BOSS BLOCK DESIGNS

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Abstract

A well-known inequality of Bose states that in a BIBD containing a parallel class of blocks $b \geq v + r - 1$. We consider BIBDs which contain m parallel classes, each containing a fixed block but otherwise disjoint. We obtain $b \geq v + r - 1 + r\frac{(m-1)(k-1)}{r-m}$. BIBDs in which equality holds, which we refer to as boss block designs, are investigated in the remainder of the paper.

1. *Introduction.*

A *balanced incomplete block design*, or BIBD, with parameters (v,b,r,k,λ) is a pair (X,β), where X is a set of size v, β is a family of b k-subsets of X such that each element of X occurs in r members of β, and each pair of elements of X occurs in λ members of β. Henceforth, we refer to elements of X as *varieties* and members of β as *blocks*. It is immediate that $vr = bk$ and $\lambda(v-1) = r(k-1)$.

A set P of blocks of β is a *parallel class* if each variety of X occurs in exactly one block of P. A BIBD is said to be *resolvable* if β can be partitioned into parallel classes. In [1], Bose proves that the parameters of a resolvable BIBD satisfy the inequality $b \geq v + r - 1$.

However, as indicated by Stanton and Sprott [6], this inequality is true for any BIBD containing a parallel class. In this paper we generalize the above inequality by supposing a BIBD contains m parallel classes each of which contains a fixed block, B, but otherwise disjoint.

In the next section, we derive an inequality for BIBDs with such a block B. In the case $m = 1$, our inequality reduces to Bose's inequality stated above.

In the following sections, we also consider designs in which our inequality is satisfied as an equality and give some examples.

2. *An inequality.*

The following is our main result.

THEOREM 2.1. *Suppose a BIBD* (X,β) *contains a block* B *which occurs in* m *parallel classes. Also, suppose no two of these parallel classes intersect*

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in any block other than B. *Then* $b \geq v + r - 1 + \frac{r(m-1)(k-1)}{r-m}$. *Further, if* $b = v + r - 1 + \frac{r(m-1)(k-1)}{r-m}$ *then* B *intersects any other block in* 0 *or* μ *varieties, where* $\mu = \frac{k^2(r-1)}{bk-mv+mk-k}$.

Proof. There are $b-m(\frac{v}{k} - 1) - 1$ blocks other than the blocks in the m parallel classes. Denote these blocks $B_1, B_2, \ldots, B_\ell$, where $\ell = b-m(\frac{v}{k} - 1) - 1$. Let $b_i = |B \cap B_i|$ for $1 \leq i \leq \ell$. Then we have

$$\sum_{i=1}^{\ell} b_i = k(r-1)$$

and

$$\sum_{i=1}^{\ell} \binom{b_i}{2} = \binom{k}{2} (\lambda - 1) .$$

Thus

$$\sum_{i=1}^{\ell} b_i^2 = k(k\lambda - k - \lambda + r).$$

Denote

$$\mu = \sum_{i=1}^{\ell} \frac{b_i}{\ell} = \frac{k^2(r-1)}{bk-mv+mk-k} .$$

Then

$$\sum_{i=1}^{\ell} (b_i - \mu)^2 = k(k\lambda - k - \lambda + r) - \frac{k^3(r-1)^2}{bk-m(v-k)-k} .$$

Clearly $0 \leq \sum_{i=1}^{\ell} (b_i - \mu)^2$. Using $\lambda(v - 1) = r(k - 1)$ and $vr = bk$, and simplifying, we obtain

$$0 \leq (v - k)[br - r^2 - r(v-k)-m(rk - 2r - v + 1 + b)].$$

Thus

$$b \geq v + r - 1 + \frac{r(m-1)(k-1)}{r - m} .$$

Also, $b = v + r - 1 + \frac{r(m-1)(k-1)}{r - m}$ if and only if $b_i = \mu$ for every $1 \leq i \leq \ell$. □

If $m = 1$, then we obtain Bose's result $b \geq v + r - 1$. A resolvable BIBD has $b = v + r - 1$ if and only if any two blocks from different parallel classes intersect in μ varieties. Such BIBDs are called *affine resolvable* and have been extensively studied. See for example, [4,5].

A comment regarding Theorem 2.1 is in order. That is the assumption that there exist $m(\frac{v}{k} - 1)$ blocks, other than B, in m parallel classes may be weakened. It is necessary only to assume that these blocks be disjoint from B, and contain each variety not in B m times. However, our main interest is in designs containing the m parallel classes as described above.

3. *Boss Block Designs.*

We now consider BIBDs which satisfy the above inequality as an equality. Thus, suppose a BIBD contains m parallel classes, each containing a fixed block B but otherwise disjoint. Further, suppose $b = v+r-1+\frac{(m-1)r(k-1)}{r-m}$. We refer to such a BIBD as an (m, μ) *boss block design*, henceforth BBIBD, and B as a *boss block*. B intersects each block not in the m parallel classes in μ varieties, where $\mu = \frac{k^2(r-1)}{bk-mv+mk-k}$. Thus, for example, an affine resolvable BIBD is a $(1, k^2/v)$ BBIBD (see Bose [1]).

We remark that if a BIBD is an (m, μ) BBIBD with boss block B, and also an (m', μ') BBIBD with boss block B', then $m = m'$ and $\mu = \mu'$. This is true since $b = v + r - 1 + \frac{(m-1)r(k-1)}{r-m}$ determines m uniquely, and then $\mu = \frac{k^2(r-1)}{bk-mv+mk-k}$ determines μ uniquely. Thus there is no possibility of ambiguity when we speak of an (m, μ) BBIBD.

We now derive a few lemmata relating the various parameters of a BBIBD.

The following equality is an immediate consequence of the definition of BBIBD.

LEMMA 3.1. *In an* (m, μ) BBIBD, $b = v + r - 1 + \frac{r(m-1)(k-1)}{r-m}$.

LEMMA 3.2. *In an* (m, μ) BBIBD, $r = m + \frac{k\lambda}{\mu}$.

Proof. Let v be a variety not in the boss block. Counting ordered pairs (C,w), with w in the boss block and $\{v,w\} \subseteq C$, we obtain $\mu(r-m) = k\lambda$.

LEMMA 3.3. *In an* (m,μ) BBIBD, $\lambda = 1 + \frac{(\mu-1)(r-1)}{k-1}$.

Proof. Let v be a variety not in the boss block. Counting ordered pairs (C, w), with w in the boss block and $\{v,w\} \subseteq C$, $C \neq B$, we obtain $(\lambda-1)(k-1) = (\mu-1)(r-1)$. □

LEMMA 3.4. *In an* (m, μ) BBIBD, $v = \frac{[(k-1)^2 + (\mu-1)]r + (k-\mu)}{(\mu-1)r + (k-\mu)}$.

Proof. In any BIBD, $v = \frac{r(k-1) + \lambda}{\lambda}$.

The result follows from Lemma 3.3.

LEMMA 3.5. *In an* (m, μ) BBIBD, $m = \frac{(r-k)(k-\mu)}{\mu(k-1)}$.

Proof. The result follows from Lemma 3.2 and Lemma 3.3.

We now derive inequalities relating μ to k.

LEMMA 3.6. *In an* (m, μ) BBIBD, $\mu \le k/2$.

Proof. Suppose first that $\mu \ge k/2 + 1$.

Then, from Lemma 3.4, $v = \frac{[(k-1)^2 + (\mu-1)]r + (k-\mu)}{(\mu-1)r + (k-\mu)}$

$$< \frac{(k^2-k)r + k/2}{(\frac{k}{2})r} = \frac{2(k-1)r + 1}{r} < 2(k-1) + 1 = 2k-1.$$

However, if a BIBD contains a parallel class $k|v$, so $v \ge 2k$. Thus we have a contradiction.

$$\text{If} \quad \mu = \frac{k+1}{2}, \text{ then } v = \frac{[(k-1)^2 + (\frac{k-1}{2})]r + \frac{k-1}{2}}{(\frac{k-1}{2})r + (\frac{k-1}{2})}$$

$$= \frac{(2k-1)r + 1}{r + 1} < 2k,$$

again yielding a contradiction.

LEMMA 3.7. *An* (m, μ) BBIBD *with* $\mu = \frac{k}{2}$ ***and*** $\mu > 1$ *has* $v = 2k$ *and the parameters of an affine resolvable* BIBD *(i.e.,* $m = 1$*).*

Proof. From Lemma 3.4,

$$v = \frac{[(k-1)^2 + (\frac{k}{2} - 1)]r + \frac{k}{2}}{(\frac{k}{2} - 1)r + \frac{k}{2}} = \frac{(2k^2 - 4k)r + rk + k}{(k - 2)r + k}$$

$$< \frac{(2k^2 - 4k)r}{(k-2)r} + \frac{rk + k}{rk + k-2r} = 2k + 1 + \frac{2r}{rk + k-2r}$$

$$\le 2k + 1 + \frac{2r}{4r + 4 - 2r} = 2k + 1 + \frac{2r}{2r + 4} < 2k + 2$$

Thus $v \le 2k + 1$. Since $k|v$ therefore $v = 2k$. Now, since $\mu = \frac{k}{2}$, it is clear that $m = 1$. (Note that $k \ge 4$ by the assumptions.)

Finally, we determine the parameters of an $(m,1)$-BBIBD.

LEMMA 3.8. *An* (m, μ) BBIBD *with* $\mu = 1$ *has parameters* $(t(k^2 - k) + k,$ $(tk + 1)(k - 1)t + 1), tk + 1, k, 1)$ *for a positive integer* t. *Also,* $m = r - k$.

Proof. By Lemma 3.3, $\lambda = 1$. Thus $v = r(k - 1) + 1$. Since $k|v$, thus $r - 1 = tk$ for some t. Since $b = \frac{vr}{k}$, the parameters have the desired form. By Lemma 3.5, $m = r - k$.

4. BBIBDs with $k \le 7$.

In this section, we determine the parameters of possible BBIBDs with $k \le 7$.

THEOREM 4.1. *A BBIBD with* $k = 2$ *is a* $(2t + 2, \binom{2t+2}{2}, 2t + 1, 2, 1)$ BIBD. *Such a* BIBD *is a* $(2t - 1, 1)$ BBIBD, *and exists for any positive integer* t.

Proof. By Lemma 3.6, $\mu = 1$, if $k = 2$. By Lemma 3.8, the parameters are as stated above. We show the desired designs exist. Given a $(2t + 2)$-set V, take β as all 2-subsets of V. Choose any block B as the boss block.

THEOREM 4.2. *A BBIBD with* $k = 3$ *is a* $(6t + 3, (2t + 1)(3t + 1), 3t + 1,$ $3, 1)$ *design, with* $m = 3t - 2$, $\mu = 1$.

Proof. This is a direct result of Lemma 3.6 and Lemma 3.8.

EXAMPLE 4.1. There exists a (15,35,7,3,1) BIBD which is a (4,1) BBIBD. 0 5 10 is the boss block.

0	5	10									
1	6	11	2	3	6	3	4	7	1	3	9
2	7	12	4	11	13	12	13	1	4	6	12
3	8	13	7	14	1	6	8	14	7	8	11
4	9	14	8	9	12	9	11	2	13	14	2
1	2	5	2	4	10	10	11	14	10	12	3
4	5	8	3	5	11	11	12	0	12	14	5
5	6	9	5	7	13	14	0	3	13	0	6
6	7	10	7	9	0	0	1	4	0	2	8
9	10	13	8	10	1						

THEOREM 4.3. *A BBIBD with* $k = 4$ *is a* $(12t + 4, (3t + 1)(4t + 1), 4t + 1, 4, 1)$ BIBD *with* $m = 4t - 3$, $\mu = 1$, *or an* $(8, 14, 7, 4, 3)$ BIBD *with* $m = 1$.

Proof. If $\mu = 1$, the result follows by Lemma 3.8. By Lemma 3.6 the only other possibility is $\mu = 2$. By Lemma 3.7, we have an $(8, 14, 7, 4, 3)$ BIBD.

EXAMPLE 4.2. There exists an $(8, 14, 7, 4, 3)$ BIBD which is a $(1, 2)$ BBIBD. This BIBD is affine resolvable.

1234	1256	1357	1458	2367	2468	3478
5678	3478	2468	2367	1458	1357	1256

THEOREM 4.4. *A BBIBD with* $k = 5$ *is a* $(20t + 5, (4t + 1)(5t + 1), 5t + 1, 5, 1)$ BIBD *with* $m = 5t - 4$, $\mu = 1$, *or a* $(15, 63, 21, 5, 6)$ BIBD *with* $m = 6$, $\mu = 2$.

Proof. By Lemma 3.6, the only possibilities are $\mu = 1$ or $\mu = 2$. If $\mu = 1$, the parameters are given by Lemma 3.8.

Thus suppose $\mu = 2$. By Lemma 3.3, $\lambda = \frac{r + 3}{4}$. By Lemma 3.5, $m = \frac{3r - 15}{8}$. By Lemma 3.4, $v = \frac{17r + 3}{r + 3}$. Thus we must have $r \equiv 5 \bmod 8$. Also, $(r + 3) \mid (17r + 3)$, so $(r + 3) \mid 48$. Thus $r = 5, 13, 21,$ or 45. But, since $5 \mid v$, the only possibility is $r = 21$. Thus the only possibility is a $(15, 63, 21, 5, 6)$ BIBD, with $m = 6$, $\mu = 2$. Such a BBIBD is given in Example 4.3 below.

EXAMPLE 4.3. There exists a $(15, 63, 21, 5, 6)$ BIBD which is a $(6, 2)$ BBIBD. The boss block is a b c d e.

$$\begin{matrix} a & b & c & d & e \\ 0_1 & 1_1 & 2_1 & 3_1 & 4_1 \\ 0_2 & 1_2 & 2_2 & 3_2 & 4_2 \end{matrix} \qquad \left\{ \begin{matrix} 2_1 & 3_1 & 0_2 & 2_2 & 3_2 \\ 0_1 & 1_1 & 4_1 & 1_2 & 4_2 \end{matrix} \right\} \bmod 5$$

c	d	0_1	2_1	3_2	(mod 5)	d	e	0_1	1_1	0_2	(mod 5)
c	e	0_1	2_1	4_1	(mod 5)	a	e	0_1	0_2	1_2	(mod 5)
a	c	0_1	1_2	2_2	(mod 5)	b	e	0_1	2_2	3_2	(mod 5)
a	d	0_1	3_2	4_2	(mod 5)	b	c	0_1	1_2	3_2	(mod 5)
a	b	0_1	1_1	3_2	(mod 5)	b	d	0_1	2_2	4_2	(mod 5)

Similar methods can be used to prove the following.

THEOREM 4.5. *The only possible BBIBDs with* $k = 6$ *or* $k = 7$, *and* $\mu \geq 2$ *are the following.*

(1) *A* (1,3) *BBIBD with parameters* (12, 22, 11, 6, 5)

(2) *A* (16,2) *BBIBD with parameters* (24, 184, 46, 6, 10).

An example of (1) is given in Section 6. The authors know of no example of (2).

We give the following theorem.

THEOREM 4.6. *For any positive integer* k *there exists only finitely many integers* v *such that a BIBD* (v,b,r,k,λ) *is an* (m,μ) *BBIBD, with* $\mu > 1$.

Proof. By Lemma 3.6, $\mu \leq \frac{k}{2}$. By Lemma 3.4, $v = \frac{[(k-1)^2 + \mu-1)]r + (k-\mu)}{(\mu-1)r + (k-\mu)}$.

Thus $$v \leq \frac{[(k-1)^2 + (\mu-1)]r + (k-\mu)}{(\mu-1)r}$$

$$\leq \frac{(k-1)^2}{(\mu-1)} + \frac{(k-\mu)}{(\mu-1)r} + 1$$

Therefore with k and μ fixed, v is bounded, since $\mu \neq 1$. Thus there are only a finite number of possible values for v.

5. *BBIBDs with* $m \leq 3$.

It has been shown that a BBIBD with $m = 1$ has parameters $(n(n\lambda - \lambda + 1), n(n\lambda + 1), n\lambda + 1, n\lambda - \lambda + 1, \lambda)$ (see, for example [6]) In this section we consider BBIBDs with $m = 2$ or 3.

THEOREM 5.1. *There exists no BBIBD with* $m = 2$.

Proof. Suppose there exists a BBIBD with $m = 2$. By Lemma 3.1, $(r-2) \mid r(k-1)$, since b is an integer. Thus if r is odd $(r-2) \mid k-1$, and if r is even, $\frac{r-2}{2} \mid (k-1)$.

If r is odd, $r \leq k+1$. By Lemma 3.3, $\lambda \geq \mu$ since in any BIBD $r \geq k$. Then Lemma 3.2 implies $r \geq 2+k$. This is a contradiction.

If r is even, then $\frac{r-2}{2} \mid (k-1)$, $r \geq 2+k$ implies $r - 2 = 2(k-1)$, or $r = 2k$.

Then, by Lemma 3.4, $v = \frac{[(k-1)^2 + (\mu-1)]2k + (k-\mu)}{(\mu-1)2k + (k-\mu)}$.

Since $k \mid v$, thus $k \mid \mu$. However, $\mu < k$, so we have a contradiction.

Similar, but more complicated calculations can be used to obtain parameters of BBIBDs with $m = 3$. The following is obtained.

THEOREM 5.2. *The only BBIBD with* $m = 3$ *is a* $(6, 15, 5, 2, 1)$ *BIBD.*

REMARK. The above design is a $(3,1)$ BBIBD by Theorem 4.1.

Proof. From Lemma 3.1, we have $(r-3)\,|\,2r(k-1)$.

First, suppose $r \equiv 2$ or $4 \pmod 6$. Then $(r-3)\,|\,(k-1)$, so $r \le k-2$. But Lemma 3.2 implies $r \ge k+3$, a contradiction.

Next, suppose $r \equiv 1$ or $5 \pmod 6$. Then $(r-3)\,|\,(2k-2)$, and as above $r \ge k+3$. Thus $r-3 = 2k-2$ so $r = 2k+1$. Substituting into Lemma 3.1 and using $bk = rv$, we obtain $v = \frac{k(4k+1)}{k+1}$. Thus $(k+1)\,|\,(4k+1)$, so $k = 2$. This gives rise to the $(6, 15, 5, 2, 1)$ BIBD.

If $r \equiv 0 \bmod 6$, then $(\frac{r}{3} - 1)\,|\,(k-1)$. Since $r \ge k+3$, we have $r = 3k$ or $r = (3k+3)/2$. Proceeding as above, we obtain in the first case that $v = (5k-1)/2k$, so $k = 1$. In the second case, $v = k(7k+5)/(k+5)$, so $s = 1, 5$ or 13. These do not give rise to possible parameters for BIBDs.

Finally, we consider $r \equiv 3 \bmod 6$. This case is handled similarly, and no BIBDs result.

6. *Further Remarks.*

As indicated in Section 2, an affine resolvable BIBD is a $(1, \frac{k^2}{v})$ BBIBD. However, the converse is not true, as the following example indicates.

EXAMPLE 6.1. There exists a $(12, 22, 11, 6, 5)$ BIBD which is a $(1,3)$ BBIBD but not affine resolvable. The block 1 2 3 4 5 6 can be taken as a boss block, and intersects all blocks except 7 8 9 10 11 12 in exactly three varieties. Yet this BIBD is not resolvable, since, for example, 1 2 3 7 8 9 occurs in no parallel class.

1	2	3	4	5	6	2	3	5	7	10	12
7	8	9	10	11	12	2	3	5	8	9	11
1	2	3	7	8	9	2	4	5	8	10	11
1	2	3	10	11	12	2	4	5	7	9	12
1	3	4	7	9	10	2	4	6	8	10	12
1	3	4	8	11	12	2	4	6	7	9	11
1	4	5	7	10	11	3	4	6	9	10	12
1	4	5	8	9	12	3	4	6	7	8	11
1	5	6	7	11	12	3	5	6	9	11	12
1	5	6	8	9	10	3	5	6	7	8	12
1	2	6	7	8	12						
1	2	6	9	10	11						

It is clear that a $(1,\mu)$BBIBD is an affine resolvable BIBD if and only if every block is a boss block. However, as the above example indicates, not every block of a BBIBD need be a boss block. In fact, it may be checked that 1 2 3 4 5 6 and 7 8 9 10 11 12 are the only boss blocks in the above BBIBD.

In some cases, a BBIBD has a unique boss block. Sufficient conditions for this to happen are given in the following theorem.

THEOREM 6.1. *A BBIBD with* $m > 1$ *and* μ *not dividing* k *has a unique boss block.*

Proof. Let B be a boss block. Any other block in the design intersects B in 0 or μ varieties. Since μ does not divide k, any parallel class must contain B. Since $m > 1$ no block other than B can be a boss block.

In Example 4.2, $m = 6$, $\mu = 2$, and $k = 5$. Thus Theorem 6.1 applies, and the block a b c d e is the only boss block in the design.

We close by describing an infinite class of boss block designs. This construction is due to Yul Inn [3].

THEOREM 6.2. *Let* $0 \leq t \leq n$ *be integers. Then there exists a BIBD, with* $v = 2^{t+n}+2^t-2^n$, $k = 2^t$ *and* $\lambda = 1$, *which is a* $(2^t-2^k+1,1)$ *BBIBD.*

Proof. Denniston [2] has shown that if $0 \leq t \leq n$ are integers, then there is a set of $v = 2^{t+n}+2^t-2^n$ points in the projective plane of order 2^n such that any line meets this set in either $k = 2^t$ or 0 points. Such a set of points is called a $\{2^t;2^n\}$-*arc*. Delete the points not in the arc, thus obtaining the desired BIBD.

Choose any block B as the boss block. Each of the points of the projective plane which were expunged from the line containing B induce a parallel class containing B. There are $m = 2^n+1-2^k$ such points, and no two of the parallel classes formed contain a common block other than B. The remaining blocks all meet B in $\mu = 1$ point. Thus we have a boss block design.

7. *Conclusion.*

Thus we have generalized Bose's inequality to obtain

$$b \geq v + r - 1 + \frac{r(m-1)(k-1)}{r - m}$$

for a BIBD containing a block B which occurs in m parallel classes no two of which contain a common block other than B. Designs which satisfy the above inequality as an equality are termed boss block designs and are studied for small values of k and m.

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