ON THE SPECTRUM OF NESTED 4-CYCLE SYSTEMS

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Abstract. We prove that if $n \equiv 1$ modulo 8 and $n \notin \{57, 65, 97, 113, 185, 265\}$, then there exists a nested 4-cycle system of order n.

1. Introduction.

We need to begin with some definitions. Let G be a graph, and let $k \geq 3$ be an integer. A *k-cycle decomposition* of G is an edge-decomposition of G into cycles of size k. We will write the k-cycle decomposition as a pair $(G, \mathcal{C})$, where $\mathcal{C}$ is the set of cycles in the edge-decomposition. A k-cycle decomposition of K_n will be called a *k-cycle system* of *order* n. Of course, a 3-cycle system is a *Steiner triple system*; and these designs exist for all orders $n \equiv 1$ or 3 modulo 6.

We will say that a k-cycle decomposition, $(G, \mathcal{C})$, can be *nested* if we can associate with each cycle $C \in \mathcal{C}$ a vertex of G, which we denote $f(C)$, such that $f(C) \notin C$, and such that the edges in $\{\{x, f(C)\}: x \in C, C \in \mathcal{C}\}$ form an edge-decomposition of G. Alternatively, we can view a nested k-cycle decomposition as an edge-decomposition of the multigraph $2G$ into wheels with k spokes, where every edge occurs in one wheel as a spoke and in one wheel on the rim.

In this paper, we are interested in nested k-cycle systems. It is easy to see that a necessary condition for the existence of a nested k-cycle system of order n is that $n \equiv 1 \bmod 2k$. The first examples of nested k-cycle systems to be studied in the literature were nested 3-cycle systems (that is, nested Steiner triple systems). It was proven in [3] that there exists a nested 3-cycle system of order n if and only if $n \equiv 1$ modulo 6. More recently, in [2], it was shown for each odd $k \geq 3$ that there exists a nested k-cycle system of order n if and only if $n \equiv 1 \bmod 2k$, with at most 13 possible exceptions.

Almost nothing is known regarding the existence of nested k-cycle systems for even values of k. In this paper, we address the first case, $k = 4$. We show that the necessary condition $n \equiv 1 \bmod 8$ is sufficient for existence, with at most 6 possible exceptions. We do this by adapting the technique used in [3] to construct nested Steiner triple systems.

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2. The construction.

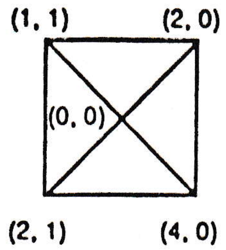
The construction we use is recursive, and depends on certain nested cycle decompositions of complete multipartite graphs. We will denote the complete multipartite graph having m parts of size 2 by $K_{(2^m)}$. We have the following construction for a nested 4-cycle decomposition of $K_{(2^m)}$ when $m \equiv 1 \bmod 4$ is a prime power. As the vertex set for $K_{(2^m)}$ we take $GF(m) \times Z_2$, and we let the parts in the multipartition be $\{y\} \times Z_2, y \in GF(m)$. Let α be a primitive element in $GF(m)$. Write $m = 4t + 1$, and define $\beta = \alpha^t$. For $0 \leq i \leq t - 1$, and for any element $a \in GF(m) \times Z_2$, define a cycle

$$C(i, a) = (a + (\alpha^i, 1);\ a + (\alpha^i\beta, 0);\ a + (\alpha^i\beta^2, 0);\ a + (\alpha^i\beta, 1)).$$

For each cycle $C(i, a)$, define the nested point to be $f(C(i, a)) = a$. Then, it is not difficult to verify that $C = \{C(a, i)\}$ is a 4-cycle decomposition of $K_{(2^m)}$ and f is a nesting of C. We record this result as

Lemma 1. *For all prime powers $m \equiv 1$ modulo 4, there exists a nested 4-cycle decomposition of the multipartite graph $K_{(2^m)}$.*

Example: A nested 4-cycle decomposition of $K_{(2^5)}$. Develop the following through $GF(5) \times Z_2$:



We also need nested 4-cycle systems of orders 9, 17, 25, and 33, which are presented in the Appendix. So, we have

Lemma 2. *There exist nested 4-cycle systems of orders 9, 17, 25, and 33.*

We now use the nested cycle decompositions of Lemmata 1 and 2 in a recursive PBD construction. A *pairwise balanced design* (or, PBD) is a pair $(X, \mathcal{A})$, where $\mathcal{A}$ is a set of subsets of X (called *blocks*) such that every pair of points of X occurs in a unique block. Let v be a positive integer and let K be a set of positive integers. Then we say that $(X, \mathcal{A})$ is a (v, K)-PBD if $v = |X|$ and $|A| \in K$ for every $A \in \mathcal{A}$.

Lemma 3. *Let $(X, \mathcal{A})$ be a $(v, \{5, 9, 13, 17\})$-PBD. Then there is a nested 4-cycle system of order $2v - 1$.*

Proof: Let x_0 be any point in X. For every block $A \in \mathcal{A}$ where $x_0 \in A$, take a nested 4-cycle system on vertex set $\{x_0\} \cup ((A \setminus \{x_0\}) \times \{1, 2\})$. (This system has 9, 17, 25, or 33 points, and hence exists by Lemma 2.) For every block $A \in \mathcal{A}$ where $x_0 \notin A$, take a nested 4-cycle decomposition of $K_{(2|A|)}$ on vertex set $A \times \{1,2\}$, having parts $\{y\} \times \{1,2\}$, $y \in A$. (This decomposition exists since $|A| = 5, 9, 13$ or 17.) If we take the union of these cycle decompositions, we obtain a nested 4-cycle system of order $2v - 1$. ∎

Hence, it is necessary only to construct $(v, \{5,9,13,17\})$-PBDs. This problem was studied in [2], where the following result was proved.

Lemma 4. *If $v \equiv 1$ modulo 4 and $v \notin \{29,33,49,57,93,133\}$, then there exists a $(v,\{5,9,13,17\})$-PBD.*

Hence, we immediately deduce

Theorem 1. *If $n \equiv 1$ modulo 8 and $n \notin \{57,65,97,113,185,265\}$, then there is a nested 4-cycle system of order n.*

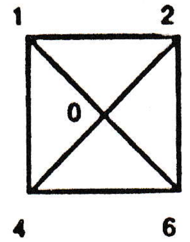
Remark: We conjecture that, if k is even, then necessary and sufficient conditions for the existence of a nested k-cycle system of order n are $n \equiv 1$ modulo $2k$.

References

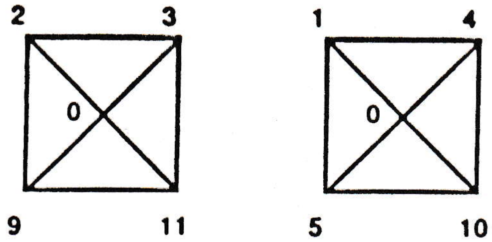
1. C.C. Lindner, C.A. Rodger, and D.R. Stinson, *Nesting of cycle systems of odd length.* (preprint).
2. R.C. Mullin, P.J. Schellenberg, S.A. Vanstone, and W.D. Wallis, *On the existence of frames*, Discrete Math. 37 (1981), 79-104.
3. D.R. Stinson, *The spectrum of nested Steiner triple systems*, Graphs and Combinatorics 1 (1985), 189-191.

Appendix

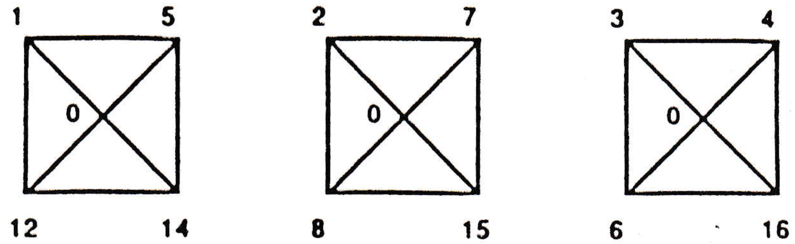
A nested 4-cycle system of order 9. Develop the following modulo 9:



A nested 4-cycle system of order 17. Develop the following modulo 17:



A nested 4-cycle system of order 25. Develop the following modulo 25:



A nested 4-cycle system of order 33. Develop the following modulo 33: