HOLEY PERPENDICULAR ARRAYS

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ABSTRACT. It is often useful to consider combinatorial designs with holes. Perpendicular arrays are an interesting and useful type of design, and may be thought of as containing holes of size one. We consider perpendicular arrays with holes of other sizes, giving several constructions and applications of these "holey" arrays.

We also prove a PBD-closure result which is of independent interest. If $v \geq 13$, $v \neq 14, 16, 18, 19, 23, 26, 27, 30, 38$, or 42, then there exists a pairwise balanced design on v points having blocks of size 4, 5, 7, 8, 9, or 12.

1. Introduction.

Let $\mathcal{G} = \{G_1, \ldots, G_n\}$ be a partition of a finite set X (the elements of $\mathcal{G}$ are called *holes*). A *holey perpendicular array* (or HPA) having strength t and partition $\mathcal{G}$, is an array P, having entries from X, which satisfies the properties:

1) P has exactly t rows
2) no column of P contains two elements from the same hole of $\mathcal{G}$
3) given any two distinct rows, i and j, of P, and given any two distinct elements of X from different holes, x and y, there is a unique column k of P such that $\{P(i,k), P(j,k)\} = \{x, y\}$.

It follows from the definition that an HPA, on symbol set X and having partition $\mathcal{G}$, contains exactly r columns, where

$$r = \left(|X|^2 - |X| - \sum_{G \in \mathcal{G}} (|G|^2 - |G|)\right)/2.$$

The *type* of an HPA having partition $\mathcal{G}$ is the multiset $\{|G| : G \in \mathcal{G}\}$. We use the notation $a^i b^j c^k \ldots$ to describe the type of an HPA, where there are i holes of size a, j holes of size b, etc. In the literature, HPAs of type 1^n have been referred to as *perpendicular arrays*. A $PA(n, s)$ denotes a HPA of strength s and type 1^n. For information concerning perpendicular arrays, we refer the reader to **[8]**, **[9]**, **[10]**, and **[11]**.

EXAMPLE 1.1: A $PA(5,4)$ (or, equivalently, an HPA of type 1^5 and strength 4)

0	1	2	3	4	0	1	2	3	4
1	2	3	4	0	2	3	4	0	1
2	3	4	0	1	4	0	1	2	3
4	0	1	2	3	3	4	0	1	2

The spectra for $PA(n,s)$ have been almost completely determined for $s \le 5$ (see [8] and [10]). The results are as follows:

(i) There exists a $PA(n,3)$ if and only if $n \ge 3$ is odd.
(ii) There exists a $PA(n,4)$ if and only if $n \ge 5$ is odd.
(iii) There exists a $PA(n,5)$ if and only if $n \ge 5$ is odd, except, possibly, $n = 39$.

In this paper, we consider HPAs with holes of size other than one. We pay particular attention to HPAs of type 2^n. We construct HPAs of type 2^n and strength 4 for all $n \ge 4$, with the possible exceptions $n = 6$, 10, 11, 14, 15, 18, 19, 23, 26, 27, 30, 38, and 42. As an application, we give a short, self-contained proof of the existence of $PA(n,4)$ for all odd integers $n \ge 5$. Also, for certain prime powers $n \equiv 1$ modulo 4, we construct HPAs of type 2^n and strength 7, using the computer.

2. Constructions for HPAs.

Our basic tool is the following recursive construction for HPAs which uses group-divisible designs. A *group-divisible design*, or GDD, is a triple $(X, \mathcal{G}, \mathcal{A})$, which satisfies the properties:

1) $\mathcal{G}$ is a partition of X into subsets (called *groups*),
2) $\mathcal{A}$ is a set of subsets of X, called *blocks*, such that a group and a block contain at most one common point, and
3) given any two points from distinct groups, say x and y, there is a unique block $A \in \mathcal{A}$ such that $x, y \in A$.

A *weighting* of a GDD $(X, \mathcal{G}, \mathcal{A})$ is a function $w: X \to Z^+ \cup \{0\}$. We can use GDDs with suitable weightings to construct large HPAs from small ones, as follows.

CONSTRUCTION 2.1. *Suppose we have a GDD $(X, \mathcal{G}, \mathcal{A})$ with a weighting w, and let $t \ge 2$ be an integer. For every block $A \in \mathcal{A}$, suppose that there is a HPA of type $\{w(x): x \in A\}$ and strength t, say $H(A)$. If we juxtapose all the arrays $H(A)$, $A \in \mathcal{A}$, horizontally, then we obtain an HPA of type $\left\{\sum_{x\in G} w(x): G \in \mathcal{G}\right\}$ and strength t.*

We obtain one very simple corollary of this construction using pairwise balanced designs (PBDs). A *pairwise balanced design* is a pair $(X, \mathcal{A})$ such that any two points occur in a unique block $A \in \mathcal{A}$.

COROLLARY 2.2. *Suppose we have a PBD $(X, \mathcal{A})$ such that there exists a $PA(|A|, t)$ for all $A \in \mathcal{A}$. If we define a set of groups $\mathcal{G} = \{\{x\}: x \in X\}$, then $(X, \mathcal{G}, \mathcal{A})$ is a GDD. If we give every point weight one, and apply Construction 2.1, then we get a $PA(|X|, t)$. (Equivalently, we are saying that the set of integers $\{m$: there exists a $PA(m,t)\}$ is PBD-closed).*

Thus, we can construct large HPAs provided we have small ones. What we need now are direct constructions for HPAs.

Most direct constructions for HPAs use difference methods. The following is an immediate generalization of the techniques described in [**10**]. Let G be an abelain group and let H be a subgroup of G, such that $g - h$ is even, where $g = |G|$ and $h = |H|$. A *holey perpendicular difference array*, or HPDA, of type $h^{g/h}$ and strength t, is a t by $(g-h)/2$ array of elements from $G \setminus H$, say $D = [d_{ij}]$, such that for any $\{i, k\} \subseteq \{1, \ldots, t\}$, we have:

$$\{\pm(d_{ij} - d_{kj}): 1 \leq j \leq (g-h)/2\} = G \setminus H.$$

Informally, we are saying that the differences obtained from any two rows produce every element of $G \setminus H$. It is easy to see that developing an HPDA through the group G will yield an HPA (cf. [**10**, Theorem 2.1]). Hence, we have

CONSTRUCTION 2.3. *If there is a holey perpendicular difference array of type $h^{g/h}$ and strength t, then there exists an HPA of type $h^{g/h}$ and strength t.*

A perpendicular difference array, $PDA(g,t)$, as defined in [**10**], is an HPDA where $H = \{0\}$, which therefore gives rise to a $PA(g,t)$.

One application of HPDAs is a finite field construction [**10**, Corollary 2.5]. Suppose m is an odd prime power, and let x be a primitive root of the Galois field $GF(m)$. Then the following $(m-1)/2$ columns comprise a $PDA(m, m)$:

$$\begin{array}{ccccl}
0 & 0 & 0 & \ldots & 0 \\
1 & x & x^2 & \ldots & x^{(n-3)/2} \\
x & x^2 & x^3 & \ldots & x^{(n-1)/2} = -1 \\
x^2 & x^3 & x^4 & \ldots & x^{(n+1)/2} = -x \\
\vdots & & & & \\
x^{n-2} & x^{n-1} & x & \ldots & x^{(n-5)/2}
\end{array}$$

Hence, we have the following corollary.

COROLLARY 2.4. *If n is an odd prime power, then there is a $PA(n,n)$.*

3. Perpendicular arrays of strength 4.

In this section, we give a short, self-contained proof of the existence of a $PA(n,4)$ for all odd $n \geq 5$. We shall use the following two small HPAs.

EXAMPLE 3.1: An HPA of type 2^4 and strength 4:

holes: $\{\{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}\}$

1	6	3	1	2	8	1	7	4	2	3	4	5	2	7	5	6	4	5	3	8	6	7	8
6	3	1	2	8	1	7	4	1	3	4	2	2	7	5	6	4	5	3	8	5	7	8	6
3	1	6	8	1	2	4	1	7	4	2	3	7	5	2	4	5	6	8	5	3	8	6	7
8	8	8	7	7	7	6	6	6	1	1	1	4	4	4	3	3	3	2	2	2	5	5	5

EXAMPLE 3.2: An HPDA of strength 4 in $Z_{10} \setminus \{0,5\}$, which gives rise to an HPA of type 2^5 and strength 4:

0	0	0	0
1	8	7	6
2	4	9	3
3	2	6	9

We use these two HPAs in our GDD construction as follows. Suppose that $n \geq 4$, $n \neq 6$, 10. Then there exist three mutually orthogonal Latin squares, or MOLS, of order n (three MOLS of order 14 were recently found by Todorov; for an existence proof of all other orders, see [**12**]). A set of three MOLS gives rise to a transversal design $TD(5,n)$, or a GDD having $5n$ points, 5 groups of size n, and n^2 blocks of size 5. Let $0 \leq t \leq n$, and delete $n-t$ points form a group of this GDD. We obtain a GDD with group-type n^4t^1, having blocks of size 4 and 5.

Now give every point weight 2, and apply the Construction 2.1 with $t = 4$. We use the two HPAs of types 2^4 and 2^5 as input HPAs, and we obtain an HPA of type $(2n)^4(2t)^1$ and strength 4.

We can now fill in the holes of this HPA with the designs $PA(2n+1, 4)$ and $PA(2t+1, 4)$, assuming they exist. For each hole G of the HPA, adjoin the columns of a PA of strength 4 on the points $G \cup \{\infty\}$, where ∞ is any new point. This produces a $PA(8n+2t+1, 4)$.

Summarizing this discussion, we have

CONSTRUCTION 3.3. *Suppose $n \geq 4$, $n \neq 6$, 10, and $0 \leq t \leq n$. If there exists a $PA(2n+1, 4)$ and a $PA(2t+1, 4)$, then there exists a $PA(8n+2t+1, 4)$.*

Assuming that we have all "small" $PA(m, 4)$ (i.e. supposing we have constructed $PA(m, 4)$ for all odd m such that $5 \leq m \leq M$, for some M), how big must M be in order that Construction 3.3 will complete the spectrum? Let us make a table of applications of the above construction.

Table 1

Construction of PAs of strength 4

n	t	$8n+2t+1$
4	$0, 2, 3, 4$	$33, 37, 39, 41$
5	$0, 2-5$	$41, 45, 47, 49, 51$
7	$0, 2-7$	$57, 61, 63, \ldots, 71$
8	$0, 2-8$	$65, 69, 71, \ldots, 81$
9	$0, 2-9$	$73, 77, 79, \ldots, 91$
11	$0, 2-11$	$89, 93, 95, \ldots, 111$
etc.		

There are no gaps from this point on. The only small numbers not covered in Table 1 are:

$$5,\ 7, \ldots, 31,\ 35,\ 43,\ 53,\ 55, \text{ and } 59.$$

So, we will be done, provided we can construct $PA(m, 4)$ for these values of m.

The finite field construction (Corollary 2.4) handles $m = 5$, 7, 9, 11, 13, 17, 19, 23, 25, 27, 29, 31, 43, 53, and 59, leaving only $m = 15$, 21, 35, and 55 to be dealt with.

The three largest of these can be killed by means of Corollary 2.2.

Since a (21,5,1) BIBD exists (i.e. there is a projective plane of order 4), and there is a $PA(5,4)$, hence there is a $PA(21,4)$.

There also exist 3 MOLS of order 7 and 11. These give rise to transversal designs $TD(5,7)$ and $TD(5,11)$. Hence there is a PBD on 35 points with blocks of size 5 and 7; and a PBD on 55 points with blocks of size 5 and 11. Since there exist $PA(m,4)$ for $m = 5$, 7, and 11, hence there exists a $PA(35,4)$ and $PA(55,4)$.

The last PA we have to construct is a $PA(15,4)$. In fact, the following $PDA(15,5)$, exhibited in [**11**], gives rise to a $PA(15,5)$:

$$\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
2 & 5 & 7 & 9 & 12 & 4 & 1 \\
6 & 3 & 14 & 10 & 7 & 13 & 4 \\
10 & 6 & 1 & 11 & 2 & 7 & 12
\end{array}$$

Finally, it is easy to see that there does not exist a $PA(m,4)$ for any even m, or for $m = 3$. Hence, we have established the following

THEOREM 3.4. *There exists a $PA(m,4)$ if and only if $m \geq 5$ is odd.*

4. HPAs of type 2^n and strength 4.

In this section, we study the existence of HPAs of type 2^n of strength 4. We have already constructed HPAs of types 2^4 and 2^5 (and strength 4). We also have direct constructions for the following cases.

LEMMA 4.1. *There exists an HPA of type 2^n and strength 4 for $n = 7$, 8, 9, and 12.*

PROOF: First, we construct an HPA of type 2^7 and strength 4. The points are $\{2, 3, \ldots, 15\}$ and the holes are $\{\{2,3\},\{4,5\},\ldots,\{14,15\}\}$. We start with the following 7 columns:

$$\begin{array}{ccccccc}
4 & 2 & 4 & 2 & 4 & 2 & 6 \\
14 & 4 & 9 & 8 & 10 & 12 & 10 \\
7 & 6 & 12 & 10 & 13 & 14 & 9 \\
10 & 8 & 15 & 15 & 2 & 7 & 12
\end{array}$$

Next, we obtain 28 columns, replacing each column by the four columns obtained by the action of the permutation group $\langle (2\ 3)\ (4\ 14\ 5\ 15)\ (6\ 11\ 7\ 10)\ (8\ 12\ 9\ 13)\rangle$. Finally, we obtain 84 columns, by replacing each column

$$\begin{array}{c} a \\ b \\ c \\ d \end{array} \quad \text{by three columns:} \quad \begin{array}{ccc} a & b & c \\ b & c & a \\ c & a & b \\ d & d & d. \end{array}$$

The resulting array is the desired HPA.

Next, we present an HDPA of type 2^8 and strength 4 in $Z_{16} \setminus \{0, 8\}$:

0	0	0	0	0	0	0
1	2	3	4	5	6	7
14	4	9	15	6	13	11
10	15	7	5	3	12	2

An HPA of type 2^9 is obtained from the following HPDA in $Z_{18} \setminus \{0, 9\}$:

0	0	0	0	0	0	0	0
1	2	3	4	5	6	7	8
16	1	14	6	11	10	15	3
12	16	8	7	3	5	17	14

Finally, we display an HPDA of type 2^{12} in $Z_{24} \setminus \{0, 12\}$:

0	0	0	0	0	0	0	0	0	0	0
1	2	3	4	5	6	7	8	9	10	11
22	20	7	15	14	1	21	6	16	11	19
18	11	23	22	19	4	15	7	14	21	8

This completes the proof. □

Now, if we apply Construction 2.1, giving every point weight 2, we see that the set

$$HPA4 = \{n: \quad \text{there exists an HPA of type } 2^n \text{ and strength } 4\}$$

is PBD-closed. Since we have $\{4, 5, 7, 8, 9, 12\} \subseteq$ HPA4, it will be of interest to study the set $B(4, 5, 7, 8, 9, 12)$, i.e. the PBD-closure of $\{4, 5, 7, 8, 9, 12\}$ (in general, $B(K)$ denotes the set of all integers v such that there exists a PBD all of whose block sizes are in K).

Hanani has shown [**6**, Lemma 6.3] that $B(4, 5, 8, 9, 12) = \{n: n \equiv 0$ or $1 \pmod 4\}$. Hence, it is necessary only to consider $n \equiv 2$ or $3 \pmod 4$. Next, we use Brouwer's result [**2**] that $B(4, 7) = \{n \equiv 1 \pmod 3\} \setminus \{10, 19\}$. Note also that $10, 19 \notin B(4, 5, 7, 8, 9, 12)$ ([**4**]).

Hence, we have

LEMMA 4.2. *$n \in B(4, 5, 7, 8, 9, 12)$ unless $n \equiv 2, 3, 6$, or 11 (modulo 12), or $n = 10, 19$.*

The following construction is useful to handle the remaining cases.

LEMMA 4.3. *Suppose $v \equiv 4$ modulo 12, and $0 \le t \le (v-1)/3$. Then $v+t \in B(4,\ 5,\ t)$.*

PROOF: Adjoin t infinite points to t parallel classes of a resolvable $(v,\ 4,\ 1)$-BIBD (these were shown to exist in [**7**]). □

Using $t = 7$, we handle all $n \equiv 11$ modulo 12, $n \ge 35$.

Using $t = 22$, we have $n \in B(4,\ 5,\ 22)$, for all $n \equiv 2$ modulo 12, $n \ge 98$. Since $22 \in B(4,\ 7)$, we have $n \in B(4,\ 5,\ 7)$ for these n.

Using $t = 35$, we have $n \in B(4,\ 5,\ 35)$, for all $n \equiv 3$ modulo 12, $n \ge 147$. Since $35 \in B(5,\ 7)$, we have $n \in B(4,\ 5,\ 7)$ for these n.

Finally, using $t = 50$, we have $n \in B(4,\ 5,\ 50)$, for all $n \equiv 6$ modulo 12, $n \ge 210$. Since $50 \in B(7,\ 8)$, we have $n \in B(4,\ 5,\ 7,\ 8)$ for these n.

Many of the remaining values can be obtained by truncating some points from some groups of a transversal design. We use the following well-known construction.

LEMMA 4.4. *Suppose there is a $TD(j+k,m)$, and $0 \le u_i \le m$, for $1 \le i \le k$. Then $jm + \sum_{1 \le i \le k} u_i \in B(j, j+1, \ldots, j+k, u_1, \ldots, u_k, m)$.*

We also use the following variant to handle one case.

LEMMA 4.5. *Suppose there is a $TD(j+k,m)$. Then $jm+k \in B(j, j+1, j+k, m)$.*

PROOF: Let A be any block of the TD. Delete all points in groups $j+1, \ldots, j+k$, except those points on A. □

Many small values in fact are not in $B(4,\ 5,\ 7,\ 8,\ 9,\ 12)$: in [**4**], Drake and Larson show that $n \notin B(4,\ 5,\ 7,\ 8,\ 9,\ 12)$ if $n = 14,\ 15,\ 18,\ 23,\ 26$, or 27. We are not sure if 30, 38, or $42 \in B(4,\ 5,\ 7,\ 8,\ 9,\ 12)$ (see [**5**] for an investigation of the structure of a possible PBD on 30 points with blocks from $\{4,\ 5,\ 7,\ 8,\ 9,\ 12\}$). However, we can apply Lemmata 4.4 and 4.5 to eliminate all the other possible exceptions. (The requisite TDs can all be found in Beth, Jungnickel, and Lenz [**1**].)

Table 2

Construction of PBDs with blocks of size 4, 5, 7, 8, 9, and 12

n	authority	equation	block sizes
39	Lemma 4.4	4^*8+7	4, 5, 7, 8
50	Lemma 4.4	7^*7+1	7, 8
51	Lemma 4.5	4^*12+3	4, 5, 7, 12
54	Lemma 4.4	7^*7+5	5, 7, 8
62	Lemma 4.4	7^*8+5+1	5, 7, 8, 9
63	Lemma 4.4	7^*9	7, 9
66	Lemma 4.4	7^*8+5+5	5, 7, 8, 9
74	Lemma 4.4	7^*9+7+4	4, 7, 8, 9
75	Lemma 4.4	7^*9+7+5	5, 7, 8, 9
78	Lemma 4.4	7^*9+7+8	7, 8, 9
86	Lemma 4.4	$7^*12+1+1$	7, 8, 9, 12
87	Lemma 4.4	4^*20+7	4, 5, 7, 20
90	Lemma 4.4	$7^*12+5+1$	5, 7, 8, 9, 12
99	Lemma 4.4	$7^*12+8+7$	7, 8, 9, 12
102	Lemma 4.4	$7^*13+7+4$	4, 7, 8, 9, 13
111	Lemma 4.4	$7^*13+12+8$	7, 8, 9, 12, 13
114	Lemma 4.4	$7^*16+1+1$	7, 8, 9, 16
123	Lemma 4.4	7^*17+4	4, 7, 8, 17
126	Lemma 4.4	7^*17+7	7, 8, 17
135	Lemma 4.4	7^*17+16	7, 8, 16, 17
138	Lemma 4.4	$7^*17+12+7$	7, 8, 9, 12, 17
150	Lemma 4.4	4^*32+22	4, 5, 22, 32
162	Lemma 4.4	4^*35+22	4, 5, 22, 35
174	Lemma 4.4	4^*35+34	4, 5, 34, 35
186	Lemma 4.4	4^*41+22	4, 5, 22, 41
198	Lemma 4.4	4^*44+22	4, 5, 22, 44

Summarizing the results above, we have the following PBD-closure result.

THEOREM 4.6. *Let* $A=\{2, 3, 6, 10, 11, 14, 15, 18, 19, 23, 26, 27\}$ *and* $B=\{30, 38, 42\}$. *Then* $Z^+\setminus(A\cup B)\subseteq B(4, 5, 7, 8, 9, 12)\subseteq Z^+\setminus A$.

5. HPAs of type 2^n and strength 7.

In this section, we indicate how it is possible to construct HPAs of type 2^n and strength 7, for some prime powers $n \equiv 5$ modulo 8. The technique is basically that used in [**3**]. For such an n, let x be a primitive element in $GF(n)$. The non-zero elements of $GF(n)$ form a cyclic group with respect to multiplication, with generator x. Let C_0 be the (multiplicative) subgroup of size $(n-1)/4$, and let C_0, C_1, C_2, and C_3 be its cosets (these are referred to as *cyclotomic classes*).

We construct our HPA on point set $GF(n) \times \{0, 1\}$, with partition $\{\{y\} \times \{0, 1\}: y \in GF(n)\}$. (For convenience, we will write an ordered pair (y, i) as y_i, $y \in GF(n)$, $i = 0, 1$.) We give a set of 8 columns. Multiplying by every element in C_0, we obtain $2n-2$ columns, and then developing through $GF(n)$, we have $2n(n-1)$ columns, which, at least, is the correct number.

Hence, it is sufficient to find a set of 8 columns that work. It is most convenient to pick a subscripting pattern ahead of time, and then use a backtracking algorithm to finish the job. Let S denote the following 7×8 $0-1$ matrix:

$$\begin{matrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{matrix}$$

If A denotes any 7×8 matrix with entries from $GF(n)$, let $A \oplus S$ be the matrix whose $i-j$ entry is a_s, where $a = A[i, j]$ and $s = S[i, j]$. $A \oplus S$ are the 8 columns that we use to generate the HPA of type 2^n and strength 7. It is not difficult to formulate the set of conditions that will ensure that a given $A \oplus S$ will work. The differences obtained from any two rows should comprise one representative from each of the four cyclotomic classes, for each of the $2 \times 2 = 4$ combinations of subscripts.

On the computer, it is not difficult to find matrices A so that $A \oplus S$ satisfies these conditions. We found the following matrices A:

$n = 29$:

0	0	0	0	0	0	0	0
1	2	1	2	1	2	1	2
2	1	4	3	3	3	16	4
3	3	5	4	5	8	13	14
4	8	2	12	4	14	23	22
5	4	3	16	14	17	25	8
9	19	8	18	21	25	4	28

$n = 37$:

0	0	0	0	0	0	0	0
1	2	1	2	1	2	1	2
2	1	3	3	6	3	14	8
3	3	2	8	29	7	15	3
4	8	4	7	8	16	2	22
5	4	11	18	14	24	7	30
6	10	17	21	5	6	12	21

$n = 53$:

0	0	0	0	0	0	0	0
1	2	1	2	1	2	1	2
3	37	37	3	37	3	3	37
4	4	39	39	39	39	4	4
2	7	2	5	22	4	17	13
5	1	24	51	43	36	21	22
6	3	8	50	42	1	43	20

$n = 61$:

0	0	0	0	0	0	0	0
1	2	1	2	1	2	1	2
2	1	3	3	7	3	11	8
3	3	2	8	5	6	3	26
4	8	4	10	9	28	10	5
5	4	10	9	11	40	20	38
6	5	11	39	55	46	37	41

$n = 101$:

0	0	0	0	0	0	0	0
1	2	1	2	1	2	1	2
2	1	4	3	3	3	19	4
3	3	5	4	5	4	8	38
4	7	2	12	7	5	24	64
5	4	3	8	17	13	11	18
6	5	6	40	55	41	69	43

Hence, we have the following result.

THEOREM 5.1. *There exists an HPA of type 2^n and strength 7 for $n = 29$, 37, 53, 61, and 101.*

6. Conclusion.

We have introduced the concept of holey perpendicular arrays. We hope that this idea will be fruitful in the study of perpendicular arrays, Latin squares, and related designs.

REFERENCES

1. Th. Beth, D. Jungnickel and H. Lenz, "Design Theory", Bibliographisches Institut, Zurich, 1985.
2. A.E. Brouwer, *Optimal packings of K_4's into a K_n*, J. Comb. Theory **A 26** (1979), 258-279.
3. J.H. Dinitz and D.R. Stinson, *MOLS with holes*, Discrete Math.**44** (1983), 145-154.
4. D.A. Drake and J.A. Larson, *Pairwise balanced designs whose line sizes do not divide six*, J. Comb. Theory **A 34** (1983), 266-300.
5. D.A. Drake and J.A. Larson, *A quest for certain linear spaces on thirty points*, J. Stat. Plan. and Inf. **10** (1984), 241-255.
6. H. Hanani, *The existence and construction of balanced incomplete block designs*, Ann. Math. Statist. **32** (1961), 361-386.
7. H. Hanani, D.K. Ray-Chaudhuri and R.M. Wilson, *On resolvable designs*, Discrete Math. **3** (1972), 75-97.
8. C.C. Lindner, R.C. Mullin and G.H.J. van Rees, *Separable orthogonal arrays*, Utilitas Math. (to appear).
9. C.C. Lindner and D.R. Stinson, *The spectrum for the conjugate invariant subgroups of perpendicular arrays*, Ars Combinatoria **18** (1984), 51-60.
10. R.C. Mullin, P.J. Schellenberg, G.H.J. van Rees and S.A. Vanstone, *On the construction of perpendicular arrays*, Utilitas Math. **18** (1980), 141-160.
11. P.J. Schellenberg, G.H.J. van Rees and S.A. Vanstone, *Four pairwise orthogonal Latin squares of order 15*, Ars Combinatoria **6** (1978), 141-150.
12. W.D. Wallis, *Three orthogonal Latin squares*, Cong. Num. **42** (1984), 69-86.

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