table 5 (continued)

58	247	Lemma 5.4
58	259	start with a TD(5, 17), and delete 1 point from a group, constructing a {4, 5}-GDD of type $17^4 16^1$. Give every point weight 3 and apply FC, building a {4}-GDD of type $51^4 48^1$. Adjoin a = 7 new points, filling in (58, 7; {4})-IPBDs and a (55, 7; {4})-IPBD.
67	262	start with a TD(5, 20), and delete 15 points from a group, constructing a {4, 5}-GDD of type $20^4 5^1$. Give every point weight 3 and apply FC, building a {4}-GDD of type $60^4 15^1$. Adjoin a = 7 new points, filling in (67, 7; {4})-IPBDs and a (22, 7; {4})-IPBD.
67	274	start with a TD(5, 20), and delete 11 points from a group, constructing a {4, 5}-GDD of type $20^4 9^1$. Give every point weight 3 and apply FC, building a {4}-GDD of type $60^4 27^1$. Adjoin a = 7 new points, filling in (67, 7; {4})-IPBDs and a (34, 7; {4})-IPBD.
70	259	Lemma 5.4
70	271	start with a TD(5, 20), and delete 13 points from a group, constructing a {4, 5}-GDD of type $20^4 7^1$. Give every point weight 3 and apply FC, building a {4}-GDD of type $60^4 21^1$. Adjoin a = 10 new points, filling in (70, 10; {4})-IPBDs and a (31, 10; {4})-IPBD.
79	274	Lemma 4.8
79	286	Lemma 4.8
82	271	Lemma 5.4
82	283	apply Lemma 3.11 with m = 12, u = 19, to construct a {4}-GDD of type $36^4 72^1 57^1$. Adjoin a = 10 new points, filling in (46, 10; {4})-IPBDs and a (67, 10; {4})-IPBD.
91	286	Lemma 4.8
91	298	Lemma 4.8
94	295	Lemma 4.9

Hence, we have

Lemma 7.1. *Suppose* $w \equiv 7$ *or* 10 *modulo 12,* $v \equiv 7$ *or* 10 *modulo 12,* $v - w$ *is odd, and* $3w + 1 \le v \le 9w + 4$. *Then there is a* $(v, w; \{4\})$*-IPBD.*

8. IPBDs where $v - w \equiv 0$ modulo12.

First, we shall consider the situation when $v \ge 4w - 30$, $w \equiv 7$ or 10 modulo 12, and $v - w \equiv 0$ modulo 12. We construct many of the designs in this section by giving all points in a GDD weight 12, and then filling in groups with 7 or 10 new points. We will use the $(v, 7, \{4\})$-IPBDs and $(v, 10; \{4\})$-IPBDs from Theorems 1.2 and 1.3. These constructions are analogous to ones given in Section 6.

Lemma 8.1.

i) *Suppose* $(X, \mathcal{G}, \mathcal{A})$ *is a GDD in which every block has size at least 4, and every group has size at least 2. Let* $a = 7$ *or* 10. *Then, for every* $G \in \mathcal{G}$, *there is a* $(12|X| + a, 12|G| + a, \{4\})$*-IPBD.*

ii) *Suppose* $(X, \mathcal{G}, \mathcal{A})$ *is a GDD in which every block has size at least 4, and every group has size at least 2, except for one group* G_0, *which has size 1.*

Let $a = 7$ *or* 10. *Then there is a* $(12|X| + a, 12 + a, \{4\})$*-IPBD.*

iii) *Suppose* $(X, \mathcal{G}, \mathcal{A})$ *is a GDD in which every block has size at least 4, and every group has size at least 4. Let* $a = 19$ *or* 22. *Suppose that there exists a* $(12|G| + a, a; \{4\})$*-IPBD, for every* $G \in \mathcal{G}$. *Then, for every* $G \in \mathcal{G}$, *there is a* $(12|X| + a, 12|G| + a, \{4\})$*-IPBD.*

Proof: Give every point weight 12 and apply FC, using $\{4\}$-GDDs of type 12^k, $k \geq 4$. Then adjoin a new points. When $a = 7$ or 10, the necessary designs exist by Theorems 1.2 and 1.3. ∎

Then, analogous to Lemma 6.4, we have

Lemma 8.2.

i) *Suppose there is a* $TD(k,t)$, *where* $k \geq 4$. *Let* $a = 7$ *or* 10. *Then for all* s *such that* $4t \leq s \leq kt$, $s \neq 4t + 1$, *there is a* $(12s + a, 12t + a, \{4\})$*-IPBD.*

ii) *Suppose there is a* $TD(k,n)$, *where* $k \geq 5$. *Let* $a = 7$ *or* 10 *and let* $0 \leq t \leq n$. *Then for all* s *such that* $4n + t \leq s \leq (k-1)n + t$, $s \neq 4n + t + 1$, *there is a* $(12s + a, 12t + a, \{4\})$*-IPBD.*

One can easily verify

Lemma 8.3. *If* $n \in T_6$ *and* $n \geq 7$, *then there exists an* $n_1 > n$ *such that* $n_1 \in T_6$ *and* $4n_1 \leq 5n - 1$.

As a consequence of the two preceding lemmata, we get

Lemma 8.4. *Suppose* $t \geq 1$, $n \in T_6$ *and* $n \geq \max\{7, t\}$. *Then, for all* $s \geq 4n + t$, $s \neq 4n + t + 1$, *and for* $a = 7, 10$, *there is a* $(12s + a, 12t + a, \{4\})$*-IPBD.*

Similarly, we have

Lemma 8.5. *Suppose* $t \in T_6$ *and* $t \geq 7$. *Then, for all* $s \geq 4t$, $s \neq 4t + 1$, *and for* $a = 7, 10$, *there is a* $(12s + a, 12t + a, \{4\})$*-IPBD.*

Some missing cases will be supplied by

Lemma 8.6. *Suppose there is a* $TD(6, t+1)$, *where* $t \geq 6$. *Let* $a = 7$ *or* 10, *and let* $r = 1, 2, 3,$ *or* 5. *Then there is a* $(12(5t + r) + a, 12t + a, \{4\})$*-IPBD.*

Proof: When $r = 1, 2,$ or 3, delete $4 - r$ points from a group of a $TD(5, t+1)$. From another group of the TD, delete a point x, and take the blocks (and group) through x as groups of a new GDD. When $r = 5$, delete all the points in a block B of a $TD(6, t+1)$. For some group G, delete $t - 5$ further points in G. Take the blocks (and group) through any point $x \in B \backslash G$ as groups of a new GDD. Apply Lemma 8.1. ∎

Another class of missing cases is dealt with as follows.

Lemma 8.7. *Suppose* $s = 4t + 1, t \geq 5$. *Let* $a = 7$ *or* 10. *Then there is a* $(12s + a, 12t + a, \{4\})$*-IPBD.*

Proof: Start with a $TD(5, 4t - 4)$, and delete $4t - 20$ points from one group. Give all points weight 3, and apply FC. This yields a $\{4\}$-GDD of type $(12t - 12)^4(48)^1$. Now adjoin $12 + a$ new points, filling in $(12t + a, 12 + a; \{4\})$-IPBDs and a $(60 + a, 12 + a; \{4\})$-IPBD (these are $(v, 19; \{4\})$-IPBDs or $(v, 22; \{4\})$-IPBDs, and will be shown to exist in this section). ∎

Theorem 8.8. *Suppose* $t \geq 7$ *and* $s \geq 4t$. *Let* $a = 7$ *or* 10. *Then there is a* $(12s + a, 12t + a, \{4\})$*-IPBD.*

Proof: If $t \geq 7$, $t \in T_6$, then apply Lemmata 8.5 and 8.7. If $t \geq 7, t \notin T_6$, then proceed as follows. Apply Lemma 8.4 with $n = t + 1$ to handle $s \geq 5t + 4$, $s \neq 5t + 5$. When $s \in \{5t + 1, 5t + 2, 5t + 3, 5t + 5\}$, apply Lemma 8.6. When $t \neq 10, 4t \leq s \leq 5t, s \neq 4t + 1$, delete points from a group of a TD$(5, t)$ and apply Lemma 8.1. When $s = 4t + 1$, apply Lemma 8.7. It remains only to handle the cases when $t = 10, 4t \leq s \leq 5t, s \neq 4t + 1$. The case $t = 10, s = 40$ is done by Lemma 8.1 i), using a TD$(4, 10)$. The cases $t = 10, 42 \leq s \leq 50$ are done by Lemma 8.1 iii), by deleting points from two groups of a TD$(6, 9)$ (the required IPBDs are given in Table 7). This completes the proof. ∎

We still have to handle the cases corresponding to $t \leq 6$. Some further constructions are given in Table 6.

Table 6

t	s	Construction
1	21	apply Lemma 8.2 ii) with n = 5, k = 6
1	$23 \leq s \leq 26$	apply Lemma 8.2 ii) with n = 5, k = 6
2	22	apply Lemma 8.2 ii) with n = 5, k = 6
2	$24 \leq s \leq 27$	apply Lemma 8.2 ii) with n = 5, k = 6
3	23	apply Lemma 8.2 ii) with n = 5, k = 6
3	24	apply Lemma 8.1, using a {4}-GDD of type $3^4 6^2$
3	$25 \leq s \leq 28$	apply Lemma 8.2 ii) with n = 5, k = 6
4	16	apply Lemma 8.2 i) with k = 5
4	$18 \leq s \leq 20$	apply Lemma 8.2 i) with k = 5
4	$22 \leq s \leq 24$	delete 0, 1, or 2 points from a group of a {5}-GDD of type 4^6 and apply Lemma 8.1 i)
4	$26 \leq s \leq 29$	apply Lemma 8.2 ii) with n = 5, k = 6
4	30, 31	adjoin 2 or 3 infinite points to a RBIBD (28, 4, 1) and apply Lemma 8.1 i)
5	20	apply Lemma 8.2 i) with k = 6
5	21	Lemma 8.7
5	$22 \leq s \leq 30$	apply Lemma 8.2 i) with k = 6
5	$32 \leq s \leq 34$	a resolvable TD(5, 7) gives rise to a {5, 7}-GDD of type 5^7. Delete 1, 2, or 3 points from a group of this GDD and apply Lemma 8.1 i)

6	24	apply Lemma 8.1 i), using a {4}-GDD of type $3^4 6^2$
6	25, 26	apply Lemma 8.1 iii), deleting 0 or 1 point from a TD(5, 5) (see Table 7)
6	27	apply Lemma 8.1 i), using a {4}-GDD of type $3^1 6^4$
6	29, 30	apply Lemma 8.1 iii), deleting a point from 1 or 2 groups of a TD(6, 5) (see Table 7)
6	$31 \leq s \leq 33$	Lemma 8.6
6	35	Lemma 8.6

By Lemma 8.4, we already have constructed $(12s + a, 12t + a, \{4\})$-IPBDs for $s \geq 28 + t$, $s \neq 29 + t$. So, it remains to do the cases corresponding to the following values of t and s.

$$\begin{aligned}
&t = 1, \quad 5 \leq s \leq 20, \quad s = 22, 27, 28, 30 \\
&t = 2, \quad 8 \leq s \leq 21, \quad s = 23, 28, 29, 31 \\
&t = 3, \quad 12 \leq s \leq 22, \quad s = 29, 30, 32. \\
&t = 4, \quad s = 17, 21, 25, 33 \\
&t = 5, \quad s = 31. \\
&t = 6, \quad s = 28.
\end{aligned}$$

At this point, we eliminate several of the more difficult cases.

Lemma 8.9. *There is a* $(91, 19; \{4\})$*-IPBD and a* $(94, 22; \{4\})$*-IPBD.*

Proof: Adjoin infinite points to the groups, and to 5 parallel classes of a resolvable $\{4\}$-GDD of type 3^8, constructing a $\{4, 5\}$-GDD of type $4^6 6^1$. Give every point weight 3 and apply FC, obtaining a $\{4\}$-GDD of type $12^6 18^1$. Adjoin $a = 1$ or 4 new points, filling in $(12 + a, a; \{4\})$-IPBDs. ∎

Lemma 8.10. *There is a* $(187, 19; \{4\})$*-IPBD and a* $(190, 22; \{4\})$*-IPBD.*

Proof: Delete a point from a TD$(5, 7)$, constructing a $\{5, 7\}$-GDD of type $4^7 6^1$. Give every point on the group of size 6 weight 3, and give every other point weight 6. Apply FC, obtaining a $\{4\}$-GDD of type $24^7 18^1$. Adjoin $a = 1$ or 4 new points, filling in $(24 + a, a; \{4\})$-IPBDs. ∎

Lemma 8.11. *There is a* $(235, 19; \{4\})$*-IPBD.*

Proof: Use Lemma 4.8 to construct a $(235, 67; \{4\})$-IPBD. Now fill in a $(67, 19; \{4\})$-IPBD (see Table 7). ∎

Lemma 8.12. *There is a* $(331, 19; \{4\})$*-IPBD and a* $(334, 22; \{4\})$*-IPBD.*

Proof: Adjoin w points to a resolvable $(88, 4, 1)$-BIBD. Then, apply Lemma 3.9 with $(v, w) = (110, 22)$ and $(111, 23)$ to build a $(331, 67; \{4\})$-IPBD and a $(334, 70; \{4\})$-IPBD. Now fill in a $(67, 19; \{4\})$-IPBD and a $(70, 22; \{4\})$-IPBD, respectively (see Table 7). ∎

Lemma 8.13. *There is a* $(238, 22; \{4\})$*-IPBD and a* $(238, 70; \{4\})$*-IPBD.*

Proof: Start with a $\{4\}$-GDD of type $6^5 12^1 15^1$ (Theorem 2.7). Give every point weight 4 and apply FC, constructing a $\{4\}$-GDD of type $24^5 48^1 60^1$. Now, adjoin $a = 10$ new points, filling in $(34, 10; \{4\})$-IPBDs and a $(58, 10; \{4\})$-IPBD. This yields a $(238, 70; \{4\})$-IPBD. If we construct a $(70, 22; \{4\})$-IPBD on the hole (Table 7), we get a $(238, 22; \{4\})$-IPBD. ▮

Lemma 8.14. *There is a* $(211, 43; \{4\})$*-IPBD and a* $(214, 46; \{4\})$*-IPBD.*

Proof: Delete a point from a TD$(5, 7)$, constructing a $\{5, 7\}$-GDD of type $4^7 6^1$. Give every point weight 6 and apply FC, obtaining a $\{4\}$-GDD of type $24^7 36^1$. Adjoin $a = 7$ or 10 new points, filling in $(24 + a, a; \{4\})$-IPBDs. ▮

Lemma 8.15. *There is a* $(355, 43; \{4\})$*-IPBD and a* $(358, 46; \{4\})$*-IPBD.*

Proof: Adjoin 5 infinite points to a resolvable $\{4\}$-GDD of type 3^8, constructing a $\{4, 5\}$-GDD of type $3^8 5^1$. Give every point weight 12 and apply FC, obtaining a $\{4\}$-GDD of type $36^8 60^1$. Adjoin $a = 7$ or 10 new points, filling in $(36 + a, a; \{4\})$-IPBDs and a $(60 + a, a; \{4\})$-IPBD. ▮

Lemma 8.16. *There is a* $(379, 67; \{4\})$*-IPBD and a* $(382, 70; \{4\})$*-IPBD.*

Proof: Adjoin infinite points to the groups, and to 5 parallel classes of a resolvable $\{4\}$-GDD of type 3^8, constructing a $\{4, 5\}$-GDD of type $4^6 6^1$. Give every point weight 12 and apply FC, obtaining a $\{4\}$-GDD of type $48^6 72^1$. Adjoin $a = 19$ or 22 new points, filling in $(48 + a, a; \{4\})$-IPBDs and a $(72 + a, a; \{4\})$-IPBD (Table 7). ▮

We now eliminate all the remaining cases in Table 7. Most are done by means of the singular indirect product. Also, several make use of $\{4\}$-GDDs constructed by completing resolvable $\{3\}$-GDDs given in Theorem 2.2 (when we complete a resolvable $\{3\}$-GDD of type t^u, we get a $\{4\}$-GDD of type $t^u r^1$, where $r = t(u-1)/2$).

Table 7

t	s	a	w	v	construction
1	5	7	19	67	$\{4\}$-GDD of type $12^4 18^1$
1	6	7	19	79	apply SIP, $79 = 4(22-3)+3$, $19 = 4(7-3)+3$
1	7	7	19	91	Lemma 8.9
1	8	7	19	103	apply SIP, $103 = 4(31-7)+7$, $19 = 4(10-7)+7$
1	9	7	19	115	apply SIP, $115 = 4(31-3)+3$, $19 = 4(7-3)+3$
1	10	7	19	127	apply SIP, $127 = 4(34-3)+3$, $19 = 4(7-3)+3$
1	11	7	19	139	adjoin 6 points to a resolvable (40, 4, 1)-BIBD and apply Lemma 3.9 with $(v, w) = (46, 6)$
1	12	7	19	151	apply SIP, $151 = 4(43-7)+7$, $19 = 4(10-7)+7$
1	13	7	19	163	apply SIP, $163 = 4(43-3)+3$, $19 = 4(7-3)+3$
1	14	7	19	175	apply SIP, $175 = 4(46-3)+3$, $19 = 4(7-3)+3$
1	15	7	19	187	Lemma 8.10
1	16	7	19	199	apply SIP, $199 = 4(55-7)+7$, $19 = 4(10-7)+7$

Table 7 (continued)

1	17	7	19	211	apply SIP, 211 = 4(55 – 3) + 3, 19 = 4(7 – 3) + 3
1	18	7	19	223	apply SIP, 223 = 4(58 – 3) + 3, 19 = 4(7 – 3) + 3
1	19	7	19	235	Lemma 8.11
1	20	7	19	247	apply SIP, 247 = 4(67 – 7) + 7, 19 = 4(10 – 7) + 7
1	22	7	19	271	apply SIP, 271 = 4(70 – 3) + 3, 19 = 4(7 – 3) + 3
1	27	7	19	331	Lemma 8.12
1	28	7	19	343	apply SIP, 343 = 4(91 – 7) + 7, 19 = 4(10 – 7) + 7
1	30	7	19	367	apply SIP, 367 = 4(94 – 3) + 3, 19 = 4(7 – 3) + 3
1	5	10	22	70	{4}-GDD of type $4^{12}22^1$
1	6	10	22	82	apply SIP, 82 = 4(22 – 2) + 2, 22 = 4(7 – 2) + 2
1	7	10	22	94	Lemma 8.9
1	8	10	22	106	apply SIP, 106 = 4(31 – 6) + 6, 22 = 4(10 – 6) + 6
1	9	10	22	118	apply SIP, 118 = 4(31 – 2) + 2, 22 = 4(7 – 2) + 2
1	10	10	22	130	apply SIP, 130 = 4(34 – 2) + 2, 22 = 4(7 – 2) + 2
1	11	10	22	142	adjoin 7 points to a resolvable (40, 4, 1)-BIBD and apply Lemma 3.9 with (v, w) = (47, 7)
1	12	10	22	154	apply SIP, 154 = 4(43 – 6) + 6, 22 = 4(10 – 6) + 6
1	13	10	22	166	apply SIP, 166 = 4(43 – 2) + 2, 22 = 4(7 – 2) + 2
1	14	10	22	178	apply SIP, 178 = 4(46 – 2) + 2, 22 = 4(7 – 2) + 2
1	15	10	22	190	Lemma 8.10
1	16	10	22	202	apply SIP, 202 = 4(55 – 6) + 6, 22 = 4(10 – 6) + 6
1	17	10	22	214	apply SIP, 214 = 4(55 – 2) + 2, 22 = 4(7 – 2) + 2
1	18	10	22	226	apply SIP, 226 = 4(58 – 2) + 2, 22 = 4(7 – 2) + 2
1	19	10	22	238	Lemma 8.13
1	20	10	22	250	apply SIP, 250 = 4(67 – 6) + 6, 22 = 4(10 – 6) + 6
1	22	10	22	274	apply SIP, 274 = 4(70 – 2) + 2, 22 = 4(7 – 2) + 2
1	27	10	22	334	Lemma 8.12
1	28	10	22	346	apply SIP, 346 = 4(91 – 6) + 6, 22 = 4(10 – 6) + 6
1	30	10	22	370	apply SIP, 370 = 4(94 – 2) + 2, 22 = 4(7 – 2) + 2
2	8	7	31	103	{4}-GDD of type 24^4
2	9	7	31	115	apply SIP, 115 = 4(31 – 3) + 3, 31 = 4(10 – 3) + 3
2	10	7	31	127	apply SIP, 127 = 4(34 – 3) + 3, 31 = 4(10 – 3) + 3
2	11	7	31	139	apply SIP, 139 = 4(40 – 7) + 7, 31 = 4(13 – 7) + 7
2	12	7	31	151	{4}-GDD of type 24^6
2	13	7	31	163	apply SIP, 163 = 4(43 – 3) + 3, 31 = 4(10 – 3) + 3
2	14	7	31	175	apply SIP, 175 = 4(46 – 3) + 3, 31 = 4(10 – 3) + 3
2	15	7	31	187	apply SIP, 187 = 4(52 – 7) + 7, 31 = 4(13 – 7) + 7
2	16	7	31	199	{4}-GDD of type 24^8
2	17	7	31	211	apply SIP, 211 = 4(55 – 3) + 3, 31 = 4(10 – 3) + 3
2	18	7	31	223	apply SIP, 223 = 4(58 – 3) + 3, 31 = 4(10 – 3) + 3
2	19	7	31	235	apply SIP, 235 = 4(64 – 7) + 7, 31 = 4(13 – 7) + 7
2	20	7	31	247	{4}-GDD of type 24^{10}
2	21	7	31	259	apply SIP, 259 = 4(67 – 3) + 3, 31 = 4(10 – 3) + 3
2	23	7	31	283	apply SIP, 283 = 4(76 – 7) + 7, 31 = 4(13 – 7) + 7
2	28	7	31	343	{4}-GDD of type 24^{14}
2	29	7	31	355	apply SIP, 355 = 4(91 – 3) + 3, 31 = 4(10 – 3) + 3
2	31	7	31	379	apply SIP, 379 = 4(100 – 7) + 7, 31 = 4(13 – 7) + 7
2	8	10	34	106	{4}-GDD of type 24^4
2	9	10	34	118	apply SIP, 118 = 4(31 – 2) + 2, 34 = 4(10 – 2) + 2
2	10	10	34	130	apply SIP, 130 = 4(34 – 2) + 2, 34 = 4(10 – 2) + 2

Table 7 (continued)

2	11	10	34	142	apply SIP, 142 = 4(40 – 6) + 6, 34 = 4(13 – 6) + 6
2	12	10	34	154	{4}-GDD of type 24^6
2	13	10	34	166	apply SIP, 166 = 4(43 – 2) + 2, 34 = 4(10 – 2) + 2
2	14	10	34	178	apply SIP, 178 = 4(46 – 2) + 2, 34 = 4(10 – 2) + 2
2	15	10	34	190	apply SIP, 190 = 4(52 – 6) + 6, 34 = 4(13 – 6) + 6
2	16	10	34	202	{4}-GDD of type 24^8
2	17	10	34	214	apply SIP, 214 = 4(55 – 2) + 2, 34 = 4(10 – 2) + 2
2	18	10	34	226	apply SIP, 226 = 4(58 – 2) + 2, 34 = 4(10 – 2) + 2
2	19	10	34	238	apply SIP, 238 = 4(64 – 6) + 6, 34 = 4(13 – 6) + 6
2	20	10	34	250	{4}-GDD of type 24^{10}
2	21	10	34	262	apply SIP, 262 = 4(67 – 2) + 2, 34 = 4(10 – 2) + 2
2	23	10	34	286	apply SIP, 286 = 4(76 – 6) + 6, 34 = 4(13 – 6) + 6
2	28	10	34	346	{4}-GDD of type 24^{14}
2	29	10	34	358	apply SIP, 358 = 4(91 – 2) + 2, 34 = 4(10 – 2) + 2
2	31	10	34	382	apply SIP, 382 = 4(100 – 6) + 6, 34 = 4(13 – 6) + 6
3	12	7	43	151	apply SIP, 151 = 4(40 – 3) + 3, 43 = 4(13 – 3) + 3
3	13	7	43	163	{4}-GDD of type $24^5 36^1$ (give every point in a {4}-GDD of type $6^5 9^1$ weight 4 and apply FC)
3	14	7	43	175	apply SIP, 175 = 4(49 – 7) + 7, 43 = 4(16 – 7) + 7
3	15	7	43	187	apply SIP, 187 = 4(49 – 3) + 3, 43 = 4(13 – 3) + 3
3	16	7	43	199	apply SIP, 199 = 4(52 – 3) + 3, 43 = 4(13 – 3) + 3
3	17	7	43	211	Lemma 8.14
3	18	7	43	223	apply SIP, 175 = 4(61 – 7) + 7, 43 = 4(16 – 7) + 7
3	19	7	43	235	apply SIP, 235 = 4(61 – 3) + 3, 43 = 4(13 – 3) + 3
3	20	7	43	247	apply SIP, 247 = 4(64 – 3) + 3, 43 = 4(13 – 3) + 3
3	21	7	43	259	{4}-GDD of type $36^5 72^1$ (give every point in a {4}-GDD of type $3^5 6^1$ weight 12 and apply FC)
3	22	7	43	271	apply SIP, 271 = 4(73 – 7) + 7, 43 = 4(16 – 7) + 7
3	29	7	43	355	Lemma 8.15
3	30	7	43	367	apply SIP, 367 = 4(97 – 7) + 7, 43 = 4(16 – 7) + 7
3	32	7	43	391	apply SIP, 391 = 4(100 – 3) + 3, 43 = 4(13 – 3) + 3
3	12	10	46	154	apply SIP, 154 = 4(40 – 2) + 2, 46 = 4(13 – 2) + 2
3	13	10	46	166	{4}-GDD of type $24^5 36^1$ (give every point in a {4}-GDD of type $6^5 9^1$ weight 4 and apply FC)
3	14	10	46	178	apply SIP, 178 = 4(49 – 6) + 6, 46 = 4(16 – 6) + 6
3	15	10	46	190	apply SIP, 190 = 4(49 – 2) + 2, 46 = 4(13 – 2) + 2
3	16	10	46	202	apply SIP, 202 = 4(52 – 2) + 2, 46 = 4(13 – 2) + 2
3	17	10	46	214	Lemma 8.14
3	18	10	46	226	apply SIP, 226 = 4(61 – 6) + 6, 46 = 4(16 – 6) + 6
3	19	10	46	238	apply SIP, 238 = 4(61 – 2) + 2, 46 = 4(13 – 2) + 2
3	20	10	46	250	apply SIP, 250 = 4(64 – 2) + 2, 46 = 4(13 – 2) + 2
3	21	10	46	262	{4}-GDD of type $36^5 72^1$ (give every point in a {4}-GDD of type $3^5 6^1$ weight 12 and apply FC)
3	22	10	46	274	apply SIP, 274 = 4(73 – 6) + 6, 46 = 4(16 – 6) + 6
3	29	10	46	358	Lemma 8.15
3	30	10	46	370	apply SIP, 370 = 4(97 – 6) + 6, 46 = 4(16 – 6) + 6
3	32	10	46	394	apply SIP, 394 = 4(100 – 2) + 2, 46 = 4(13 – 2) + 2
4	17	7	55	211	apply SIP, 211 = 4(58 – 7) + 7, 55 = 4(19 – 7) + 7
4	21	7	55	259	apply SIP, 259 = 4(76 – 15) + 15, 55 = 4(25 – 15) + 15
4	25	7	55	307	apply SIP, 307 = 4(88 – 15) + 15, 55 = 4(25 – 15) + 15

4	33	7	55	403	apply SIP, 403 = 4(112 – 15) + 15, 55 = 4(25 – 15) + 15
4	17	10	58	214	apply SIP, 214 = 4(58 – 6) + 6, 58 = 4(19 – 6) + 6
4	21	10	58	262	apply SIP, 262 = 4(76 – 14) + 14, 58 = 4(25 – 14) + 14
4	25	10	58	310	apply SIP, 310 = 4(88 – 14) + 14, 58 = 4(25 – 14) + 14
4	33	10	58	406	apply SIP, 406 = 4(112 – 14) + 14, 58 = 4(25 – 14) + 14
5	31	7	67	379	Lemma 8.16
5	31	10	70	382	Lemma 8.16
6	28	7	79	343	apply Lemma 3.9 with (v, w) = (114, 26)
6	28	10	82	346	apply Lemma 3.9 with (v, w) = (115, 27)

The constructions given in Tables 6 and 7 prove the following.

Lemma 8.17. *Suppose* $w \equiv 7$ *or* 10 *modulo 12,* $v \equiv w$ *modulo 12, and* $v \geq 4w - 30$*. Then there is a* $(v, w; \{4\})$*-IPBD.*

We have several cases to consider when $w \leq 82$, $3w + 1 < v < 4w - 30$. Constructions are given in Table 8. As in Table 7, most of the required $\{4\}$-GDDs are obtained by completing resolvable $\{3\}$-GDDs.

Table 8

w	v	construction
43	139	{4}-GDD of type $12^8 42^1$
46	142	{4}-GDD of type $4^{24} 46^1$
55	175	{4}-GDD of type $12^{10} 54^1$
55	187	apply SIP, 187 = 4(49 – 3) + 3, 55 = 4(16 – 3) + 3
58	178	{4}-GDD of type $4^{30} 58^1$
58	190	apply SIP, 190 = 4(49 – 2) + 2, 58 = 4(16 – 2) + 2
67	211	{4}-GDD of type $12^{12} 66^1$
67	223	apply SIP, 223 = 4(58 – 3) + 3, 67 = 4(19 – 3) + 3
67	235	Lemma 4.8
70	214	{4}-GDD of type $4^{36} 70^1$
70	226	apply SIP, 226 = 4(58 – 2) + 2, 70 = 4(19 – 2) + 2
70	238	Lemma 8.13
79	247	{4}-GDD of type $12^{14} 78^1$
79	259	apply Lemma 3.11 with m = 12, u = 12 to build a {4}-GDD of type $36^5 72^1$. Adjoin a = 7 new points.
79	271	apply Lemma 3.11 with m = 12, u = 16 to build a {4}-GDD of type $36^4 72^1 48^1$. Adjoin a = 7 new points.
79	283	apply Lemma 3.11 with m = 12, u = 20 to build a {4}-GDD of type $36^4 72^1 60^1$. Adjoin a = 7 new points.
82	250	{4}-GDD of type $4^{42} 82^1$
82	262	apply Lemma 3.11 with m = 12, u = 12 to build a {4}-GDD of type $36^5 72^1$. Adjoin a = 10 new points.
82	274	apply Lemma 3.11 with m = 12, u = 16 to build a {4}-GDD of type $36^4 72^1 48^1$. Adjoin a = 10 new points.
82	286	apply Lemma 3.11 with m = 12, u = 20 to build a {4}-GDD of type $36^4 72^1 60^1$. Adjoin a = 10 new points.

So, we have proved

Lemma 8.18. *Suppose* $w \equiv 7$ *or* 10 *modulo 12,* $w \leq 82$, *and* $v \equiv w$ *modulo 12. Then there is a* $(v, w; \{4\})$*-IPBD.*

Finally, we consider the interval $3w + 1 < v < 4w - 30$, $v - w \equiv 0$ modulo 12, $w \geq 91$.

We use the following four corollaries to Lemma 3.11.

Lemma 8.19. *Suppose* $w \equiv 7$ *modulo 24,* $w \geq 55$, $w \not\equiv 175, 271,$ *or* 319, $v \equiv 7$ *or* 10 *modulo 12, and* $(7w - 35)/2 \leq v \leq 4w - 21$. *Then there is a* $v, w; \{4\})$*-IPBD.*

Proof: Apply Lemma 3.11 with $m = (w - 7)/6$ and $u = (v - 18m - 7)/3$. Then a TD$(6, m)$ exists, and $m \leq u < 2m$. This builds a $\{4\}$-GDD of type $(3m)^4\ (6m)^1\ (3u)^1$. Note that every group size is $\equiv 0$ or 3 modulo 12, so we can fill in groups with $a = 7$ new points. ∎

Similarly, we have the following three variations, using $a = 10, 19,$ and 22 new points.

Lemma 8.20. *Suppose* $w \equiv 10$ *modulo 24,* $w \geq 58$, $w \not\equiv 178, 274,$ *or* 322, $v \equiv 7$ *or* 10 *modulo 12, and* $(7w - 50)/2 \leq v \leq 4w - 30$. *Then there is a* $(v, w; \{4\})$*-IPBD.*

Lemma 8.21. *Suppose* $w \equiv 19$ *modulo 24,* $w \geq 115$, $w \not\equiv 187, 283,$ *or* 331, $v \equiv 7$ *or* 10 *modulo 12, and* $(7w - 95)/2 \leq v \leq 4w - 57$. *Then there is a* $(v, w; \{4\})$*-IPBD.*

Lemma 8.22. *Suppose* $w \equiv 22$ *modulo 24,* $w \geq 118$, $w \not\equiv 190, 286,$ *or* 334, $v \equiv 7$ *or* 10 *modulo 12, and* $(7w - 110)/2 \leq v \leq 4w - 66$. *Then there is a* $(v, w; \{4\})$*-IPBD.*

After application of Lemmata 4.8, 4.9, and 8.19 - 8.22, several cases remain. Some of these are disposed of in Table 9.

Table 9

w	v	construction
175	655, 667	apply Lemma 3.13 with m = 16, u = 28, u' = 16 and 18. Adjoin a = 7 new points.
178	670	apply Lemma 3.13 with m = 16, u = 28, u' = 18. Adjoin a = 10 new points.
271	1015, 1027, 1039, 1051	apply Lemma 3.13 with m = 24, u = 44, u' = 28, 30, 32, and 34. Adjoin a = 7 new points.
274	1030, 1042, 1054	apply Lemma 3.13 with m = 24, u = 44, u' = 30, 32, and 34. Adjoin a = 10 new points.
283	1063, 1075	apply Lemma 3.13 with m = 24, u = 46, u' = 34 and 36. Adjoin a = 7 new points.
286	1078	apply Lemma 3.13 with m = 24, u = 46, u' = 36. Adjoin a = 10 new points.
319	1195	apply SIP, 1195 = 4(304 – 7) + 7, 319 = 4(85 – 7) + 7
331	1243	apply SIP, 1243 = 4(316 – 7) + 7, 331 = 4(88 – 7) + 7

Lemma 8.23. *Suppose* $w = 319$ *and* $v = 1207, 1219, 1231$, *or* 1243; *or* $w = 322$ *and* $v = 1210, 1222, 1234$, *or* 1246. *Then there is a* $(v, w; \{4\})$ *-IPBD.*

Proof: If $w = 319$, apply Lemma 3.10 with $m = 80$ and $u = 56, 60, 64$, or 68. This constructs a $\{4\}$-GDD of type $240^4\ (3u)^1$. Then, adjoin $a = 79$ new points, filling in $(319, 79; \{4\})$-IPBDs and a $(3u + 79, 79; \{4\})$-IPBD. If $w = 322$, adjoin $a = 82$ new points. ▮

Lemma 8.24. *Suppose* $w = 331$ *and* $v = 1255$ *or* 1267; *or* $w = 334$ *and* $v = 1258$ *or* 1270. *Then there is a* $(v, w; \{4\})$ *-IPBD.*

Proof: If $w = 331$, apply Lemma 3.10 with $m = 84$ and $u = 56$ or 60. This constructs a $\{4\}$-GDD of type $252^4\ (3u)^1$. Then, adjoin $a = 79$ new points, filling in $(331, 79; \{4\})$-IPBDs and a $(3u + 79, 79; \{4\})$-IPBD. If $w = 334$, adjoin $a = 82$ new points. ▮

Also, for all $w \equiv 19$ modulo 24, we need to handle $v = 4w - 45$ and $4w - 33$; and for all $w \equiv 22$ modulo 24, we need to handle $v = 4w - 54$ and $4w - 42$.

Lemma 8.25. *Suppose* $w \equiv 19$ *modulo 24 and* $v = 4w - 45$; *or* $w \equiv 22$ *modulo 24,* $v = 4w - 54$, *and* $w \geq 67$. *Then there is a* $(v, w; \{4\})$ *-IPBD.*

Proof: For $w \equiv 19$ modulo 24, $w \leq 163$, we have $(15w - 17)/4 \geq 4w - 45$. Similarly, for $w \equiv 22$ modulo 24, $w \leq 190$, we have $(15w - 26)/4 \geq 4w - 54$. Hence, Lemma 4.8 or Lemma 4.9 applies in these cases. Hence, we can assume that $w \geq 187$. If $w \equiv 19$ modulo 24, apply Lemma 3.10 with $m = (w - 55)/3$ and $u = 40$. This constructs a $\{4\}$-GDD of type $(w - 55)^4\, 120^1$. Now, adjoin $a = 55$ new points, filling in $(w, 55; \{4\})$-IPBDs and a $(175, 55; \{4\})$-IPBD. If $w \equiv 22$ modulo 24, apply Lemma 3.10 with $m = (w - 58)/3$ and $u = 40$, and adjoin $a = 58$ new points. ▮

Lemma 8.26. *Suppose* $w \equiv 19$ *modulo 24 and* $v = 4w - 33$, *or* $w \equiv 22$ *modulo 24 and* $v = 4w - 42$, *and* $w \geq 91$. *Then there is a* $(v, w; \{4\})$ *-IPBD.*

Proof: For $w \equiv 19$ modulo 24, $w \leq 115$, we have $(15w - 17)/4 \geq 4w - 33$. Similarly, for $w \equiv 22$ modulo 24, $w \leq 142$, we have $(15w - 26)/4 \geq 4w - 42$. Hence, Lemma 4.8 or Lemma 4.9 applies in these cases. Hence, we can assume that $w \geq 139$. If $w \equiv 19$ modulo 24, apply Lemma 3.10 with $m = (w - 43)/3$ and $u = 32$. This constructs a $\{4\}$-GDD of type $(w - 43)^4\, 96^1$. Now, adjoin $a = 43$ new points, filling in $(w, 43; \{4\})$-IPBDs and a $(139, 43; \{4\})$-IPBD. If $w \equiv 22$ modulo 24, apply Lemma 3.10 with $m = (w - 46)/3$ and $u = 32$, and adjoin $a = 46$ new points. ▮

Now, summarizing previous results, we can prove

Lemma 8.27. *Suppose* $w \equiv 7$ *or* 10 *modulo 12,* $v \equiv w$ *modulo 12 and* $v > 3w$. *Then there is a* $(v, w; \{4\})$ *-IPBD.*

Proof: When $v \geq 4w - 30$, Lemma 8.17 applies, and when $w \leq 82$, Lemma 8.18 applies. Hence, assume $3w + 1 \leq v \leq 4w - 33$ and $w \geq 91$. These cases are covered by Lemmata 4.8, 4.9, 8.19 - 8.26, and Table 9. ∎

Theorem 8.28. *Suppose* $w \equiv 7$ *or* 10 *modulo 12,* $v \equiv 7$ *or* 10 *modulo 12 and* $v > 3w$. *Then there is a* $(v, w; \{4\})$ *-IPBD.*

Proof: If $v-w$ is even, apply Lemma 8.27. If $v-w$ is odd and $v \leq 9w+4$, Lemma 7.1 applies. If $v - w$ is odd and $v > 9w+4$, then there exists a $(3w+1, w; \{4\})$-IPBD by Lemma 4.10, and a $(v, 3w+1; \{4\})$-IPBD, since $v - (3w+1)$ is even. Hence, the desired $(v, w; \{4\})$-IPBD exists. ∎

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Appendix

A $\{4\}$-GDD of type $12^2 6^4$

points: $(\mathbf{Z}_{12} \times \{1,2,3\}) \cup \{a_0, a_1, b_0, b_1\} \cup \{\infty_i: 1 \le i \le 8\}$

groups: $\{\mathbf{Z}_{12} \times \{3\}\} \cup \{\{a_0, a_1, b_0, b_1\} \cup \{\infty_i: 1 \le i \le 8\}\} \cup$
$\{\{0+i, 2+i, 4+i, 6+i, 8+i, 10+i\} \times \{j\}: i=0,1, j=1,2\}$

blocks: develop the following modulo 12 (second coordinates are written as subscripts; develop subscripts on a and b modulo 2):

$\{0_1, 1_1, 0_2, 3_2\}$ $\{0_3, a_0, 0_1, 3_1\}$ $\{0_3, a_1, 0_2, 5_2\}$ $\{0_3, b_0, 1_1, 6_1\}$
$\{0_3, b_1, 1_2, 2_2\}$ $\{0_3, \infty_1, 2_1, 6_2\}$ $\{0_3, \infty_2, 4_1, 9_2\}$ $\{0_3, \infty_3, 5_1, 11_2\}$
$\{0_3, \infty_4, 7_1, 4_2\}$ $\{0_3, \infty_5, 8_1, 3_2\}$ $\{0_3, \infty_6, 9_1, 10_2\}$ $\{0_3, \infty_7, 10_1, 8_2\}$
$\{0_3, \infty_8, 11_1, 7_2\}$

A $\{4\}$-GDD of type $18^2 3^{12}$

points: $(\mathbf{Z}_{18} \times \{1,2,3\}) \cup \{a_0, a_1, a_2, b_0, b_1, c_0, c_1, d_0, d_1, e_0, e_1\} \cup$
$\{\infty_i: 1 \le i \le 7\}$

groups: $\{\mathbf{Z}_{18} \times \{3\}\} \cup \{\{a_0, a_1, a_2, b_0, b_1, c_0, c_1, d_0, d_1, e_0, e_1\} \cup$
$\{\infty_i: 1 \le i \le 7\}\} \cup \{\{0+i, 6+i, 12+i\} \times \{j\}: i=0,1,2,3,4,5,$
$j=1,2\}$

blocks: develop the following modulo 18 (second coordinates are written as subscripts; develop subscripts on a modulo 3, and on b, c, d, and e modulo 2):

$\{0_1, 9_1, 0_2, 9_2\}$ $\{0_1, 8_1, 1_2, 5_2\}$ $\{1_1, 5_1, 0_2, 8_2\}$ $\{0_3, a_0, 0_1, 6_2\}$
$\{0_3, a_1, 9_2, 11_2\}$ $\{0_3, a_2, 1_1, 3_1\}$ $\{0_3, b_0, 6_1, 9_1\}$ $\{0_3, b_1, 3_2, 8_2\}$
$\{0_3, c_0, 8_1, 15_1\}$ $\{0_3, c_1, 4_2, 15_2\}$ $\{0_3, d_0, 4_1, 17_1\}$ $\{0_3, d_1, 0_2, 17_2\}$
$\{0_3, e_0, 10_1, 11_1\}$ $\{0_3, e_1, 7_2, 10_2\}$ $\{0_3, \infty_1, 2_1, 16_2\}$ $\{0_3, \infty_2, 16_1, 2_2\}$
$\{0_3, \infty_3, 5_1, 13_2\}$ $\{0_3, \infty_4, 13_1, 5_2\}$ $\{0_3, \infty_5, 12_1, 14_2\}$ $\{0_3, \infty_6, 14_1, 12_2\}$
$\{0_3, \infty_7, 7_1, 1_2\}$

A $\{4\}$-GDD of type $6^5 12^1 15^1$

points: $(\mathbf{Z}_{15} \times \{1,2,3\}) \cup \{a_0, a_1, a_2, b_0, b_1, b_2\} \cup \{\infty_i: 1 \le i \le 6\}$

groups: $\{\mathbf{Z}_{15} \times \{3\}\} \cup \{\{a_0, a_1, a_2, b_0, b_1, b_2\} \cup \{\infty_i: 1 \le i \le 6\}\} \cup$
$\{\{0+i, 5+i, 10+i\} \times \{1,2\}: i=0,1,2,3,4\}$

blocks: develop the following modulo 15 (second coordinates are written as subscripts; develop subscripts on a and b modulo 3):

$\{0_1, 3_1, 1_2, 7_2\}$ $\{0_3, 0_1, 2_1, 6_1\}$ $\{0_3, 1_2, 2_2, 14_2\}$ $\{0_3, a_0, 8_1, 9_1\}$
$\{0_3, a_1, 3_2, 7_2\}$ $\{0_3, a_2, 3_1, 6_2\}$ $\{0_3, b_0, 4_1, 11_1\}$ $\{0_3, b_1, 0_2, 8_2\}$
$\{0_3, b_2, 14_1, 11_2\}$ $\{0_3, \infty_1, 1_1, 10_2\}$ $\{0_3, \infty_2, 5_1, 4_2\}$ $\{0_3, \infty_3, 7_1, 13_2\}$
$\{0_3, \infty_4, 10_1, 12_2\}$ $\{0_3, \infty_5, 12_1, 5_2\}$ $\{0_3, \infty_6, 13_1, 9_2\}$