On the existence of Kirkman triple systems containing Kirkman subsystems

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Abstract. The obvious necessary conditions for the existence of a Kirkman triple system of order $v$ containing a Kirkman subsystem of order $w$ are $v \equiv w \equiv 3 \pmod{6}$, $v \geq 3w$. We show that these conditions are sufficient.

1. Introduction

A pairwise balanced design (or, PBD) is a pair $(X, A)$, such that $X$ is a set of elements (called points) and $A$ is a set of subsets of $X$ (called blocks), such that every unordered pair of points is contained in a unique block of $A$. If $v$ is a positive integer and $K$ is a set of positive integers, then we say that $(X, A)$ is a $(v, K)$-PBD if $|X| = v$, and $|A| \in K$ for every $A \in A$. The integer $v$ is called the order of the PBD.

Using this notation, we can define a Steiner triple system of order $v,$ which we denote STS($v$), to be a $(v, \{3\})$-PBD. It is of course well-known that an STS($v$) exists if and only if $v \equiv 1$ or $3 \pmod{6}$.

Let $(X, A)$ be a PBD. If a set of points $Y \subseteq X$ has the property that, for any $A \in A$, either $|Y \cap A| \leq 1$ or $A \subseteq Y$, then we say that $Y$ is a subdesign or flat of the PBD. The order of the subdesign is $|Y|$. The subdesign $Y$ is proper if $Y \neq X$. If $Y$ is a subdesign, then we can delete all blocks $A \subseteq Y$ and replace them by a single block, $Y$, and the result is a PBD. Also, any block or point of a PBD is itself a subdesign.

The problem of constructing Steiner triple systems containing subsystems was studied by Doyen and Wilson in [2]. The obvious necessary conditions for the existence of an STS($v$) containing an STS($w$) as a subsystem are $v \geq 2w + 1$, $v \equiv 1$ or $3 \pmod{6}$, $w \equiv 1$ or $3 \pmod{6}$. In [2], it is shown that these necessary conditions are sufficient.

A parallel class in a PBD is a set of blocks that form a partition of the point set. A PBD is resolvable if the block set can be partitioned into parallel classes. A Kirkman triple system of order $v$, or KTS($v$), is defined to be a resolvable STS($v$). In [14], Ray-Chaudhuri and Wilson showed that there exists a KTS($v$) if and only if $v \equiv 3 \pmod{6}$.

In this paper, we are interested in KTS($v$) which contain KTS($w$) as subsystems. We say that a KTS($w$) is a subsystem of a KTS($v$) only if the parallel classes of the KTS($w$) are induced by the parallel classes of the KTS($v$). We shall describe the subsystem as a sub-KTS($w$). The obvious necessary conditions for the existence of a KTS($v$) containing a sub-KTS($w$) is $v \geq 3w$, $v \equiv w \equiv 3 \pmod{6}$. This problem has been studied in several recent papers, and the following results have been proved.

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Theorem 1.1 [15, 12]. For all \( v \equiv w \equiv 3 \mod 6, v \geq 4w - 9 \), there exists a KTS(\( v \)) containing a sub-KTS(\( w \)).

Theorem 1.2 [11]. For all \( v \equiv 3 \mod 6, v = 3w, 3w + 6 \) or \( 3w + 12 \), there exists a KTS(\( v \)) containing a sub-KTS(\( w \)), except possibly when \( w = 45, 51, 63, \) or \( 87 \) and \( v = 3w + 12 \).

Theorem 1.3 [12]. Suppose \( v \equiv w \equiv 3 \mod 6, v \geq 3w \), and \( v - w = 12s + 6 \) or \( 12s + 12 \), where \( s \in \{0, 1, 2, 3, 4, 5, 6, 7, 20, 24, 25, 28, 29, 30, 31, 36, 40, 44, 45, 52, 59, 60, 63, 64, 65\} \) or \( s \geq 68 \). Then there exists a KTS(\( v \)) containing a sub-KTS(\( w \)).

There are precisely 884 ordered pairs \((v, w)\), where \( v \equiv w \equiv 3 \mod 6 \) and \( v \geq 3w \), which are not covered by any of the three theorems above. These are listed in the Appendix. In this paper, we eliminate all of these possible exceptions.

2. Constructions for Kirkman triple systems containing subsystems

We need to define several types of designs. First, we define a useful generalization of a PBD called a group-divisible design. A group-divisible design (or, GDD), is a triple \((X, G, A)\), which satisfies the following properties:

1. \( G \) is a partition of \( X \) into subsets called groups;
2. \( A \) is a set of subsets of \( X \) (called blocks) such that a group and a block contain at most one common point;
3. every pair of points from distinct groups occurs in a unique block.

The group-type (or type) of a GDD \((X, G, A)\) is the multiset \([|G| : G \in G]\).

We usually use an “exponential” notation to describe group-types: a group-type \(1^{i}2^{j}3^{k} \ldots\) denotes \(i\) occurrences of 1, \(j\) occurrences of 2, etc. As with PBDs, we will say that a GDD is a K-GDD if \(|A| \in K\) for every \(A \in A\). As well, we say that a GDD is resolvable if the blocks can be partitioned into parallel classes.

Now, we define the idea of a GDD with a hole. Informally, an incomplete GDD, or IGDD, is a GDD from which a sub-GDD is missing (this is the “hole”). We give a formal definition. An IGDD is a quadruple \((X, Y, G, A)\) which satisfies the following properties:

1. \(X\) is a set of points, and \(Y \subseteq X\);
2. \(G\) is a partition of \(X\) into groups;
3. \(A\) is a set of blocks, each of which intersects each group in at most one point;
4. no block contains two members of \(Y\);
5. every pair of points \(\{x, y\}\) from distinct groups, such that at least one of \(x, y\) is in \(X\setminus Y\), occurs in a unique block of \(A\).

We say that an IGDD \((X, Y, G, A)\) is a K-IGDD if \(|A| \in K\) for every block \(A \in A\). The type of the IGDD is defined to be the multiset of ordered pairs
\{(|G|, |G \cap Y|): G \in \mathcal{G}\}. As with GDDs, we shall use an exponential notation to describe types. Note that if \( Y = \emptyset \), then the IGDD is a GDD.

We have already defined PBDs with subdesigns. If we allow the subdesign to be missing (i.e., a hole), we have an incomplete PBD, as follows. An incomplete PBD (or IPBD) is a triple \((X, Y, A)\), where \( X \) is a set of points, \( Y \subseteq X \), and \( A \) is a set of blocks which satisfies the following properties:

1. for any \( A \in A \), \(|A \cap Y| \leq 1\);
2. any two points \( x, y \), not both in \( Y \), occur in a unique block.

Hence, \( Y \) is the hole. Note that \((X, Y, A)\) is an IPBD if and only if \((X, A \cup \{Y\})\) is a PBD. We say that \((X, Y, A)\) is a \((v, w; K)\)-IPBD if \(|X| = v, |Y| = w\), and \(|A| \in K\) for every \( A \in A \).

We also employ a more general type of incomplete PBD. We are interested in the situation when we have two subdesigns, of given sizes, which intersect in a third subdesign of a given size. However, as usual, the subdesigns need not be present, i.e., we allow holes. We will refer to these designs as \(\omega\)-IPBDs, in order to suggest the structure of the holes. We give a formal definition. An incomplete \(\omega\)-PBD is a tuple \((X, Y_1, Y_2, A)\), where \(Y_1 \subseteq X, Y_2 \subseteq X\), and \(A\) is a set of blocks such that every pair of points \(\{x, y\}\) occurs in a unique block, unless \(\{x, y\} \subseteq Y_1\) or \(\{x, y\} \subseteq Y_2\), in which case the pair occurs in no block. We say that the \(\omega\)-IPBD is a \((v; w_1, w_2; w_3; K)\)-\(\omega\)-IPBD if \(|X| = v, |Y_1| = w_1, |Y_2| = w_2, |Y_1 \cap Y_2| = w_3\), and \(|A| \in K\) for every \(A \in A\).

Our main application of \(\omega\)-IPBDs involves using them to fill in the groups of IGDDs. This construction was presented in [17].

Construction 2.1 Filling in groups Let \(K\) be a set of positive integers, and let \(ba0\). Suppose that the following designs exist:

1. a \(K\)-IGDD of type \(\{(t_1, u_1), (t_2, u_2), \ldots, (t_n, u_n)\}\);
2. a \((t_i + b; u_i + a, b; a; K)\)-\(\omega\)-IPBD, for \(1 \leq i \leq n - 1\); and
3. a \((t_n + b; u_n + a)\)-IPBD.

Then there exists a \((t + b, u + a; K)\)-IPBD, where \(t = \sum t_i\) and \(u = \sum u_i\).

In this paper, we also make extensive use of Kirkman triple systems with holes of various types. We refer to these as incomplete KTS, or IKTS. If we have a KTS(\(v\)) containing a sub-KTS(\(w\)), we can remove the subsystem, leaving a hole. We shall denote the resulting incomplete system by \((v, w)\)-IKTS. (Of course, if we have a \((v, w)\)-IKTS, where \(v \equiv w \equiv 3\) modulo 6, then we can fill a KTS(\(w\)) into the hole, constructing a KTS(\(v\)) containing a sub-KTS(\(w\)).) Next, suppose we have a KTS(\(v\)) which contains sub-KTS(\(w_1\)) and sub-KTS(\(w_2\)) which intersect in a sub-KTS(\(w_3\)). If we remove these subsystems, we obtain an incomplete system which we denote by \((v; w_1, w_2; w_3)\)-\(\omega\)-IKTS.

We also make extensive use of an object which can be thought of as a resolvable GDD having a spanning set of holes. A \(K\)-frame is a \(K\)-GDD \((X, \mathcal{H}, A)\), in which the set of blocks \(A\) can be partitioned into holey parallel classes, each of
which is a partition of $X/H$, for some $H \in \mathcal{H}$. The members of $\mathcal{H}$ are called holes. We refer to a $\{3\}$-frame as a Kirkman frame.

As before, the type of a frame $(X, \mathcal{H}, A)$ is defined to be the multiset $\{|H|: H \in \mathcal{H}\}$. It can be shown (see [15] and [9]) that for each hole $H$ of a $\{k\}$-frame, there are $|H|/(k - 1)$ holey parallel classes which partition $X \setminus H$. It is also interesting to note that a $\{k\}$-frame of type $\{t_1, t_2, \ldots, t_n\}$ is equivalent to a $\{k + 1\}$-IGDD of type $\{(kt_1/(k - 1), t_1/(k - 1)), (kt_2/(k - 1), t_2/(k - 1)), \ldots, (kt_n/(k - 1), t_n/(k - 1))\}$.

If we fill in the holes of Kirkman frames, we can construct KTS with sub-KTS, as follows.

**Construction 2.2 Filling In Holes** Let $a \geq 0$, and suppose that there exists a Kirkman frame of type $\{t_1, t_2, \ldots, t_n\}$; a KTS($t_i + a$) containing a sub-KTS($a$), for $1 \leq i \leq n - 1$; and a KTS($t_n + a$). Then there exists a KTS($t + a$), where $t = \sum t_i$, containing a sub-KTS($t_n + a$).

We also use incomplete (Kirkman) frames, which bear the same relationship to Kirkman frames as IGDDs do to GDDs. An incomplete $K$-frame is a $K$-IGDD $(X, Y, \mathcal{H}, A)$ in which the set of blocks $A$ can be partitioned into holey parallel classes, each of which is a partition of $X/H$, for some $H \in \mathcal{H}$, or a partition of $X \setminus (H \cup Y)$, for some $H \in \mathcal{H}$. It can be shown that for each hole $H$, there are $|H \cap Y|/2$ holey parallel classes which partition $X \setminus (H \cup Y)$, and $|H \setminus Y|/2$ holey parallel classes which partition $X \setminus H$.

We also construct KTS containing sub-KTS by filling in the holes of incomplete Kirkman frames with $o$-IKTS.

**Construction 2.3 Generalized Filling In Holes** Let $b \geq a \geq 0$. Suppose that the following designs exist:

1. an incomplete Kirkman frame of type $\{(t_1, u_1), (t_2, u_2), \ldots, (t_n, u_n)\}$;
2. a $(t_i + b; u_i + a, b; a)$-$o$-IKTS, for $1 \leq i \leq n - 1$; and
3. a $(t_n + b; u_n + a; a)$-IKTS.

Then there exists a $(t + b; u + a)$-IKTS, where $t = \sum t_i$ and $u = \sum u_i$.

We also observe that if we fill in all but one group of a Kirkman frame (Construction 2.2), we obtain an IKTS, and if we fill in all but two groups, we obtain a $o$-IKTS.

It will be necessary to build families of IGDDs. Our basic construction for IGDDs is a recursive one. We refer to it as the “Fundamental IGDD Construction” (see [8] and [11]).

**Construction 2.4 Fundamental IGDD Construction** Suppose $(X, Y, \mathcal{G}, A)$ is an IGDD, and let $t, s: X \rightarrow Z^+ \cup \{0\}$ be functions such that $t(x) \leq s(x)$, for every $x \in X$. For every block $A \in A$, suppose that we have a $K$-IGDD of type $\{(s(x), t(x)): x \in A\}$. Suppose also that we have a $K$-IGDD of type
\{(\sum_{x \in \mathbb{G}} s(x), \sum_{x \in \mathbb{G}} t(x)) : G \in \mathcal{G}\}. Then there exists a \(K\)-IGDD of type \\
\{(\sum_{x \in \mathbb{G}} s(x), \sum_{x \in \mathbb{G}} t(x)) : G \in \mathcal{G}\}.

We use a similar IGDD construction to build frames and incomplete frames (see [15]).

**Construction 2.5 Fundamental Frame Construction** Suppose \((X, Y, \mathcal{G}, A)\) is an IGDD, and let \(s : X \rightarrow \mathbb{Z}_+ \cup \{0\}\) be a function. For every block \(A \in A\), suppose that we have a Kirkman frame of type \(\{(s(x) : x \in A)\}\). Then there exists an incomplete Kirkman frame of type \(\{(\sum_{x \in \mathbb{G}} s(x), \sum_{x \in \mathbb{G}} t(x)) : G \in \mathcal{G}\}\).

As applications of the above, we mention a family of constructions which are called the product constructions. These utilize (incomplete) transversal designs, which we now define. A transversal design \(TD(k, n)\) is a \(k\)-GDD of type \(n^k\). It is well-known that a \(TD(k, n)\) is equivalent to \(k - 2\) mutually orthogonal Latin squares (MOLS) of order \(n\). We also define a \(TD(k, n) - TD(k, m)\) (an incomplete transversal design) to be a \(k\)-IGDD of group-type \((n, m)^k\).

The most general product construction is referred to as the **Generalized Singular Indirect Product**, or GSIP.

**Construction 2.6 Generalized Singular Indirect Product** Suppose \(u, t, v, w, a, \text{ and } b\) are non-negative integers such that \(0 \leq b - a \leq v - w, a \leq w \leq v\). Suppose that the following designs exist:

1. a Kirkman frame of type \(tu\);
2. a \(TD(u, (v - b) / t) - TD(u, (w - a) / t)\);
3. a \((v; w, b; a)\)-IKTS; and
4. a \((b, a)\)-IKTS.

Then there exists a \((u(v - b) + b, u(w - a) + a)\)-IKTS.

**Proof:** Start with the given incomplete TD, give every point weight \(t\), and apply the Fundamental Frame Construction. We get an incomplete Kirkman frame of type \((v - b, w - a)^u\). Now fill in the holes. \(\blacksquare\)

When \(b = a\), we obtain the **Singular Indirect Product**, or SIP.

**Construction 2.7 Singular Indirect Product** Suppose \(u, t, v, w, \text{ and } a\) are non-negative integers such that \(0 \leq a \leq w \leq v\). Suppose that the following designs exist:

1. a Kirkman frame of type \(tu\);
2. a \(TD(u, (v - a) / t) - TD(u, (w - a) / t)\); and
3. a \((v; w)\)-IKTS.

Then there exists a \((u(v - a) + a, u(w - a) + a)\)-IKTS.

When \(b = a = w\), we obtain the **Singular Direct Product**, or SDP.

**Construction 2.8 Singular Direct Product** Suppose \(u, t, v\) and \(w\) are non-negative integers such that \(w \leq v\). Suppose that the following designs exist:

1. a Kirkman frame of type \(tu\);
(2) a TD\((u, (v - w)/t); \)
(3) a \((v, w)\)-IKTS; and
(4) a KTS\((u)\).

Then, there is a \((u(v - w) + w, w)\)-IKTS and a \((u(v - w) + w, v)\)-IKTS.

3. Applications of the constructions

We now describe two recursive constructions for producing Kirkman triple systems containing Kirkman subsystems, which will eliminate all but 31 of the 884 exceptions. These constructions will make use of Kirkman frames constructed in [15].

Lemma 3.1. There exists a Kirkman frame of type \(t^u\) if and only if \(t\) is even, \(u \geq 4\), and \(t(u - 1) \equiv 0 \mod 3\).

Proof: See [15, Theorem 4.5].

We shall use certain incomplete TDs.

Lemma 3.2. For all positive integers \(v\) and \(w\) such that \(v \geq 3w\) and \((v, w) \neq (6, 1)\), there is a TD\((4, v) - TD(4, w)\).

Proof: See [4].

We shall also require some particular classes of \(\alpha\)-IKTS.

Lemma 3.3. For all \(m \geq 0\), there exists a \((18m + 9; 6m + 3, 9; 3)\)-IKTS.

Proof: For \(m = 0\), the design exists trivially. For \(m > 0\), start with a TD\((4, 6m + 3) - TD(4, 3)\) (Lemma 3.2). Delete all the points in one group. Then, on two of the remaining groups, fill in \((6m + 3, 3)\)-IKTS.

Lemma 3.4. For all \(m \geq 2\), there exists a \((18m + 15; 6m + 3, 15; 3)\)-IKTS.

Proof: For \(m = 2\), use a Kirkman frame of type \(12^4\), filling in two holes with \((15, 3)\)-IKTS. For \(m \geq 3\), start with a resolvable \(3\)-GDD of type \(6^{m+1}\) (see [10] and [1]). Adjoin infinite points to the parallel classes of this GDD, to construct a \(4\)-GDD of type \(6^{m+1}(3m)\). Give every point weight 2 and apply the Fundamental Frame Construction. This produces a frame of type \(12^{m+1}(6m)\). Now fill in all but one hole of size 12 with \((15, 3)\)-IKTS.

Construction 3.1 Suppose there exists a TD\((6, m)\), \(0 \leq t \leq m\), and \(0 \leq u \leq m\). Let \(a \geq 0\). Suppose there exist KTS\((6m + a)\) and KTS\((6m + 6t + a)\), each containing a KTS\((a)\), and a KTS\((6m + 6u + a)\). Then there exists a KTS\((36m + 6t + 6u + a)\) containing a sub-KTS\((6m + 6u + a)\).

Proof: Start with a TD\((6, m)\), and give the points in the first four groups weight 6; give \(t\) of the points in the 5th group weight 12, and give the remaining points in the 5th group weight 6; and give \(u\) of the points in the 6th group weight 12,
and give the remaining points in the 6th group weight 6. In order to apply the Fundamental Frame Construction, we need Kirkman frames of types $6^6$, $6^512^1$ and $6^412^2$, which are obtained as follows. A frame of type $6^6$ exists by Lemma 3.1. We get a frame of type $6^512^1$ by applying the Fundamental Construction to a $\{4\}$-GDD of type $3^56^1$, giving every point weight 2 (this GDD is produced by adjoining infinite points to 6 parallel classes of a KTS(15)). Similarly, we get a frame of type $6^412^2$ by applying the Fundamental Construction to a $\{4\}$-GDD of type $3^46^2$, giving every point weight 2 (this GDD is exhibited in the Appendix of [12]). Hence, we build a Kirkman frame of type $(6m)^4(6m+6t)^1(6m+6u)^1$ (for any $0 \leq t \leq m$, $0 \leq u \leq m$). We now add an infinite points, and fill in KTS containing sub-KTS(a), and the KTS$(6m+6u+a)$.  

**Construction 3.2** Suppose there is a TD$(6,m)$, $0 \leq t \leq m$, $0 \leq u \leq m$, and $a = 3$ or 6. Then there exists a KTS$(72m+18t+12u+2a+3)$ containing a sub-KTS$(24m+6t+3)$. 

**Proof:** Start with a TD$(6,m)$, delete $m-t$ points from the 5th group, and delete $m-u$ points from the 6th group. Then, give the points in the first five groups weights $(9, 3)$, and give the points in the 6th group weights $(6, 0)$. In order to apply the Fundamental IGDD construction, we need $\{4\}$-IGDDs of types $(9, 3)^4$, $(9, 3)^5$, $(9, 3)^46^1$, and $(9, 3)^56^1$. The first two IGDDs are equivalent to frames of types $6^4$ and $6^5$ respectively (see the remark preceding Construction 2.2). The last two IGDDs are presented in the Appendix of [11]. Then, we obtain a $\{4\}$-IGDD of type $(9m, 3m)^4(9t, 3t)^1(6u)^1$. Next, assign every point weight 2, and apply the Fundamental Frame Construction. This produces an incomplete frame of type $(18m, 6m)^4(18t, 6t)^1(12u)^1$. Next, we will fill $\circ$-IKTS into the holes of the frame, using Construction 2.3. We adjoin a total of $2a+3$ points, 3 of which are incorporated into the sub-KTS.

If $a = 3$, then we fill in $(18m+9; 6m+3, 9, 3)\sim-IKTS$, $(18t+9; 6t+3, 9, 3)\sim-IKTS$, and $(12u+9, 3)\sim-IKTS$. These exist from Lemma 3.3.

If $a = 6, t > 1$, then we fill in $(18m+15; 6m+3, 15, 3)\sim-IKTS$, $(18t+15; 6t+3, 15, 3)\sim-IKTS$, and $(12u+15, 3)\sim-IKTS$. These exist from Lemma 3.4.

If $a = 6, t = 1, u \neq 1, 2$, then we instead use $(18m+15; 6m+3, 15, 3)\sim-IKTS$, $(12u+15; 3, 15, 3)\sim-IKTS$, and $(18t+15, 9)\sim-IKTS$.

If $a = 6, t = 1, u = 1$ or 2, we proceed slightly differently. We start with a TD$(5,m)$ and delete $m-u-1$ points from the 5th group. Give all points weight $(9, 3)$, except for $u$ points in the fifth group, which get weights $(6, 0)$. Proceeding as before, we obtain an incomplete Kirkman frame of type $(18m, 6m)^4(12u+18, 6)^1$. Now, fill in $(18m+15; 6m+3, 15, 3)\sim-IKTS$, and $(12u+33, 9)\sim-IKTS$.

This covers all cases, so the proof is complete.  

By computer, we established that Constructions 3.1 and 3.2 eliminate all but 31 ordered pairs \((v, w)\), which are presented in Table 1. Appropriate applications of Constructions 3.1 and 3.2 for the remaining 853 ordered pairs are given in the research report [13].

<table>
<thead>
<tr>
<th>Table 1</th>
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<tbody>
<tr>
<td>The 31 exceptions remaining after application of Constructions 3.1 and 3.2</td>
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<tr>
<td>--------------------------------------------------</td>
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<tr>
<td>(141, 39)</td>
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<tr>
<td>(165, 51)</td>
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<tr>
<td>(189, 57)</td>
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<tr>
<td>(255, 69)</td>
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<tr>
<td>(279, 75)</td>
</tr>
<tr>
<td>(285, 81)</td>
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<td>(297, 93)</td>
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</table>

15 of the exceptions in Table 1 can be eliminated by the product constructions. We list these in Table 2. In all applications, \(t = 2\) and \(u = 4\), so we are using Kirkman frames of type \(2^4\). The requisite incomplete TDs exist by Lemma 3.2.

At this point, 16 ordered pairs remain as possible exceptions. We eliminate most of these using a well-known PBD construction. First, we define the set \(K_{1,3} = \{k \equiv 1 \mod 3, k \geq 4\}\).

**Lemma 3.5.** Suppose there is a \((v, w; K_{1,3})\)-IPBD. Then there is a \((2v + 1, 2w + 1, 2^4)\)-IKTS.

**Proof:** The IPBD gives rise to a GDD of type \(w^1 1^v w\). Give every point weight 2, and apply the Fundamental Frame Construction. This produces a frame of type \((2w)^1 2^w 1^v w\). Now fill KTS(3) into the holes of size 2. 

**Lemma 3.6.** Suppose there is a resolvable \(\{4\}\)-GDD of type \(t^u\), where \(t \equiv 0 \mod 3\), and let \(0 \leq s \leq (t(u - 1))/3\). Then there is a \((6tu + 6s + 9, 2tu + 3)\)-IKTS.

**Proof:** Adjoin infinite points to \(s\) of the parallel classes of the GDD. This produces a \(\{4, 5\}\)-GDD of type \(t^u 1^s\) in which every block of size 5 hits the group of size \(s\). Assign weights \((3, 1)\) to every point of the original GDD, and assign weights \((3, 0)\) to the \(s\) infinite points. Apply the Fundamental IGDD construction, using \(\{4\}\)-IGDDs of types \((3, 1)^4 (3, 0)^1\) and \((3, 1)^4\). (These arise from deleting a block from \(\{4\}\)-GDDs of types \(3^4\) and \(3^5\), respectively.) We construct in this way a \(\{4\}\)-IGDD of type \((3t, t)^u (3a)^1\).

Next, we will construct in the groups of this IGDD with \(\omega\)-IPBDS using Construction 2.1. Set \(a = 1\) and \(b = 4\). We use \((3t + 4; t + 1, 4; 1; K_{1,3})\)-IPBDS, which are constructed by adjoining \(t + 1\) infinite points to the parallel classes of a KTS(2t + 3). For the last group, we use a block of size \(3s + 4\). This gives us a \((3tu + 3s + 4, tu + 1; K_{1,3})\)-IPBD. Now, apply Lemma 3.5.
Table 2

<table>
<thead>
<tr>
<th>u</th>
<th>w</th>
<th>construction</th>
<th>ingredients</th>
<th>remarks</th>
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</thead>
<tbody>
<tr>
<td>153</td>
<td>45</td>
<td>SDP</td>
<td>153 = 4(45 - 9) + 9</td>
<td>(45, 9)-IKTS</td>
</tr>
<tr>
<td>165</td>
<td>45</td>
<td>SIP</td>
<td>165 = 4(45 - 5) + 5</td>
<td>TD(4, 20) - TD(4, 5)</td>
</tr>
<tr>
<td>171</td>
<td>51</td>
<td>SIP</td>
<td>171 = 4(45 - 3) + 3</td>
<td>TD(4, 21) - TD(4, 6)</td>
</tr>
<tr>
<td>177</td>
<td>51</td>
<td>SDP</td>
<td>177 = 4(51 - 9) + 9</td>
<td>(51, 9) - IKTS</td>
</tr>
<tr>
<td>183</td>
<td>51</td>
<td>GSI</td>
<td>183 = 4(57 - 15) + 15</td>
<td>(57, 15, 15; 3)-o-IKTS</td>
</tr>
<tr>
<td>201</td>
<td>57</td>
<td>SDP</td>
<td>201 = 4(57 - 9) + 9</td>
<td>TD(4, 21) - TD(4, 6)</td>
</tr>
<tr>
<td>207</td>
<td>63</td>
<td>SDP</td>
<td>207 = 4(63 - 15) + 15</td>
<td>TD(4, 21) - TD(4, 6)</td>
</tr>
<tr>
<td>261</td>
<td>69</td>
<td>SIP</td>
<td>261 = 4(69 - 5) + 5</td>
<td>TD(4, 21) - TD(4, 6)</td>
</tr>
<tr>
<td>267</td>
<td>75</td>
<td>SIP</td>
<td>267 = 4(69 - 3) + 3</td>
<td>TD(4, 21) - TD(4, 6)</td>
</tr>
<tr>
<td>273</td>
<td>75</td>
<td>SDP</td>
<td>273 = 4(75 - 9) + 9</td>
<td>TD(4, 21) - TD(4, 6)</td>
</tr>
<tr>
<td>279</td>
<td>75</td>
<td>GSI</td>
<td>279 = 4(81 - 15) + 15</td>
<td>(81, 15, 21; 3)-o-IKTS</td>
</tr>
<tr>
<td>285</td>
<td>87</td>
<td>SDP</td>
<td>285 = 4(87 - 21) + 21</td>
<td>(87, 21)-IKTS</td>
</tr>
</tbody>
</table>

In a similar fashion, we have

**Lemma 3.7.** Suppose there is a resolvable \( \{4\}\)-GDD of type \( t^u \), where \( t \equiv 1 \mod{3} \), and let \( 0 \leq s \leq t(u - 1)/3 \). Then there is a \( (6tu + 6s + 3, 2tu + 1) \)-IKTS.

Proof: As before, construct a \( \{4\}\)-IGDD of type \( (3t, t)^u(3s) \). Then, set \( a = 0 \) and \( b = 1 \). We fill in \( (3t + 1, t; K_{1,3}) \)-IPBDs (which are constructed by adjoining \( t \) infinite points to the parallel classes of a KTS\( (2t + 1) \)) and a block of size \( 3s + 1 \) (if \( s > 0 \)). We get a \( (3tu + 3s + 1, tu; K_{1,3}) \)-IPBD. Now, apply Lemma 3.5. A slightly different application of the same idea gives us
Lemma 3.8. Suppose there is a \(\{4\}\)-frame of type \(t^n\), where \(t \equiv 0 \mod 3\), and \(0 \leq s \leq t/3\). Also, suppose there is a \((3t + 3s + 4, t + 1; K_{1,3})\)-IPBD. Then there exists a \((6tu + 6s + 9, 2tu + 3)\)-JKTS.

Proof: Adjoin \(s\) points to a hole of the frame, constructing a \(\{4, 5\}\)-IGDD of type \(tu^{-1}(t + s, s)\). Assign weights \((3, 1)\) to every point of the original GDD, and assign weights \((3, 0)\) to the \(s\) infinite points. Apply the Fundamental IGDD Construction, resulting in a \(\{4\}\)-IGDD of type \((3t, t)^{u-1}(3t + 3s, 3t)\). Then, set \(a = 1\) and \(b = 4\). We use \((3t + 4; t + 1, 4; 1; K_{1,3})\)-IPBDs, which are constructed as in Lemma 3.6. For the last group, we use the \((3t + 3s + 4, t + 1; K_{1,3})\)-IPBD. This gives us a \((3tu + 3s + 4, tu + 1; K_{1,3})\)-IPBD. Apply Lemma 3.5.

We list several applications of these constructions in Table 3.

<table>
<thead>
<tr>
<th>(u)</th>
<th>(w)</th>
<th>(v)</th>
<th>(w)</th>
<th>(\text{ingredients})</th>
<th>(\text{remarks})</th>
</tr>
</thead>
<tbody>
<tr>
<td>141</td>
<td>39</td>
<td>3.5</td>
<td></td>
<td>((70, 19; {4}))-IPBD</td>
<td>see [7]</td>
</tr>
<tr>
<td>147</td>
<td>45</td>
<td>3.6</td>
<td>(t = 3, u = 8, s = 2)</td>
<td>((73, 22; {4, 7, 10}))-IPBD</td>
<td>see Appendix</td>
</tr>
<tr>
<td>159</td>
<td>45</td>
<td>3.5</td>
<td></td>
<td>((79, 22; {4}))-IPBD</td>
<td>see [6]</td>
</tr>
<tr>
<td>165</td>
<td>51</td>
<td>3.6</td>
<td>(t = 3, u = 8, s = 2)</td>
<td>resolvable ({4})-GDD of type (3^8)</td>
<td>see [5]</td>
</tr>
<tr>
<td>189</td>
<td>51</td>
<td>3.6</td>
<td>(t = 3, u = 8, s = 6)</td>
<td>resolvable ({4})-GDD of type (3^8)</td>
<td>see [5]</td>
</tr>
<tr>
<td>189</td>
<td>57</td>
<td>3.7</td>
<td>(t = 4, u = 7, s = 3)</td>
<td>resolvable ({4})-GDD of type (4^7)</td>
<td>see [3]</td>
</tr>
<tr>
<td>195</td>
<td>57</td>
<td>3.7</td>
<td>(t = 4, u = 7, s = 4)</td>
<td>resolvable ({4})-GDD of type (4^7)</td>
<td>see [3]</td>
</tr>
<tr>
<td>201</td>
<td>63</td>
<td>3.8</td>
<td>(t = 6, u = 5, s = 2)</td>
<td>({4})-frame of type (6^5)</td>
<td>see [16]</td>
</tr>
<tr>
<td>261</td>
<td>75</td>
<td>3.6</td>
<td>(t = 9, u = 4, s = 6)</td>
<td>resolvable TD((4, 9))</td>
<td></td>
</tr>
<tr>
<td>267</td>
<td>81</td>
<td>3.7</td>
<td>(t = 4, u = 10, s = 4)</td>
<td>resolvable ({4})-GDD of type (4^{10})</td>
<td>see [3]</td>
</tr>
<tr>
<td>285</td>
<td>81</td>
<td>3.7</td>
<td>(t = 4, u = 10, s = 7)</td>
<td>resolvable ({4})-GDD of type (4^{10})</td>
<td>see [3]</td>
</tr>
<tr>
<td>297</td>
<td>93</td>
<td>3.8</td>
<td>(t = 9, u = 5, s = 3)</td>
<td>({4})-frame of type (9^5)</td>
<td>see [16]</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>TD((4, 7))</td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td>TD((4, 9))</td>
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<td></td>
<td>TD((4, 10))</td>
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<td>TD((4, 7))</td>
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<td>TD((4, 9))</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>TD((4, 10))</td>
<td></td>
</tr>
</tbody>
</table>

We can eliminate the four remaining exceptions by ad hoc means.

Lemma 3.9. There is a KTS\((279)\) containing a sub-KTS\((87)\).

Proof: Start with a \(\{4\}\)-GDD of type \(6^49^1\) (this is obtained by adjoining infinite points to the 9 parallel classes of a resolvable \(\{3\}\)-GDD of type \(6^4\), which is constructed in [10]). Give every point weight 8, and apply the frame construction. This produces a frame of type \(48^472^1\). Now adjoin 15 infinite points, filling in \((63, 15)\)-IKTS and KTS\((87)\).

Lemma 3.10. There is a KTS\((255)\) containing a sub-KTS\((69)\).

Proof: Start with a TD\((5, 4)\). Give every point weight 12, except for one point which gets weight 6, and apply the frame construction. We fill in frames of type \(12^5\) and \(12^46^1\) (this latter frame is obtained by applying the frame construction to
a \{4\}-GDD of type 6^4 3^1, giving the points weight 2). This produces a frame of type 48^4 42^1. Now adjoin 21 infinite points, filling in (69,21)-IKTS, (63,21)-IKTS, and KTS(69).

Lemma 3.11. There is a KTS(273) containing a sub-KTS(87).

Proof: A \{4\}-IGDD of type (9,3)^4 6^1 is given in the Appendix of [11]. The design is presented as a \{3,4\}-GDD of type 6^4 whose blocks of size three can be partitioned into twelve holey parallel classes. Three of these holey parallel classes correspond to each of the first four groups. It is easy to verify that the blocks of size four can be partitioned into two holey parallel classes having the fifth group as their hole. Hence, we can construct from this design a \{4,5\}-IGDD of type (9,3)^4(8,2)^1. Now, apply Construction 2.5, giving every point weight 6. We obtain a Kirkman frame of type (54,18)^4(48,12)^1. Then apply Construction 2.3 with b = 9 and a = 3, filling in (63;21,9;3)-o-IKTS (which is obtained by applying Lemma 3.5 to a (31,10;\{4\})-IPBD) and a (57,15)-IKTS.

Lemma 3.12. There is a KTS(291) containing a sub-KTS(87).

Proof: Remove a point from the hole of the (73,22;\{4,7,10\})-IPBD given in the appendix, producing a \{4,7\}-GDD of type 3^4 9^1 21^1. Give every point weight four and apply the Fundamental Frame Construction. Then fill in KTS(15), a KTS(39) and a KTS(87).

As a result of all the above constructions, we have our main result.

Theorem. There is a KTS(\nu) containing a sub-KTS(\omega) if and only if \nu \equiv \omega \equiv 3 \mod 6 and \nu \geq 3 \omega.

References


Appendix
The 884 exceptions remaining after Theorems 1.1, 1.2 and 1.3

<table>
<thead>
<tr>
<th>$w$</th>
<th>values of $v$</th>
</tr>
</thead>
<tbody>
<tr>
<td>39</td>
<td>141</td>
</tr>
<tr>
<td>45</td>
<td>147 153 159 165</td>
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<tr>
<td>51</td>
<td>165 171 177 183 189</td>
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<tr>
<td>57</td>
<td>189 195 201 207 213</td>
</tr>
<tr>
<td>63</td>
<td>201 207 213 219 225 231 237</td>
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<tr>
<td>69</td>
<td>225 231 237 243 249 255 261</td>
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<tr>
<td>75</td>
<td>243 249 255 261 267 273 279 285</td>
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<tr>
<td>81</td>
<td>261 267 273 279 285 291 297 303 309</td>
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<tr>
<td>87</td>
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<tr>
<td>99</td>
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</tr>
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<td>423 453 459 465 471 525</td>
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<td>141</td>
<td>459 465 471 477 531 537 543 549</td>
</tr>
<tr>
<td>147</td>
<td>465 471 477 483 537 543 549 555 561 567 573</td>
</tr>
</tbody>
</table>

14
A (73, 22; \{4, 7, 10\})-IPBD

Points:  \( \mathbb{Z}_{42} \cup \{a, b_0, b_1, c_0, c_1, d_0, d_1, e_0, e_1\} \cup \{\infty_i: 0 \leq i \leq 20\} \cup \{\infty\} \).

Blocks:  Develop the following modulo 42, where subscripts on \( \infty \) are developed modulo 21 and subscripts on letters are developed modulo 2.

\[
\begin{align*}
0, & \ 6, \ 12, \ 18, \ 24, \ 30, \ 36 & \infty, & \ 0, \ 14, \ 28 \\
\infty, & \ 0, \ 14, \ 28 & \infty_0, & \ a, \ 11, \ 22 \\
\infty_0, & \ b_0, \ 2, \ 29 & \infty_0, & \ c_0, \ 3, \ 34 \\
\infty_0, & \ d_0, \ 4, \ 17 & \infty_0, & \ e_0, \ 6, \ 41 \\
\infty_0, & \ 0, \ 22, \ 26 & \infty_0, & \ 28, \ 31, \ 36 \\
\infty_0, & \ 30, \ 39, \ 40 & \infty_0, & \ 12, \ 14, \ 37
\end{align*}
\]