The equivalence of certain incomplete transversal designs and frames

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ABSTRACT

The main result of this note is to show that two classes of designs are equivalent to each other: a certain class of frames and a certain class of incomplete transversal designs. The existence of an incomplete transversal design $TD(k+1,kw) - TD(k+1,w)$ implies the existence of a frame of block-size $k$ and type $((k-1)\cdot w)^{k+1}$, and conversely, the existence of a frame of block-size $k$ and type $t^{k+1}$ implies the existence of an incomplete transversal design $TD(k+1,t(k-1)) - TD(k+1,t/(k-1))$. Several examples are given.

1. Introduction.

The main result of this note is to show that two classes of designs are equivalent to each other: a certain class of frames and a certain class of incomplete transversal designs. We need to define some terminology before stating our result.

A group-divisible design (or GDD) is a triple $(X,G,A)$, which satisfies the following properties:

1. $G$ is a partition of $X$ into subsets called groups
2. $A$ is a set of subsets of $X$ (called blocks) such that a group and a block contain at most one common point
3. every pair of points from distinct groups occurs in a unique block.

The group-type, or type, of a GDD $(X,G,A)$ is the multiset $\{ |G| : G \in G \}$. We usually use an "exponential" notation to describe group-types: a group-type $1^i2^j3^k\cdots$ denotes $i$ occurrences of 1, $j$ occurrences of 2, etc. We will say that a GDD has block-size $k$ if $|A| = k$ for every $A \in A$.

If $(X,G,A)$ is a GDD of block-size $k$ and $G \in G$, then we say that a

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set $P \subseteq A$ of blocks is a **holey parallel class** with hole $G$ provided that $P$ consists of $(|X| - |G|)/k$ disjoint blocks that partition $X \setminus G$. We write $h(P) = G$ to denote that $G$ is the hole of $P$. If we can partition the set of blocks $A$ into a set $P$ of holey parallel classes, then we say that $(X, G, P)$ is a **frame** with block-size $k$.

We can think of a frame as being a resolvable BIBD with holes, exactly as a GDD is a BIBD with holes. (All the frames in this paper are "one-dimensional" objects. In other papers, the term "frame" has usually referred to square arrays (i.e. "two-dimensional" objects) in which the rows, and the columns, constitute a resolution, or partition of the block set, into holey parallel classes. Further, these two resolutions are required to be "orthogonal").

The following result was proved in the case $k = 3$ in [7], and the general proof is essentially the same.

**Theorem 1.1.** Let $(X, G, P)$ be a frame with block-size $k$. For every group $G \in G$, there are exactly $|G \setminus (k-1)$ holey parallel classes $P \in P$ with $h(P) = G$.

Frames with block-size 3 are studied in [7] and are used to prove new results on the existence of subdesigns in Kirkman triple systems.

A **transversal design** $TD(k, n)$ is a GDD with $kn$ points, $k$ groups of size $n$, and $n^2$ blocks of size $k$. It follows that every group and every block of a transversal design intersect in a point. It is well-known that a $TD(k, n)$ is equivalent to $k - 2$ mutually orthogonal Latin squares (MOLS) of order $n$.

We also need to define the idea of incomplete transversal designs. Informally, a $TD(k, n) - TD(k, m)$ (an **incomplete** transversal design) is a transversal design from which a sub-transversal design is missing. (This concept was introduced by J. Horton in [5]. He used the notation $IA(n, m, k, )$.) We give a formal definition. A $TD(k, n) - TD(k, m)$ is a quadruple $(X, G, H, A)$ which satisfies the following properties:

1. $X$ is a set of cardinality $kn$
2. $G = \{G_i: 1 \leq i \leq n\}$ is a partition of $X$ into $k$ groups of size $n$
3. $H = \{H_i: 1 \leq i \leq n\}$, where each $H_i \subseteq G_i$, and $|H_i| = m$, $1 \leq i \leq n$
4. $A$ is a set of $n^2 - m^2$ blocks of size $k$, each of which intersects each group in a point
5. every pair of points $\{x, y\}$ from distinct groups, such that at least one of $x, y$ is in $\bigcup_{1 \leq i \leq n}(G_i - H_i)$, occurs in a unique block of $A$.

Transversal designs are of fundamental importance in constructions.
for designs, and incomplete transversal designs have proved to be a very useful generalization. For some constructions and applications of these designs, we refer the reader to [4], [5], [6], and [8].

Our main result which will be proved in Section 2 is the following.

**Theorem 1.2.** The existence of an incomplete TD \( (k+1,kw) \) implies the existence of a frame of block-size \( k \) and type \( ((k-1)\cdot w)^{k+1} \), and conversely, the existence of a frame of block-size \( k \) and type \( t^{k+1} \) implies the existence of a TD \( (k+1,t/(k-1)) \).

It is easy to see that, if an incomplete TD \( (k+1,v) \) exists, then \( v \geq kw \), and analogously, if there exists a frame of block-size \( k \) and type \( t^u \), then \( u \geq k + 1 \). Thus, the designs referred to in Theorem 1.2 are "extremal" in some sense.

As well, we can construct certain "separable" designs as a consequence of these designs. A symmetric 1-design \( S(1,k,v) \) is a pair \( (X,A) \), where \( X \) is a set of \( v \) points, and \( A \) is a set of \( v \) \( k \)-subsets of \( X \) (blocks) such that every point occurs in precisely \( k \) blocks. We have the following result which will be proved in Section 2.

**Theorem 1.3.** The existence of a frame of block-size \( k \) and type \( t^{k+1} \) (or the equivalent incomplete TD) implies the existence of a GDD \( (X,G,A) \) of type \( t^{k+1} \) in which the set of blocks \( A \) can be partitioned into \( t/(k-1) \) sets of blocks \( A_1, \ldots, A_{t/(k-1)} \), such that each \( (X,A_i) \) is a symmetric \( S(1,k,tk) \), \( 1 \leq i \leq t/(k-1) \).

We do not know under what conditions the converse of Theorem 1.3 is true; this is discussed further in Section 4.

We prove Theorems 1.2 and 1.3 in Section 2. Then in Section 3, we give several examples, some old and some new.

### 2. Proofs of the Theorems.

We now give proofs of Theorems 1.2 and 1.3.

**Proof of Theorem 1.2.** Let \( (X,G,H,A) \) be a TD \( (k+1,kw) \) - TD \( (k+1,w) \), where \( G = \{ G_i: 1 \leq i \leq k+1 \} \) and \( H = \{ H_i: 1 \leq i \leq k+1 \} \), where each \( H_i \subseteq G_i, 1 \leq i \leq k+1 \). Denote \( H_i = \{ \alpha_{ij}: 1 \leq j \leq w \}, 1 \leq i \leq k+1 \), and let \( J_i = G_i \setminus H_i, 1 \leq i \leq k+1 \). Let \( Y = \bigcup_{1 \leq i \leq k+1} J_i \) and \( J = \{ J_i: 1 \leq i \leq k+1 \} \). We shall construct a frame, \( (Y,J,P) \).

By simple counting, it follows that \( |A \cap (\bigcup_{1 \leq i \leq k+1} H_i)| = 1 \), for every \( A \in A \). For every \( \alpha_{ij} \), we define a holey parallel class.
\[ P_{ij} = \{ A \setminus \{ \omega_j \} : \omega_j \in A \in \mathbf{A} \}. \]
Then,
\[ P = \{ P_{ij} : 1 \leq j \leq w, 1 \leq i \leq k+1 \}. \]
It is straightforward to check that \((Y,J,P)\) is a frame of block-size \(k\) and type \(((k-1)w)^{k+1}\).

Conversely, suppose we start with a frame of block-size \(k\) and type \(t^{k+1}\), \((X,G,P)\), where \(G = \{ G_i : 1 \leq i \leq k+1 \}\) and \(P = \{ P_{ij} : 1 \leq i \leq k+1, i \leq j \leq t/(k-1) \}\). We associate with each \(P_{ij}\) a new point \(\omega_{ij}\). Now, define \(H_i = \{ \omega_{ij} : i \leq j \leq t/(k-1) \}, 1 \leq i \leq k+1, \)
\(H = \{ H_i : 1 \leq i \leq k+1 \}, \quad J = \{ J_i = H_i \cup G_i : 1 \leq i \leq k+1 \}\), and \(Y = \bigcup_{1 \leq i \leq k+1} J_i\).

We construct a \(TD(k+1,t(k-1)) - TD(k+1,t/(k-1)), (Y,J,H,A)\), where the blocks are \(A = \{ A \cup \omega_i : A \in P_{ij}, 1 \leq j \leq w, 1 \leq i \leq k+1 \}\). Again it is easy to verify that we have the desired incomplete \(TD\).

We now give a proof of Theorem 1.3.

**Proof of Theorem 1.3.** We start with a frame of block-size \(k\) and type \(t^{k+1}\), \((X,G,P)\), where \(G = \{ G_i : 1 \leq i \leq k+1 \}\) and \(P = \{ P_{ij} : 1 \leq i \leq k+1, i \leq j \leq t/(k-1) \}\). For \(1 \leq j \leq t/(k-1)\), we define \(A_j = \bigcup_{1 \leq i \leq t^{k+1}} P_{ij}\). Then it is easy to see that each \(A_j\) is a symmetric 1-design, as desired. For, the number of blocks in each \(A_j\) is \((k+1)(tk/k) = t(k+1)\), and each point occurs \(k\) blocks of each \(A_j\).

We also have the following consequence of Theorem 1.3.

**Corollary 2.4.** If there exists a frame of block-size \(k\) and type \(t^{k+1}\), a \(TD(m,t)\), and a \(TD(m,k)\), then there exists a \(TD(m,t(k+1))\).

**Proof.** Construct the separable design in Theorem 1.3, and apply Theorem 4 of Bose, Shrikhande, and Parker [1].

3. Examples.

In this section we give several interesting examples.

**Example 3.1.** There is a \(TD(4,6) - TD(4,2)\), which is equivalent to a frame of block-size 3 and type \(4^1\). This incomplete \(TD\) was first found by Euler and has been rediscovered several times since (see, for example, [5]). It is particularly interesting in view of the non-existence of a \(TD(4,6)\).

**Example 3.2.** There is a \(TD(5,8) - TD(5,2)\) or, equivalently, a frame of block-size 4 and type \(6^3\).
Proof. We construct the frame. In [3], a GDD of block-size 4 and type 6 is constructed, and this GDD gives rise to the following frame:

\[ X = \mathbb{Z}_{15} \times \{0,1\} \text{ and } G = \{(i_0,i_1,(5+i)\overline{0}),(5+i)\overline{1},(10+i)\overline{0},(10+i)\overline{1} \mid 0 \leq i \leq 4\}. \]

We start with two holey parallel classes:

\[
\begin{array}{cccccccc}
9_0 & 12_0 & 13_0 & 1 & 13_0 & 4_0 & 6_0 & 12_1 \\
14_0 & 2_0 & 3_0 & 6_1 & 3_0 & 9_0 & 11_0 & 2_1 \\
4_0 & 7_0 & 8_0 & 11_1 & 8_0 & 14_0 & 1_0 & 7_1 \\
9_1 & 12_1 & 8_1 & 11_0 & 13_1 & 4_1 & 11_1 & 2_0 \\
14_1 & 2_1 & 13_1 & 1_0 & 3_1 & 9_1 & 1_1 & 7_0 \\
4_1 & 7_1 & 3_1 & 6_0 & 8_1 & 14_1 & 6_1 & 12_0 \\
\end{array}
\]

The remaining 8 classes are obtained by adding 1, 2, 3, and 4, reducing modulo 15.

Example 3.3. There is a TD(5,12) - TD(5,3) or, equivalently, a frame of block-size 4 and type 96.

Proof. We construct a TD(4,12) - TD(5,3). Denote \( Y = \mathbb{Z}_9 \cup \{\infty_1,\infty_2,\infty_3\} \), \( X = Y \times \{1,2,3,4,5\} \), \( G = \{G_i = Y \times \{i\} : 1 \leq i \leq 5\} \), and \( H = \{H_i = \{\infty_1,\infty_2,\infty_3\} \times \{i\} : 1 \leq i \leq 5\} \). We give a set of 15 base blocks, which are developed through \( Z_9 \). For convenience, we omit the second coordinate of each ordered pair; each element in column \( i \) of the following array has second coordinate \( i \), \( 1 \leq i \leq 5 \).

\[
\begin{array}{cccc}
\infty_1 & 0 & 0 & 0 \\
\infty_2 & 1 & 0 & 4 \\
\infty_3 & 0 & 1 & 2 \\
0 & \infty_1 & 0 & 2 \\
0 & 0 & \infty_1 & 5 \\
0 & \infty_2 & 1 & 7 \\
0 & 1 & \infty_2 & 8 \\
0 & \infty_3 & 7 & 1 \\
0 & 7 & \infty_3 & 6 \\
0 & 2 & 5 & \infty_1 \\
0 & 5 & 2 & 0 \\
0 & 8 & 3 & \infty_2 \\
0 & 3 & 8 & 4 \\
0 & 4 & 6 & \infty_3 \\
0 & 6 & 4 & 3 \\
\end{array}
\]
Example 3.4. There is a \( TD(6,10) - TD(6,2) \) or, equivalently, a frame of block-size 5 and type 8$^8$.

Proof. This incomplete \( TD \) was found by Brouwer [2]. He also observed that it gave rise to a separable design, and hence there is a \( TD(6,48) \) (Corollary 2.4).

4. Remarks.

As an open problem, we ask under what conditions the converse of Theorem 1.3 is true. We make a couple of observations.

Suppose we begin with a \( GDD(X,G,A) \) of block-size \( k \) and type \( t^{k+1} \) in which the set of blocks \( A \) can be partitioned into \( t/(k-1) \) sets of blocks \( A_1, \ldots, A_{t/(k-1)} \), such that each \( (X,A_i) \) is a symmetric \( S(1,k,tk) \), \( 1 \leq i \leq t/(k-1) \). Suppose \( G = \{G_j; 1 \leq j \leq k+1\} \). Each \( A_i \) consists of a set of \( t(k+1) \) blocks that contain every point \( k \) times. Given any \( G \in G \), there are \( t \cdot k \) blocks of each \( A_i \) that meet \( G \), and \( t \) that don't. Hence, we can partition each \( A_i \) into \( k+1 \) sets \( P_{ij} \), such that each \( P_{ij} \) consists of \( t \) blocks disjoint from \( G_j \), \( 1 \leq j \leq k+1 \). We would like each \( P_{ij} \) to be a holey parallel class; then we would have the desired frame. However, this need not happen, as indicated by the following example.

Example 4.1. We give a \( GDD \) of block-size 2 and type 3$^3$, with the blocks partitioned into three 1-designs. The groups are \( \{1,2,3\}, \{4,5,6\}, \{7,8,9\} \), and the blocks are as follows (for brevity, we write a block \( \{a,b\} \) as \( ab \)):

\[
\begin{array}{cccccccc}
A_1 & 15 & 19 & 59 & 34 & 38 & 48 & 26 & 27 & 67 \\
A_2 & 16 & 18 & 68 & 24 & 29 & 35 & 57 & 39 & 47 \\
A_3 & 14 & 17 & 25 & 28 & 58 & 36 & 69 & 49 & 37 \\
\end{array}
\]

If we partition \( A_2 \) as described above, we obtain \( P_{21} = \{68,57,47\} \), \( P_{22} = \{18,29,39\} \), and \( P_{23} = \{16,24,35\} \). Unfortunately, these are not holey parallel classes. In this example, it is possible to partition the blocks into holey parallel classes in a different way, to produce a frame of block-size 2 and type 3$^3$.

It would be interesting to find examples of separable \( GDDs \) in which there is no way to partition the blocks into holey parallel classes.

References.


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