On Resolvable Group-divisible Designs with Block Size 3

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ABSTRACT

In this paper, we consider a generalization of nearly Kirkman triple systems (NKTS). We can view a NKTS as being a resolvable group divisible design (GDD) of block-size three and group-size two. This suggests the study of resolvable GDDs of block-size three, having other group-sizes.

We are able to construct many new examples of these designs. In doing so, we make essential use of a type of design called a "frame", which can be thought of as a Kirkman triple system with "holes".

1. Introduction.

We need to begin with some definitions.

A *pairwise balanced design* (or, PBD) is a pair (X,**A**), such that **A** is a set of subsets (called *blocks*) of X, each of cardinality at least two, such that every unordered pair of *points* (i.e. elements of X) is contained in a unique block in **A**. If v is a positive integer and K is a set of positive integers, each of which is greater than or equal to 2, then we say that (X,**A**) is a (v,K)-PBD if $|X| = v$, and $|A| \in K$ for every $A \in$ **A**.

If $K = \{k\}$, then a (v,K)-PBD is referred to as a (v,k,1)-BIBD *(balanced incomplete block design)*. A (v,3,1)-BIBD is called a *Steiner triple system;* these designs exist for all v = 1 or 3 modulo 6.

A *group-divisible design* (or, GDD), is a triple (X,**G**,**A**), which satisfies the following properties:

a) **G** is a partition of X into subsets called *groups*

2) **A** is a set of subsets of X (called *blocks)* such that a group and a block contain at most one common point

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3) every pair of points from distinct groups occurs in a unique block.

The group-type, or *type,* of a GDD $(X,\mathbf{G},\mathbf{A})$ is the multiset $\{|G|: G \in \mathbf{G}\}$. We usually use an "exponential" notation to describe group-types: a group-type $1^i 2^j 3^k \cdots$ denotes i occurrences of 1, j occurrences of 2, etc. We will say that a GDD is a K-GDD if $|A| \in K$ for every $A \in \mathbf{A}$.

A *parallel class* in a PBD or GDD is a set of blocks that partitions the set of points. If we can partition the blocks into parallel classes, we say that the design is *resolvable.* Clearly, if a parallel class of blocks of size k exists in a design, then k must divide v.

A resolvable $(v,3,1)$-BIBD is called a *Kirkman triple system* and denoted KTS(v). It was conjectured over a century ago that a KTS(v) exists if and only if $v \equiv 3$ modulo 6, but this was proven only in 1971 by Ray-Chaudhuri and Wilson [7] (see also [4]). A related class of GDDs was defined by Kotzig and Rosa [6]: a 3-GDD of group-type 2^u is referred to as a *nearly Kirkman triple system,* and denoted NKTS($2u$). The results of [1] and [6] establish the existence of *NKTS*(v) for all $v \equiv 0$ modulo 6, $v \geq 18$, $v \notin \{84,102,172\}$. In [2], Brouwer constructed *NKTS*(102) and *NKTS*(172). Finally, a purported *NKTS*(84) was presented in [5]; however this design is not an *NKTS*, and it appears that the construction cannot be salvaged. However, we shall construct an *NKTS*(84) in Section 2, thus completing the spectrum.

One obvious generalization of NKTS is to consider resolvable 3-GDDs with other group sizes. It is not difficult to prove that all groups in a resolvable k-GDD must be the same size (see [11, Lemma 1.1] for a proof), so we consider resolvable 3-GDDs of group type g^u. We denote such a design by RGDD(g^u). First, we clearly must have $u \geq 3$. We have noted that $g \cdot u \equiv 0$ modulo 3. Also, since every point occurs in a block with every point not in the same group, we must have $g \cdot (u-1) \equiv 0$ modulo 2. We show that these two necessary numerical conditions are sufficient for existence, with a few exceptions and a few unsolved cases.

2. Kirkman frames and recursive constructions.

We use a particular type of design called a frame as an essential tool in recursive constructions for *RGDD*s.

If $(X,\mathbf{G},\mathbf{A})$ is a k-GDD and $G \in \mathbf{G}$, then we say that a set $P \subset \mathbf{A}$ of blocks is a *holey parallel class* with *hole* G provided that P consists of $(|X|-|G|)/k$ disjoint blocks that partition $X \backslash G$. If we can partition the set of blocks $\mathbf{A}$ into a set $\mathbf{P}$ of holey parallel classes, then we say that $(X,\mathbf{G},\mathbf{P})$ is a *k-frame.* We can think of a frame as being a resolvable BIBD with holes, exactly as a GDD is a BIBD with holes.

We will be using 3-frames, which we refer to as *Kirkman frames.*

These are studied in [8], in which they are used to prove new results on the existence of subdesigns in Kirkman triple systems. In the case where all the holes have the same size, their existence was completely determined, as recorded in Theorem 2.1.

Theorem 2.1. *There exists a Kirkman frame of type t^u if and only if t is even, $u \geq 4$, and $t(u-1) \equiv 0$ modulo 3.*

Kirkman frames are related to the problems of resolvable 3-GDDs and resolvable coverings of pairs by triples by means of the following simple constructions.

Theorem 2.2. *Suppose there is a Kirkman frame of type $t_1^{u_1} t_2^{u_2} \cdots t_j^{u_j}$, and let $t \mid t_i$, for $1 \leq i \leq j$. Suppose also that there exist RGDD($t^{t_i/t+1}$), for $1 \leq i \leq j$. Then there exists an RGDD(t^u), where $u = 1 + \sum_{1 \leq i \leq j} t_i \cdot u_i / t$.*

In applying these two theorems, it will be useful to have constructed some Kirkman frames. We do this now. We use a recursive construction for frames, which is found in [8, Construction 3.1].

Theorem 2.3. *Let $(X, \mathbf{G}, \mathbf{A})$ be a GDD, and let $w: X \rightarrow \mathbf{Z}^+ \cup \{0\}$ (we say that w is a* weighting). *For every $A \in \mathbf{A}$, suppose there is a frame of type $\{w(x): x \in A\}$. Then there is a frame of type $\{\sum_{x \in G} w(x): G \in \mathbf{G}\}$.*

We now mention some useful corollaries of this construction.

Corollary 2.4. *Suppose r and m are integers and $0 \leq r \leq 4m$. Then there is a Kirkman frame of type $24^{3m+1}(6r)^1$, and one of type $48^{3m+1}(12r)^1$.*

Proof. We begin with a resolvable $(12m+4,4,1)$-BIBD, which exists by [4]. Adjoin "infinite" points to r of the $4m + 1$ parallel classes, thus creating a $\{4,5\}$-GDD of group-type $4^{3m+1} r^1$. Apply Theorem 2.3, giving every point weight 6, noting that Kirkman frames of types 6^4 and 6^5 exist (Theorem 2.1). The first frame results. If we instead use weight 12, the second frame is obtained.

Corollary 2.5. *Suppose $m \geq 4$, $m \neq 6$ or 10, and $0 \leq r \leq m$. Then there is a Kirkman frame of type $(6m)^4(6r)^1$, and one of type $(12m)^4(12r)^1$.*

Proof. There is a TD(5,m), by [9] and [10]. Delete $m - r$ points from one group to yield a {4,5}-GDD of type m^4r^1. Now, apply Theorem 2.3, giving every point weight 6. We have the required input frames, of types 6^4 and 6^5. The first frame is constructed. Again, to construct the second frame, use weight 12.

As mentioned in the introduction, an *NKTS*(84) remains to be constructed. We accomplish this using frames.

First, a group-divisible design with group-type 6^59^1 and blocks of size 4 is presented in the Appendix. This gives rise to a Kirkman frame as follows.

Lemma 2.6. *There exists a Kirkman frame of type* 12^518^1.

Proof. Apply Theorem 2.3 to the above *GDD*, giving every point weight 2.

The *NKTS*(84) is constructed from the frame of type 12^518^1 by means of a slight generalization of Theorem 2.2. We use the 78 points in the frame, together with 6 ideal points. For each hole of size 12, we fill in an incomplete *NKTS*(18) "missing" an *NKTS*(6) (this is Brouwer's ingredient "*C*" in [2]). For the hole of size 18, fill in an *NKTS*(24). (Each hole is filled in with the relevant design on the point set of that hole, plus the 6 ideal points.)

The resulting design is an *NKTS*(84). We record this as

Theorem 2.7. *There exists an NKTS*(84).

We have observed that we can construct resolvable *GDD*s by filling in the holes of Kirkman frames. We have a few other simple recursive constructions.

Theorem 2.8. *Suppose there is a RGDD*(g^u), *RGDD*(w^3), *and RGDD*(w^g). *Then there is a RGDD*($(3w)^{gu/3}$).

Proof. From the *RGDD*(g^u), we construct a {3,g}$-$*GDD* of type $3^{gu/3}$ by taking as groups a parallel class of blocks if size 3. This new *GDD* is *uniformly resolvable:* the blocks can be partitioned into parallel classes, each of which consists of blocks of only one size. Now give every point weight w, replacing every block by a *RGDD*(w^3) or *RGDD*(w^g).

Theorem 2.9. *Suppose there is a* $RGDD(g^u)$, *and a* $RGDD(h^3)$. *Then there is a* $RGDD((gh)^u)$.

Proof. Start with a $RGDD(g^u)$, and take h copies of each point. Then replace each block by the blocks of a $RGDD(h^3)$ in which the groups are the copies of the three points in the block.

The other construction fills in the groups of an $RGDD$.

Theorem 2.10. *Suppose there is a* $RGDD(g^u)$ *and a* $RGDD(h^v)$, *where* $h \cdot v = g$. *Then there is a* $RGDD(h^{u \cdot v})$.

Proof. Obvious.

3. Resolvable *GDD*s.

Before constructing the resolvable *GDD*s, we first note the necessary numerical conditions for existence. We have the following.

Lemma 3.1. *Suppose there is a* $RGDD(g^u)$. *Then the following conditions hold:*

if $g \equiv 1$ *or* 5 *modulo* 6, *then* $u \equiv 3$ *modulo* 6;

if $g \equiv 3$ *modulo* 6, *then* $u \equiv 1$ *modulo* 2;

if $g \equiv 2$ *or* 4 *modulo* 6, *then* $u \equiv 0$ *modulo* 3; *and*

if $g \equiv 0$ *modulo* 6, *then there are no congruential conditions on* u.

Proof. We observed in the introduction that $3 \mid (g \cdot u)$ and $g(u-1)$ is even. The results follow.

We can now prove our existence results. We split the proof into several cases.

Lemma 3.2. *There is a* $RGDD(g^3)$ *if and only if* $g \neq 2,6$.

Proof. A $RGDD(g^3)$ is equivalent to a pair of orthogonal Latin squares of order g, which are well-known to exist if and only if $g \neq 2,6$.

Lemma 3.3. *If* $g \equiv 1$ *or* 5 *modulo* 6, *then there is an* $RGDD(g^u)$ *if and only if* $u \equiv 3$ *modulo* 6.

Proof. If $g = 1$, then an RGDD(1^u) is just a Kirkman triple system of order u, which exist for all $u \equiv 3$ modulo 6. If $g > 1$, $g \equiv 1$ or 5 modulo 6, then we apply Theorem 2.9, obtaining RGDD(g^u) from RGDD(1^u).

Lemma 3.4. *For $g \equiv 3$ modulo 6, then there is an RGDD(g^u) if and only if $u \equiv 1$ modulo 2.*

Proof. If $g = 3$, then we obtain an RGDD(3^u) from a Kirkman triple system of order $3u$, by taking one parallel class as groups of our RGDD. This construction works for all odd u. If $g > 3$, $g \equiv 3$ modulo 6, then obtain RGDD(g^u) from RGDD(3^u) by means of Theorem 2.9.

Lemma 3.5. *There is an RGDD(2^u) if and only if $u \equiv 0$ modulo 3, $u \geq 9$.*

Proof. These are NKTS($2u$).

Lemma 3.6. *There is an RGDD(4^u) if and only if $u \equiv 0$ modulo 3.*

Proof. There is an RGDD(4^3) by Lemma 3.2, and we present RGDD(4^6) and RGDD(4^{12}) in the Appendix. We can construct RGDD(4^{18}) from RGDD(24^3) and RGDD(4^6) by applying Theorem 2.10 (an RGDD(24^3) exists by Lemma 3.2).

If $u \equiv 3$ modulo 6, $u > 3$, then we apply Theorem 2.2. We construct an RGDD(4^u) from a Kirkman frame of type $8^{(u-1)/2}$, filling in RGDD(4^3).

If $u \equiv 0$ modulo 6, $u > 18$, then there is a 4-GDD having group-type $2^{(u-6)/2}5^1$, by Brouwer [3]. Apply Theorem 2.3, giving every point weight 4, obtaining a Kirkman frame of type $8^{(u-6)/2}20^1$. Now, apply Theorem 2.2, filling in RGDD(4^3) and RGDD(4^6).

Lemma 3.7. *If $g \equiv 2$ or 4 modulo 6, $g \geq 8$, then there is a RGDD(g^u) if and only if $u \equiv 0$ modulo 3, except possibly when $u = 6$ and $g \equiv 2$ or 10 modulo 12.*

Proof. If $u = 3$, then Lemma 3.2 applies. If $u \equiv 0$ modulo 3, $u \geq 9$, then we construct *RGDD*(g^u) using Theorem 2.9, from *RGDD*(2^u). For $u = 6$, $g \equiv 4$ or 8 modulo 12 ($g \neq 8$), we can likewise construct *RGDD*(g^6) from *RGDD*(4^6), using Theorem 2.9. An *RGDD*(8^6) is presented in the Appendix. This leaves only the cases indicated.

The designs we have yet to construct are RGDD(g^u) where $g \equiv 0$ modulo 6. There is no restriction on u in these cases. First, we consider $g = 6$.

Lemma 3.8. *If $u \geq 4$, $u \neq 11$ or 14, then there exists an RGDD*(6^u). *Also, there does not exist an RGDD*(6^3).

Proof. First, suppose $u \equiv 0$ modulo 3. If $u = 3$, the RGDD does not exist (Lemma 3.2). We present an RGDD(6^6) in the Appendix. If $u \geq 9$, obtain an RGDD(6^u) from an RGDD(2^u) by applying Theorem 2.9.

Next, we consider $u \equiv 1$ modulo 3. We present RGDD(6^4), *RGDD*(6^7) and *RGDD*(6^{10}) in the Appendix. If $u \geq 13$, then we apply Theorem 2.2, using a Kirkman frame of type $18^{(u-1)/3}$, filling in RGDD(6^4).

We further subdivide the case $u \equiv 2$ modulo 3 into subcases modulo 12.

If $u \equiv 2$ modulo 12, $u \geq 26$, then we apply Corollary 2.5 with $m = (u-6)/4$ and $r = 5$. This produces a Kirkman frame of type $(6(u-6)/4)^4 30^1$. Now we apply Theorem 2.2, filling in RGDD($6^{(u-2)/4}$) (which exists since $(u-2)/4 \equiv 0$ modulo 3) and an RGDD(6^6). The case $u = 14$ is unsolved.

An RGDD(6^5) is presented in the Appendix. If $u \equiv 5$ modulo 12, $u \geq 17$, then we apply Theorem 2.2, using a Kirkman frame of type $24^{(u-1)/4}$. We fill in RGDD(6^5).

Next, consider $u \equiv 8$ modulo 12. First, an RGDD(6^8) is presented in the Appendix. If $u \geq 20$, then we apply Corollary 2.4 with $m = (u-8)/12$ and $r = 3$, producing a Kirkman frame of type $24^{(u-4)/4} 18^1$. Then apply Theorem 2.2, filling in RGDD(6^4) and RGDD(6^5).

Finally, the case $u \equiv 11$ modulo 12 is similar. The case $u = 11$ is a possible exception. If $u \geq 35$, then we can apply Corollary 2.4 with $m = (u-11)/12$ and $r = 6$, constructing a Kirkman frame of type $24^{(u-7)/4} 36^1$. We then fill in RGDD(6^5) and RGDD(6^7) (Theorem 2.2). If $u = 23$, we proceed as follows. Begin with an RGDD(12^4) (appendix). Adjoining a group at infinity of size 18, we obtain a 4-GDD of group type $12^4 18^1$. Giving every point weight 2, and applying Theorem 2.3, we obtain a Kirkman frame of type $24^4 36^1$. Now, fill in the holes with RGDD(6^5) and RGDD(6^7), thereby constructing RGDD(6^{23}).

Since we have covered all cases, the proof is complete.

We next prove a similar result for groups of size 12. We shall use the following corollary of Theorem 2.8.

Corollary 3.9. *Suppose there is a RGDD*(6^u). *Then there is a RGDD*(12^{2u}).

Proof. Apply Theorem 2.8 with $g = 6$, $w = 4$, noting that $RGDD(4^3)$ and $RGDD(4^6)$ exist.

Lemma 3.10. *If $u \geq 3$, then there exists a $RGDD(12^u)$.*

Proof. If u is odd, then construct $RGDD(12^u)$ from $RGDD(3^u)$, using Theorem 2.9 with $h = 4$.

If u is even, $u \neq 4,6,22$, or 28, apply Corollary 3.9. We present $RGDD(12^4)$ in the Appendix. Next, we obtain a $RGDD(12^6)$ from a $RGDD(4^6)$ by applying Theorem 2.9. To construct $RGDD(12^{22})$ and $RGDD(12^{28})$, we apply Theorem 2.2, using Kirkman frames of type 36^7 and 36^9, filling in $RGDD(12^4)$.

Now we consider group-sizes that are multiples of 6 or 12.

Lemma 3.11.

1) *Suppose $g \equiv 6$ or 30 modulo 36, $g \geq 30$. If $u \geq 3$ and $u \neq 14$, then there exists a $RGDD(g^u)$.*

2) *Suppose $g = 18$. If $u \geq 3$ and $u \neq 11$ or 14, then there exists a $RGDD(g^u)$.*

3) *Suppose $g \equiv 0,12,18$, or 24 modulo 36, $g \geq 24$. If $u \geq 3$, then there exists a $RGDD(g^u)$.*

Proof. The cases where $u = 3$ were done in Lemma 3.6, so we can assume $u \geq 4$.

Suppose that $u \geq 4$, $u \neq 11$ or 14. These designs are constructed from $RGDD(6^u)$ and $RGDD(12^u)$ using Theorem 2.9.

Next, we consider $u = 11$. If $g \equiv 0 \bmod 6$, $g > 18$, we construct $RGDD(g^{11})$ from $RGDD(3^{11})$, using Theorem 2.9.

Finally, suppose $u = 14$. Let $g = 3w$, where $w \equiv 0,4,6$, or 8 modulo 12, $w \geq 8$. Then, we apply Theorem 2.8 using $RGDD(6^7)$, $RGDD(w^3)$, and $RGDD(w^6)$ (this last $RGDD$ exists by Lemma 3.7).

This covers all the required cases.

Our main existence result is obtained by gathering together all the results we have proved so far.

Theorem 3.12. *The necessary congruential conditions for existence of a $RGDD(g^u)$ are sufficient, with the exceptions of $RGDD(2^3)$, $RGDD(2^6)$, and $RGDD(6^3)$, and with the following possible exceptions $RGDD(g^u)$:*

(i) *$g \equiv 6$ or 30 modulo 36 and $u = 14$;*

(ii) $g = 6$ *or* 18 *and* $u = 11$*; and*

(iii) $g \equiv 2$ *or* 10 *modulo* 12 *and* $u = 6$.

We observe that all the exceptions in (i) and (ii) could be eliminated with the construction of $RGDD(6^{11})$ and $RGDD(6^{14})$.

Addendum. It has come to our attention that the problem of constructing resolvable GDDs with block-size three has been considered independently by Eric Mendelsohn and Shen Hao. They have also proved some of the results contained in this paper. They will report their results in a forthcoming paper.

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Appendix

A *GDD* of type $6^5 9^1$ and block-size 4.

Points: $(Z_5 \times \{1,2,3,4,5,6,7\}) \cup \{\infty_i: 1 \leq i \leq 4\}$.

Groups: $\{\{i\} \times \{1,2,3,4,5,6\}: i \in Z_5\}$ $\cup \{((\{0,1,2,3,4\} \times \{7\}) \cup \{\infty_i: 1 \leq i \leq 4\}\}$.

Blocks: develop the following blocks modulo 5:

{(0,7),(0,1),(2,1),(1,4)}	{(0,7),(0,2),(1,2),(3,3)}
{(0,7),(0,3),(2,3),(4,6)}	{(0,7),(2,4),(3,4),(0,5)}
{(0,7),(1,5),(4,5),(3,2)}	{(0,7),(1,6),(2,6),(3,1)}
{(0,7),(1,1),(4,3),(2,5)}	{(0,7),(2,2),(4,4),(0,6)}
{(0,7),(4,1),(1,3),(3,5)}	{(0,7),(4,2),(0,4),(3,6)}
{(0,1),(1,1),(2,2),(4,2)}	{(0,3),(1,3),(2,4),(4,4)}
{(0,5),(1,5),(2,6),(4,6)}	
$\{(0,1),(4,3),(3,5),\infty_1\}$	$\{(0,2),(4,4),(1,6),\infty_1\}$
$\{(0,1),(1,3),(2,6),\infty_2\}$	$\{(0,2),(3,4),(4,5),\infty_2\}$
$\{(0,1),(3,4),(2,5),\infty_3\}$	$\{(0,2),(4,3),(2,6),\infty_3\}$
$\{(0,2),(1,3),(2,5),\infty_4\}$	$\{(0,1),(2,4),(1,6),\infty_4\}$

A resolvable GDD of type 4^6.

Points: $(\mathbf{Z}_{10} \times \{1,2\}) \cup \{\infty_i: 1 \leq i \leq 4\}$.

Groups: $\{\{0+i,5+i\} \times \{1,2\}: i = 0,1,2,3,4\} \cup \{\{\infty_i: 1 \leq i \leq 4\}\}$.

Resolution classes: develop the following modulo 10

{(1,1),(7,1),(8,2)}	{(3,1),(4,1),(6,1)}
{(1,2),(7,2),(8,1)}	{(3,2),(4,2),(6,2)}
$\{(0,1),(2,2),\infty_1\}$	$\{(2,1),(0,2),\infty_2\}$
$\{(5,1),(9,2),\infty_3\}$	$\{(9,1),(5,2),\infty_4\}$

A resolvable GDD of type 4^{12}.

Points: $(\mathbf{Z}_{22} \times \{1,2\}) \cup \{\infty_i: 1 \leq i \leq 4\}$.

Groups: $\{\{0+i,11+i\} \times \{1,2\}: 0 \leq i \leq 10\} \cup \{\{\infty_i: 1 \leq i \leq 4\}\}$.

Resolution classes: develop the following modulo 22

$\{(0,1),(1,1),(2,2)\}$	$\{(3,1),(5,1),(8,2)\}$
$\{(0,2),(1,2),(2,1)\}$	$\{(3,2),(5,2),(8,1)\}$
$\{(4,1),(9,1),(16,2)\}$	$\{(7,1),(14,1),(20,2)\}$
$\{(4,2),(9,2),(16,1)\}$	$\{(7,2),(14,2),(20,1)\}$
$\{(6,1),(10,1),(18,1)\}$	$\{(12,1),(15,1),(21,1)\}$
$\{(6,2),(10,2),(18,2)\}$	$\{(12,2),(15,2),(21,2)\}$
$\{(11,1),(19,2),\infty_1\}$	$\{(19,1),(11,2),\infty_2\}$
$\{(13,1),(17,2),\infty_3\}$	$\{(17,1),(13,2),\infty_4\}$

A resolvable GDD of type 6^4.

Points: $\mathbf{Z}_8 \times \{1,2,3\}$.

Groups: $\{\{0+i,4+i\} \times \{1,2,3\}: i = 0,1,2,3\}$

Resolution classes: First, we form five classes by developing each of the following five blocks modulo 8.

$$\{(0,1),(1,2),(7,3)\}$$
$$\{(0,1),(6,2),(1,3)\}$$
$$\{(0,1),(3,2),(5,3)\}$$
$$\{(0,1),(5,2),(2,3)\}$$
$$\{(0,1),(7,2),(6,3)\}.$$

Next, we obtain four more parallel classes, by developing the following parallel class modulo 8 (since adding 4 to every element leaves this parallel class fixed, we get an orbit of length 4):

$\{(2,1),(4,1),(5,1)\}$	$\{(6,1),(0,1),(1,1)\}$
$\{(4,2),(6,2),(7,2)\}$	$\{(0,2),(2,2),(3,2)\}$
$\{(5,3),(7,3),(0,3)\}$	$\{(1,3),(3,3),(4,3)\}$
$\{(7,1),(1,2),(2,3)\}$	$\{(3,1),(5,2),(6,3)\}$

A resolvable GDD of type 6^5.

Points: $(\mathbf{Z}_{12} \times \{1,2\}) \cup (\{a\} \times \mathbf{Z}_2) \cup \{\infty_i: 1 \leq i \leq 4\}$.

Groups: $\{\{0+i,2+i,4+i,6+i,8+i,10+i\} \times \{j\}: i = 0,1; j = 1,2\} \cup \{(\{a\} \times \mathbf{Z}_2) \cup \{\infty_i: 1 \leq i \leq 4\}\}$.

Resolution classes: develop the following modulo 12 (note: the second coordinate of elements in $(\{a\} \times \mathbf{Z}_2)$ is written as a subscript, and is evaluated modulo 2)

{(4,1),(11,1),(9,2)}	{(8,1),(9,1),(4,2)}
{(3,2),(10,2),(6,1)}	{(8,2),(11,2),(5,1)}
{(0,1),(0,2),∞_1}	{(1,1),(2,2),∞_2}
{(2,1),(1,2),∞_3}	{(3,1),(5,2),∞_4}
{(7,1),(10,1),a_0}	{(6,2),(7,2),a_1}

A resolvable GDD of type 6^6.

Points: $(\mathbf{Z}_{15} \times \{1,2\}) \cup \{\infty_i: 1 \leq i \leq 6\}$.

Groups: $\{\{0+i,5+i,10+i\} \times \{1,2\}: i = 0,1,2,3,4\} \cup \{\{\infty_i: 1 \leq i \leq 6\}\}$.

Resolution classes: develop the following modulo 15:

{(0,1),(1,1),(3,1)}	{(0,2),(1,2),(4,2)}
{(2,2),(8,2),(10,2)}	{(2,1),(6,1),(3,2)}
{(4,1),(11,1),(13,2)}	{(7,1),(13,1),(5,2)}
{(5,1),(9,2),∞_1}	{(8,1),(14,2),∞_2}
{(9,1),(12,2),∞_3}	{(10,1),(6,2),∞_4}
{(12,1),(11,2),∞_5}	{(14,1),(7,2),∞_6}

A resolvable GDD of type 6^7.

Points: $(\mathbf{Z}_{18} \times \{1,2\}) \cup \{\infty_i: 1 \leq i \leq 6\}$.

Groups: $\{\{0+i,3+i,6+i,9+i,12+i,15+i\} \times \{j\}: i = 0,1,2;\ j = 1,2\} \cup \{\{\infty_i: 1 \leq i \leq 6\}\}$.

Resolution classes: develop the following modulo 18.

{(0,1),(1,1),(5,1)}	{(0,2),(1,2),(5,2)}
{(2,1),(4,1),(2,2)}	{(3,1),(10,1),(4,2)}
{(6,1),(14,1),(3,2)}	{(7,2),(17,2),(12,1)}
{(8,2),(15,2),(16,1)}	{(14,2),(16,2),(8,1)}
{(7,1),(11,2),∞_1}	{(9,1),(12,2),∞_2}
{(11,1),(13,2),∞_3}	{(13,1),(9,2),∞_4}
{(15,1),(6,2),∞_5}	{(17,1),(10,2),∞_6}

A resolvable GDD of type 6^8.

Points: $(\mathbf{Z}_{21} \times \{1,2\}) \cup \{\infty_i: 1 \leq i \leq 6\}$.

Groups: $\{\{0+i,7+i,14+i\} \times \{1,2\}: i = 0,1,2,3,4,5,6\}$

$\cup \ \{\{\infty_i: 1 \leq i \leq 6\}\}$.

Resolution classes: develop the following modulo 21.

{(0,1),(1,1),(3,1)}	{(2,1),(6,1),(11,1)}
{(0,2),(1,2),(3,2)}	{(2,2),(6,2),(11,2)}
{(4,1),(10,1),(5,2)}	{(5,1),(13,1),(4,2)}
{(7,1),(17,1),(13,2)}	{(8,2),(16,2),(19,1)}
{(9,2),(19,2),(15,1)}	{(14,2),(20,2),(9,1)}
{(8,1),(17,2),∞_1}	{(12,1),(15,2),∞_2}
{(14,1),(12,2),∞_3}	{(16,1),(18,2),∞_4}
{(18,1),(10,2),∞_5}	{(20,1),(7,2),∞_6}

A resolvable *GDD* of type 6^{10}.

Points: $(Z_{27} \times \{1,2\}) \cup \{\infty_i: 1 \leq i \leq 6\}$.

Groups: $\{\{0+i, 9+i, 18+i\} \times \{1,2\}: \ i = 0,1,2,3,4,5,6,7,8\} \cup \{\{\infty_i: 1 \leq i \leq 6\}\}$.

Resolution classes: develop the following modulo 27

{(0,1),(1,1),(3,1)}	{(2,1),(6,1),(12,1)}
{(4,1),(9,1),(16,1)}	{(0,2),(1,2),(3,2)}
{(2,2),(6,2),(12,2)}	{(5,1),(13,1),(7,2)}
{(7,1),(18,1),(4,2)}	{(8,1),(21,1),(9,2)}
{(5,2),(17,2),(10,1)}	{(15,2),(23,2),(11,1)}
{(11,2),(25,2),(15,1)}	{(19,2),(26,2),(20,1)}
{(10,2),(21,2),(23,1)}	{(8,2),(13,2),(24,1)}
{(14,1),(22,2),∞_1}	{(17,1),(20,2),∞_2}
{(19,1),(24,2),∞_3}	{(22,1),(14,2),∞_4}
{(25,1),(18,2),∞_5}	{(26,1),(16,2),∞_6}

A resolvable *GDD* of type 8^6.

Points: $(Z_{20} \times \{1,2\}) \cup \{\infty_i: 1 \leq i \leq 8\}$.

Groups: $\{\{0+i, 5+i, 10+i, 15+i\} \times \{1,2\}: \ i = 0,1,2,3,4\} \cup \{\{\infty_i: 1 \leq i \leq 8\}\}$.

Resolution classes: develop the following modulo 20

$\{(0,1),(1,1),(3,1)\}$	$\{(2,1),(6,1),(13,1)\}$
$\{(0,2),(1,2),(3,2)\}$	$\{(2,2),(6,2),(13,2)\}$
$\{(4,1),(10,1),(7,2)\}$	$\{(5,1),(17,1),(4,2)\}$
$\{(8,2),(14,2),(16,1)\}$	$\{(9,2),(17,2),(15,1)\}$
$\{(7,1),(15,2),\infty_1\}$	$\{(8,1),(19,2),\infty_2\}$
$\{(9,1),(18,2),\infty_3\}$	$\{(11,1),(12,2),\infty_4\}$
$\{(12,1),(16,2),\infty_5\}$	$\{(14,1),(10,2),\infty_6\}$
$\{(18,1),(11,2),\infty_7\}$	$\{(19,1),(5,2),\infty_8\}$

A resolvable GDD of type 12^4.

Points: $(\mathbf{Z}_{18} \times \{1,2\}) \cup (\{a\} \times \mathbf{Z}_2) \cup (\{b\} \times \mathbf{Z}_3) \cup \{\infty_i : 1 \le i \le 7\}$.

Groups: $\{\{0+i,3+i,6+i,9+i,12+i,15+i\} \times \{1,2\}: \quad i = 0,1,2\}$ $\cup \{(\{a\} \times \mathbf{Z}_2) \cup (\{b\} \times \mathbf{Z}_3) \cup \{\infty_i : 1 \le i \le 7\}\}$.

Resolution classes: develop the following modulo 18 (note: the second coordinate of elements in $(\{a\} \times \mathbf{Z}_2) \cup (\{b\} \times \mathbf{Z}_3)$ is written as a subscript, and is evaluated modulo 2 or 3, as the case may be).

$\{(0,1),(1,1),(5,1)\}$	$\{(0,2),(1,2),(5,2)\}$
$\{(2,1),(4,1),(3,2)\}$	$\{(2,2),(4,2),(9,1)\}$
$\{(7,1),(14,2),\infty_1\}$	$\{(8,1),(12,2),\infty_2\}$
$\{(11,1),(16,2),\infty_3\}$	$\{(12,1),(10,2),\infty_4\}$
$\{(15,1),(11,2),\infty_5\}$	$\{(16,1),(8,2),\infty_6\}$
$\{(17,1),(7,2),\infty_7\}$	$\{(6,1),(14,1),b_0\}$
$\{(8,1),(12,2),b_1\}$	$\{(9,2),(17,2),b_2\}$
$\{(3,1),(10,1),a_0\}$	$\{(6,2),(13,2),a_1\}$