The equivalence of certain incomplete transversal designs and frames

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ABSTRACT

The main result of this note is to show that two classes of designs are equivalent to each other: a certain class of frames and a certain class of incomplete transversal designs. The existence of an incomplete transversal design $TD(k+1,kw) - TD(k+1,w)$ implies the existence of a frame of block-size k and type $((k-1)\cdot w)^{k+1}$, and conversely, the existence of a frame of block-size k and type t^{k+1} implies the existence of an incomplete transversal design $TD(k+1,tk/(k-1)) - TD(k+1,t/(k-1))$. Several examples are given.

1. Introduction.

The main result of this note is to show that two classes of designs are equivalent to each other: a certain class of frames and a certain class of incomplete transversal designs. We need to define some terminology before stating our result.

A *group-divisible design* (or *GDD*) is a triple $(X,\mathbf{G},\mathbf{A})$, which satisfies the following properties:

(1) $\mathbf{G}$ is a partition of X into subsets called *groups*

(2) $\mathbf{A}$ is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point

(3) every pair of points from distinct groups occurs in a unique block.

The group-type, or *type*, of a $GDD(X,\mathbf{G},\mathbf{A})$ is the multiset $\{|G|: G \in \mathbf{G}\}$. We usually use an "exponential" notation to describe group-types: a group-type $1^i 2^j 3^k \cdots$ denotes i occurrences of 1, j occurrences of 2, etc. We will say that a GDD has block-size k if $|A| = k$ for every $A \in \mathbf{A}$.

If $(X,\mathbf{G},\mathbf{A})$ is a GDD of block-size k and $G \in \mathbf{G}$, then we say that a set $P \subseteq \mathbf{A}$ of blocks is a *holey parallel class* with *hole* G provided that P consists of $(|X| - |G|)/k$ disjoint blocks that partition $X \setminus G$. We write $h(P) = G$ to denote that G is the hole of P. If we can partition the set of blocks $\mathbf{A}$ into a set $\mathbf{P}$ of holey parallel classes, then we say that $(X,\mathbf{G},\mathbf{P})$ is a *frame* with block-size k.

We can think of a frame as being a resolvable *BIBD* with holes, exactly as a *GDD* is a *BIBD* with holes. (All the frames in this paper are "one-dimensional" objects. In other papers, the term "frame" has usually referred to square arrays (i.e. "two-dimensional" objects) in which the rows, and the columns, constitute a resolution, or partition of the block set, into holey parallel classes. Further, these two resolutions are required to be "orthogonal".)

The following result was proved in the case $k = 3$ in [7], and the general proof is essentially the same.

Theorem 1.1. *Let* $(X,\mathbf{G},\mathbf{P})$ *be a frame with block-size* k. *For every group* $G \in \mathbf{G}$, *there are exactly* $|G|/(k-1)$ *holey parallel classes* $P \in \mathbf{P}$ *with* $h(P) = G$.

Frames with block-size 3 are studied in [7] and are used to prove new results on the existence of subdesigns in Kirkman triple systems.

A *transversal design* $TD(k,n)$ is a *GDD* with kn points, k groups of size n, and n^2 blocks of size k. It follows that every group and every block of a transversal design intersect in a point. It is well-known that a $TD(k,n)$ is equivalent to $k-2$ mutually orthogonal Latin squares (*MOLS*) of order n.

We also need to define the idea of incomplete transversal designs. Informally, a $TD(k,n) - TD(k,m)$ (an *incomplete* transversal design) is a transversal design from which a sub-transversal design is missing. (This concept was introduced by J. Horton in [5]. He used the notation $IA(n,m,k)$.) We give a formal definition. A $TD(k,n) - TD(k,m)$ is a quadruple $(X,\mathbf{G},\mathbf{H},\mathbf{A})$ which satisfies the following properties:

(1) X is a set of cardinality kn

(2) $\mathbf{G} = \{G_i : 1 \leq i \leq n\}$ is a partition of X into k groups of size n

(3) $\mathbf{H} = \{H_i : 1 \leq i \leq n\}$, where each $H_i \subseteq G_i$, and $|H_i| = m$, $1 \leq i \leq n$

(4) $\mathbf{A}$ is a set of $n^2 - m^2$ blocks of size k, each of which intersects each group in a point

(5) every pair of points $\{x,y\}$ from distinct groups, such that at least one of x,y is in $\bigcup_{1 \leq i \leq n}(G_i - H_i)$, occurs in a unique block of $\mathbf{A}$.

Transversal designs are of fundamental importance in constructions for designs, and incomplete transversal designs have proved to be a very useful generalization. For some constructions and applications of these designs, we refer the reader to [4], [5], [6], and [8].

Our main result which will be proved in Section 2 is the following.

Theorem 1.2. *The existence of an incomplete* $TD(k+1,kw) - TD(k+1,w)$ *implies the existence of a frame of block-size* k *and type* $((k-1)\cdot w)^{k+1}$, *and conversely, the existence of a frame of block-size* k *and type* t^{k+1} *implies the existence of a* $TD(k+1,tk/(k-1)) - TD(k+1,t/(k-1))$.

It is easy to see that, if an incomplete $TD(k+1,v) - TD(k+1,w)$ exists, then $v \geq kw$, and analogously, if there exists a frame of block-size k and type t^u, then $u \geq k+1$. Thus, the designs referred to in Theorem 1.2 are "extremal" in some sense.

As well, we can construct certain "separable" designs as a consequence of these designs. A symmetric 1-design $S(1,k,v)$ is a pair $(X,\mathbf{A})$, where X is a set of v points, and $\mathbf{A}$ is a set of v k-subsets of X (blocks) such that every point occurs in precisely k blocks. We have the following result which will be proved in Section 2.

Theorem 1.3. *The existence of a frame of block-size* k *and type* t^{k+1} *(or the equivalent incomplete TD) implies the existence of a* $GDD(X,\mathbf{G},\mathbf{A})$ *of type* t^{k+1} *in which the set of blocks* $\mathbf{A}$ *can be partitioned into* $t/(k-1)$ *sets of blocks* $\mathbf{A}_1, \ldots, \mathbf{A}_{t/(k-1)}$, *such that each* $(X,\mathbf{A}_i)$ *is a symmetric* $S(1,k,tk)$, $1 \leq i \leq t/(k-1)$.

We do not know under what conditions the converse of Theorem 1.3 is true; this is discussed further in Section 4.

We prove Theorems 1.2 and 1.3 in Section 2. Then in Section 3, we give several examples, some old and some new.

2. Proofs of the Theorems.

We now give proofs of Theorems 1.2 and 1.3.

Proof of Theorem 1.2. Let $(X,\mathbf{G},\mathbf{H},\mathbf{A})$ be a $TD(k+1,kw) - TD(k+1,w)$, where $\mathbf{G} = \{G_i\colon 1 \leq i \leq k+1\}$ and $\mathbf{H} = \{H_i\colon 1 \leq i \leq k+1\}$, where each $H_i \subseteq G_i$, $1 \leq i \leq k+1$. Denote $H_i = \{\infty_{ij}\colon 1 \leq j \leq w\}$, $1 \leq i \leq k+1$, and let $J_i = G_i \setminus H_i$, $1 \leq i \leq k+1$. Let $Y = \bigcup_{1 \leq i \leq k+1} J_i$ and $\mathbf{J} = \{J_i\colon 1 \leq i \leq k+1\}$. We shall construct a frame, $(Y,\mathbf{J},\mathbf{P})$.

By simple counting, it follows that $|A \cap (\bigcup_{1 \leq i \leq k+1} H_i)| = 1$, for every $A \in \mathbf{A}$. For every ∞_{ij}, we define a holey parallel class

$P_{ij} = \{A \backslash \{\infty_j\}: \infty_j \in A \in \mathbf{A}\}$. Then, $\mathbf{P} = \{P_{ij}: 1 \leq j \leq w, 1 \leq i \leq k+1\}$. It is straightforward to check that $(Y,\mathbf{J},\mathbf{P})$ is a frame of block-size k and type $((k-1)w)^{k+1}$.

Conversely, suppose we start with a frame of block-size k and type t^{k+1}, $(X,\mathbf{G},\mathbf{P})$, where $\mathbf{G} = \{G_i: 1 \leq i \leq k+1\}$ and $\mathbf{P} = \{P_{ij}: 1 \leq i \leq k+1, i \leq j \leq t/(k-1)\}$. We associate with each P_{ij} a new point ∞_j. Now, define $H_i = \{\infty_j: i \leq j \leq t/(k-1)\}$, $1 \leq i \leq k+1$, $\mathbf{H} = \{H_i: 1 \leq i \leq k+1\}$, $\mathbf{J} = \{J_i = H_i \cup G_i: 1 \leq i \leq k+1\}$, and $Y = \bigcup_{1 \leq i \leq k+1} J_i$.

We construct a $TD(k+1,tk/(k-1)) - TD(k+1,t/(k-1))$, $(Y,\mathbf{J},\mathbf{H},\mathbf{A})$, where the blocks are $\mathbf{A} = \{A \cup \infty_j: A \in P_{ij}, 1 \leq j \leq w, 1 \leq i \leq k+1\}$. Again it is easy to verify that we have the desired incomplete TD.

We now give a proof of Theorem 1.3.

Proof of Theorem 1.3. We start with a frame of block-size k and type t^{k+1}, $(X,\mathbf{G},\mathbf{P})$, where $\mathbf{G} = \{G_i: 1 \leq i \leq k+1\}$ and $P = \{P_{ij}: 1 \leq i \leq k+1, i \leq j \leq t/(k-1)\}$. For $1 \leq j \leq t/(k-1)$, we define $\mathbf{A}_j = \bigcup_{1 \leq i \leq k+1} P_{ij}$. Then it is easy to see that each $\mathbf{A}_j$ is a symmetric 1-design, as desired. For, the number of blocks in each $\mathbf{A}_j$ is $(k+1)\cdot(tk/k) = t(k+1)$, and each point occurs k blocks of each $\mathbf{A}_j$.

We also have the following consequence of Theorem 1.3.

Corollary 2.4. *If there exists a frame of block-size k and type t^{k+1}, a $TD(m,t)$, and a $TD(m,k)$, then there exists a $TD(m,t(k+1))$.*

Proof. Construct the separable design in Theorem 1.3, and apply Theorem 4 of Bose, Shrikhande, and Parker [1].

3. Examples.

In this section we give several interesting examples.

Example 3.1. There is a $TD(4,6) - TD(4,2)$, which is equivalent to a frame of block-size 3 and type 4^4. This incomplete TD was first found by Euler and has been rediscovered several times since (see, for example, [5]). It is particularly interesting in view of the non-existence of a $TD(4,6)$.

Example 3.2. There is a $TD(5,8) - TD(5,2)$ or, equivalently, a frame of block-size 4 and type 6^5.

Proof. We construct the frame. In [3], a *GDD* of block-size 4 and type 6^5 is constructed, and this *GDD* gives rise to the following frame:

$X = \mathbf{Z}_{15} \times \{0,1\}$ and $\mathbf{G} = \{\{i_0, i_1, (5+i)_0, (5+i)_1, (10+i)_0, (10+i)_1\}: 0 \leq i \leq 4\}$.

We start with two holey parallel classes:

9_0	12_0	13_0	1_1	13_0	4_0	6_0	12_1
14_0	2_0	3_0	6_1	3_0	9_0	11_0	2_1
4_0	7_0	8_0	11_1	8_0	14_0	1_0	7_1
9_1	12_1	8_1	11_0	13_1	4_1	11_1	2_0
14_1	2_1	13_1	1_0	3_1	9_1	1_1	7_0
4_1	7_1	3_1	6_0	8_1	14_1	6_1	12_0

The remaining 8 classes are obtained by adding 1,2,3, and 4, reducing modulo 15.

Example 3.3. There is a $TD(5,12) - TD(5,3)$ or, equivalently, a frame of block-size 4 and type 9^5.

Proof. We construct a $TD(4,12) - TD(5,3)$. Denote $Y = \mathbf{Z}_9 \cup \{\infty_1,\infty_2,\infty_3\}$, $X = Y \times \{1,2,3,4,5\}$, $\mathbf{G} = \{G_i = Y \times \{i\}: 1 \leq i \leq 5\}$, and $\mathbf{H} = \{H_i = \{\infty_1,\infty_2,\infty_3\} \times \{i\}: 1 \leq i \leq 5\}$. We give a set of 15 base blocks, which are developed through $\mathbf{Z}_9$. For convenience, we omit the second coordinate of each ordered pair; each element in column i of the following array has second coordinate i, $1 \leq i \leq 5$.

∞_1	0	0	0	0
∞_2	1	0	4	2
∞_3	0	1	2	4
0	∞_1	0	2	5
0	0	∞_1	5	2
0	∞_2	1	7	8
0	1	∞_2	8	7
0	∞_3	7	1	6
0	7	∞_3	6	1
0	2	5	∞_1	0
0	5	2	0	∞_1
0	8	3	∞_2	4
0	3	8	4	∞_2
0	4	6	∞_3	3
0	6	4	3	∞_3

Example 3.4. There is a $TD(6,10) - TD(6,2)$ or, equivalently, a frame of block-size 5 and type 8^6.

Proof. This incomplete TD was found by Brouwer [2]. He also observed that it gave rise to a separable design, and hence there is a $TD(6,48)$ (Corollary 2.4).

4. Remarks.

As an open problem, we ask under what conditions the converse of Theorem 1.3 is true. We make a couple of observations.

Suppose we begin with a $GDD(X,\mathbf{G},\mathbf{A})$ of block-size k and type t^{k+1} in which the set of blocks A can be partitioned into $t/(k-1)$ sets of blocks $\mathbf{A}_1, \ldots, \mathbf{A}_{t/(k-1)}$, such that each $(X,\mathbf{A}_i)$ is a symmetric $S(1,k,tk)$, $1 \leq i \leq t/(k-1)$. Suppose $\mathbf{G} = \{G_j\colon 1 \leq j \leq k+1\}$. Each $\mathbf{A}_i$ consists of a set of $t(k+1)$ blocks that contain every point k times. Given any $G \in \mathbf{G}$, there are $t \cdot k$ blocks of each $\mathbf{A}_i$ that meet G, and t that don't. Hence, we can partition each $\mathbf{A}_i$ into $k+1$ sets P_{ij}, such that each P_{ij} consists of t blocks disjoint from G_j, $1 \leq j \leq k+1$. We would like each P_{ij} to be a holey parallel class; then we would have the desired frame. However, this need not happen, as indicated by the following example.

Example 4.1. We give a GDD of block-size 2 and type 3^3, with the blocks partitioned into three 1-designs. The groups are $\{1,2,3\}$, $\{4,5,6\}$, $\{7,8,9\}$, and the blocks are as follows (for brevity, we write a block $\{a,b\}$ as ab):

$\mathbf{A}_1$	15	19	59	34	38	48	26	27	67
$\mathbf{A}_2$	16	18	68	24	29	35	57	39	47
$\mathbf{A}_3$	14	17	25	28	58	36	69	49	37

If we partition $\mathbf{A}_2$ as described above, we obtain $P_{21} = \{68,57,47\}$, $P_{22} = \{18,29,39\}$, and $P_{23} = \{16,24,35\}$. Unfortunately, these are not holey parallel classes. In this example, it is possible to partition the blocks into holey parallel classes in a different way, to produce a frame of block-size 2 and type 3^3.

It would be interesting to find examples of separable GDDs in which there is no way to partition the blocks into holey parallel classes.

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