ONE-FACTORIZATIONS OF REGULAR GRAPHS AND HOWELL DESIGNS OF SMALL ORDER

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1. Introduction.

A *1-factorization* of a graph G is a partition of the edge-set of G into 1-factors (=perfect matchings). Two 1-factorizations F_1, F_2 of G are *orthogonal* if any two edges of G belong to distinct 1-factors of F_2 whenever they belong to the same 1-factor of F_1.

A *Howell design* $H(s,n)$ is a square array of side s such that (i) each cell is either empty or contains a 2-subset of an n-set N, (ii) every element of N occurs in exactly one cell of each row and each column, and (iii) any 2-subset of N occurs in at most one cell of the array. Necessary and sufficient conditions for the existence of a Howell design $H(s,n)$ were recently obtained by the second author [**1, 8**].

On the other hand, not much seems to be known about the number of nonisomorphic Howell designs. It is well-known (see, e.g., [**6**]) that a Howell design $H(s,n)$ is equivalent to a pair of orthogonal 1-factorizations of (some) regular graph of degree s and order n (i.e., with n vertices). Thus when trying to enumerate Howell designs, a natural approach would appear to be to enumerate first the 1-factorizations of regular graphs, and then to examine whether they admit orthogonal mates, with a subsequent (or simultaneous) isomorphism rejection. This is essentially the approach that we adopted, and this paper is a report on our findings.

Starting with a known listing of regular graphs given in [**3**], we enumerate: 1. nonisomorphic 1-factorizations of regular graphs of order ≤ 10 and degree ≤ 7; 2. nonisomorphic pairs of orthogonal 1-factorizations of these regular graphs, i.e., nonisomorphic Howell designs.

2. General comments.

Let G be a regular graph of degree s on n vertices (n even). If there exists a pair of orthogonal 1-factorizations of G (resulting in a $H(s,n)$), we will say that G *admits a Howell design*. Although the existence question

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for Howell designs has been completely settled, the question about which regular graphs admit a Howell design does not appear to be an easy one. Since the existence of a pair of orthogonal Latin squares of order n is equivalent to the existence of a pair of orthogonal 1-factorizations of $K_{m,m}$, it follows (cf. [**5**]) that the complete bipartite graph $K_{m,m}$ admits a Howell design whenever $m \neq 2$ or 6. (Although no other regular graph of degree $\frac{1}{2}n$ on n vertices admits a Howell design if $n \leq 8$, four out of 59 other 5-regular graphs on 10 vertices do (cf. Tables 6 and 9 below), and so does at least one of (the other) 6-regular graphs on 12 vertices [**4, 5**].) Similarly, the unique $(n-1)$-regular and the unique $(n-2)$-regular graph on n vertices (i.e., the complete graph K_n, and the cocktail-party graph $K_n - F$ (F a 1-factor), respectively) admit a Howell design if $n \geq 8$, and if $n \geq 6$, respectively. This follows from the results on the existence of $H(n-1,n)$ (i.e., Room squares of order n) and $H(n-2,n)$'s [**1, 8**].

Not much else appears to be known in general. One of the reasons that we undertook the investigation of s-regular graphs on n vertices for small n ($n \leq 10$) was the hope that this will shed some light on the general question above. Although this did not quite materialize, nevertheless, we feel that the following is most likely to be true:

Conjecture. For every $k \geq 1$ there exists a number N_k such that for all $n > N_k$ (n even), every $(n-k)$-regular graph on n vertices admits a Howell design $H(n-k,n)$.

The conjecture is true, of course, for $k = 1, 2$, but to prove it already for $k = 3$ will likely not be easy.

3. Howell designs $H(s,n)$, $n \leq 8$.

The following is well-known and/or an easy exercise, and is recorded only for the sake of completeness:

There exists no $H(2,4)$, $H(3,4)$ or $H(5,6)$ [**5, 9**].

There exists a unique $H(3,6)$ and unique $H(4,6)$ (up to an isomorphism, of course); of the two 3-regular graphs on 6 vertices, it is $K_{3,3}$ which admits a Howell design (and $\bar{C}_6$ which does not).

Having dismissed the trivial case $n \leq 6$, we thus proceed to the case $n = 8$.

3.1. *There exists a unique $H(4,8)$ (up to an isomorphism).*

We may assume w.l.o.g. that the first row of a Howell design $H(4,8)$ is (12 34 56 78). Then the second row is, w.l.o.g., either (58 67 14 23)

(Case 1) or (67 18 23 45) (Case 2). We may again w.l.o.g. assume that the cell (3, 1) contains the element 3.

Case 1. The cell (3, 1) must contain the pair 36 (as 34 occurs already in the first row, and 37 in cell (3, 1) would force 28 in cell (3, 3) which in turn would force 37 in cell (4, 3) as well, a contradiction). Therefore the cell (4, 1) contains 47. The cell (3, 3) must then contain 27 (the element 7 can occur in the third column only in cell (3, 3) but since 78 occurs already in the first row, 27 is the only possibility left). Consequently, the cell (3, 4) contains 38. It is now easily seen that a unique completion of the 3rd and 4th row is now forced, to (36 18 27 45) and (47 25 38 16), respectively.

The underlying graph of the resulting Howell design is $K_{4,4}$, and, in fact, this is the well-known $HD(4,8)$ [5] from a pair of orthogonal Latin squares of order 4 (which is well-known to be unique up to an isomorphism).

Case 2. The cell (3, 1) must contain the pair 35 (as 34 occurs already in the first row, and 38 would force 45 in cell (4, 1) but 45 occurs already). Therefore the cell (4, 1) contains 48; this forces 17 in cell (4, 3) and then again 48 in cell (3, 3) a contradiction.

An alternative proof is via exhibiting all distinct 1-factorizations of 4-regular graphs on 8 vertices, and inspecting all pairs of them for orthogonality. However, except possibly in this case, this approach becomes quickly impractical (cf. Tables 6-8) for "proofs by hand" and is suitable only if one uses computer.

3.2. There exists no $H(5,8)$.

Hung and Mendelsohn [5] reported this as a result of a computer search. We think that the "non-computer proof" below may be of interest.

Assume that a Howell design $H(5,8)$ exists. We may assume w.l.o.g. that the cell (1, 1) is empty, and that the first row is (–12 34 56 78). Then w.l.o.g. the first column is either $(-13\ 24\ 57\ 68)^T$ (Case I) or $(-18\ 23\ 45\ 67)^T$ (Case II).

Case I. As the cell (2, 1) contains the pair 13, we may assume w.l.o.g. that one of the remaining three occupied cells of row 2 contains the pair 25; this forces the remaining two occupied cells in row 2 to contain the pairs 48 and 67. If 48 were to occur in cell (2, 2) then one would be unable to complete column 2 (as all three of 56, 57, 67 already occur in the square); thus 48 must occur in cell (2, 4). Similarly, if 25 were to occur in cell (2, 3), one could not complete column 3 (as all three of 67, 68, 78 already occur in the square); thus 25 must occur in cell $(2,5)$. If 67 were in cell $(2,2)$ then the only way to complete column 2 would be with pairs 38, 45, and the pair 45 would have to be in cell $(5,2)$; but then the only way to complete row 5 would be with pairs 17, 23, and the only way to complete column 4 would also be with pairs 17, 23 which is impossible; thus 67 cannot occur in cell $(2,2)$. Consequently, row 2 must be of the form (13–67 48 25). The only way to complete column 3 is with pairs 15, 28, and the pair 28 must go in cell $(4,3)$; but then the only way to complete row 4 would be with pairs 14, 36, and the only way to complete column 5 would also be with pairs 14, 36 which is impossible.

Case II. W.l.o.g., we have to distinguish four cases according to whether the remaining three pairs in the occupied cells of row 2 are 27, 35, 46 (Case IIa), 26, 35, 47 (Case IIb), 25, 36, 47 (Case IIc) or 24, 36, 57 (Case IId).

Case IIa. If the pair 35 were in cell $(2,2)$ then the only way to complete column 2 would be with pairs 47, 68, and column 2 would have to be $(12\ 35\ 47\ 68\text{–})^T$. But with 45 and 68 in cells $(4,1)$ and $(4,2)$, respectively, row 4 is impossible to complete. Thus the pair 35 cannot be in cell $(2,2)$ which implies that 35 must be in cell $(2,5)$ and 46 must be in cell $(2,2)$. Now column 2 can be completed either with the pairs 37, 58 or with the pairs 38, 57. If the pairs 37, 58 were to occur in column 2 then the pair 37 must be in cell $(4,2)$. If the pair 58 were in cell $(5,2)$ then row 5 would have to be (67 58–13 24) which in turn forces the pair 16 in cell $(4,5)$ but this implies that the two cells $(3,2)$ and $(3,5)$ in row 3 are empty, a contradiction. If the pair 58 were in cell $(3,2)$ then the only way to complete row 3 would be with pairs 16, 47 and the only way to complete row 4 would be with pairs 16, 28, a contradiction. Thus the pairs 38, 57 must occur in column 2, with 57 in cell $(3,2)$. The element 5 must now occur in cell $(4,3)$.

If 58 occurs in cell $(5,3)$ then row 5 must be of the form (67–58 13 24) which in turn forces column 4 to be of the form $(56\ 27\ 48\text{–}13)^T$ and column 5 to be of the form $(78\ 35\ 16\text{–}24)^T$, a contradiction as this would mean two empty cells in row 4. If 15 occurs in cell $(5,3)$ then column 3 must be $(34\ 27\ 68\text{–}15)^T$ which forces column 5 to be $(78\ 35\text{–}16\ 24)^T$; now if 38 would appear in cell $(4,2)$, row 5 cannot be completed, and if 38 would appear in cell $(5,2)$, two cells $(2,4)$ and $(5,4)$ in column 4 would be empty. Finally, if 25 occurs in cell $(5,3)$ then row 2 must be (18 46–27 35) which forces column 3 to be $(34\text{–}68\ 17\ 25)^T$ and the pair 26 to appear in cell $(4,5)$. But then 38 must occur in cell $(4,2)$ and 14 in cell $(3,5)$ which forces two cells $(3,4)$ and $(4,4)$ in column 4 to be empty, a contradiction.

Case IIb. If the pair 35 were in cell $(2,2)$ it would be impossible to complete column 2, thus 35 must be in cell $(2,5)$ and so 26 must be in cell $(2,3)$. Column 5 can now be completed only with pairs $16, 24$, with 24 in cell $(5,5)$ which forces cell $(4,4)$ to contain element 2. If 27 were in cell $(4,4)$ then column 3 must be $(34\ 26\ 17\text{–}58)^T$ which forces row 5 to be $(67\text{–}58\ 13\ 24)$ which in turn forces column 4 to be $(56\text{–}48\ 27\ 13)^T$; but then row 3 is impossible to complete. If 28 were in cell $(4,4)$ then the only way to complete column 3 is with pairs $17, 58$, and the only way to complete row 5 is with pairs $13, 58$ which implies that 58 must be in cell $(5,3)$, 13 must be in cell $(5,4)$, 47 must be in cell $(2,4)$, and so two cells $(2,2)$ and $(5,2)$ in column 2 would be empty, a contradiction.

Case IIc. If the pair 47 were in cell $(2,2)$ then row 2 must be $(18\ 47\ 25\text{–}36)$; the only way to complete column 2 would be with pairs $35, 68$ and the only way to complete column 3 would be with pairs $17, 68$, a contradiction. Thus 47 must be in cell $(2,4)$. If 36 were in cell $(2,2)$ then column 2 would have to be $(12\ 36\ 57\text{–}48)^T$ which would force column 4 to be $(56\ 47\text{–}28\ 13)^T$ but then row 5 would be impossible to complete. Thus 36 cannot be in cell $(2,2)$, and row 2 must be $(18\text{–}25\ 47\ 36)$. This forces column 5 to be $(78\ 36\ 15\text{–}24)^T$ but then row 3 is impossible to complete.

Case IId. If the pair 36 were in cell $(2,2)$ then column 2 must be $(12\ 36\ 47 - 58)^T$ but then row 5 is impossible to complete. Thus 36 must be in cell $(2,5)$, and 24 must be in cell $(2,4)$. This leaves as the only possibility for column 5 $(78\ 36\ 14 - 25)^T$ but then row 3 is impossible to complete.

In either case, the square cannot be completed. □

3.3 There exist exactly three nonisomorphic $H(6,8)$'s.

This was obtained both by hand and by computer, by using essentially the same approach. First all nonisomorphic 1-factorizations of the cocktail-party graph $K_8 - F$ were found (13 in total, cf. Table 2). Afterwards, for each of these all possible orthogonal mates were constructed. In the final step, the obtained set of Howell designs was tested for isomorphic copies, and duplicates were deleted.

The only difference between the hand and computer calculations that is worth noting is that when working by hand, in enumerating the nonisomorphic 1-factorizations of $K_8 - F$ a use was made of the known nonisomorphic 1-factorizations of K_8 [9]: from each of the 6 nonisomorphic 1-factorizations of K_8, a representative of each orbit of 1-factors (under the group of the 1-factorization) was omitted one at a time. This resulted in a set of 1-factorizations of $K_8 - F$ from which isomorphic duplicates were then removed. When working on a computer, all distinct 1-factorizations of $K_8 - F$ were obtained first, and isomorphic duplicates were eliminated by a "sieve" method (cf. Section 4 below).

The nonisomorphic 1-factorizations of the cocktail-party graph $K_8 - F$ are listed in Table 2, together with their *types*. The type of a 1-factorization indicates how many of all $\binom{6}{2} = 15$ pairs of its 1-factors have as their union two 4-cycles, and one 8-cycle, respectively. As seen from Table 2, in this case the type is a fairly sensitive, though not a complete invariant. We also indicate in Table 2 (in the column headed $OF(K_8)$ No.) which of the 6 nonisomorphic 1-factorizations of the complete graph K_8 results (in the numbering of [9]) if the (unique) "missing" 1-factor F is added to the 1-factorization of $K_8 - F$. Only four out of thirteen nonisomorphic 1-factorizations of $K_8 - F$ admit an orthogonal mate, and only one (No. 2, cf. Table 2) admits 2 nonisomorphic mates, one of them isomorphic to itself. The three nonisomorphic Howell designs $H(6, 8)$ are listed in Table 3.

4. One-factorizations of s-regular graphs on 10 vertices and Howell designs $H(s, 10)$ with $s \leq 7$.

When the number of vertices is increased to 10, it becomes quickly apparent that for none of the questions addressed in Section 3 is hand computation any longer feasible. Thus all results of this section were obtained by computer. A brief description of the algorithms used follows.

We used the list of connected regular graphs given in [3] (where the graphs are given by the list of their edges) which we augmented whenever applicable with the disconnected regular graphs. For a given graph, all its distinct 1-factors were generated by a simple backtrack. The duplicates from the (ordered) list of distinct 1-factors were then eliminated by a sieve-like procedure using the action of the automorphism group of the graph on its 1-factors. First, all images of the first 1-factor on the list (except itself) were deleted from the list, next all images of the second (remaining) 1-factor (except itself) were deleted, etc. Clearly, after one pass through the list only nonisomorphic 1-factors remain. Note that the procedure is computationally feasible only due to the small size of both the group order and the set of distinct 1-factors.

The procedure for generating distinct and nonisomorphic 1-factorizations is similar, as is the procedure for generating all nonisomorphic pairs of orthogonal 1-factorizations (i.e., Howell designs).

The results of calculations are listed in Tables 4-11. First of all, the numbering of graphs is as in [**3**] (for specific differences, see comments directly within the respective tables). Tables 4,5 list all regular graphs on 10 vertices of degree 3 and 4 respectively (actually, the unique disconnected graph of degree 4 consisting of two disjoint K_5's is not included in Table 5 as it has no 1-factor). Since the listing of graphs in [**3**] does not appear readily available, we have included Tables 4a and 5a with the lists of edges of the connected 3- and 4-regular graphs on 10 vertices, as it appears in [**3**]. [In the listing of the 4-regular graphs on 10 vertices in [**3**], there is a misprint in the graph No. 32: the edge 68 should apparently read 69.] Tables 6, 7 and 8 list all 5-, 6- and 7-regular graphs on 10 vertices respectively. An extra column contains information on Howell designs $H(s, 10)$ for $s = 5, 6, 7$. The totals give the following:

There exist exactly 6 *nonisomorphic* $H(5, 10)$*'s.*

There exist exactly 18 *nonisomorphic* $H(6, 10)$*'s.*

There exist exactly 901 *nonisomorphic* $H(7, 10)$*'s.*

On the other hand, there exists no set of three pairwise orthogonal 1-factorizations (i.e., no "Howell cube") of an s-regular graph on 10 vertices where $s = 5, 6$ or 7.

We may add, for the sake of completeness, that there exist exactly 257630 nonisomorphic Room squares of side 9, exactly 257 nonisomorphic Room cubes of side 9, and exactly one (up to isomorphism) 4-dimensional Room hypercube of side 9 ([**2**]). A corresponding enumeration for the graph $K_{10} - F$ was recently done by E. Seah and D.R. Stinson. There are exactly 3192 nonisomorphic 1-factorizations, exactly 18220 nonisomorphic Howell designs $H(8, 10)$, exactly 3 nonisomorphic sets of three orthogonal 1-factorizations, and exactly 1 set of four orthogonal 1-factorizations (up to isomorphism). (This set of four was first constructed by E. Lamken and S.A. Vanstone).

Tables 9 and 10 list all nonisomorphic $H(5, 10)$'s and $H(6, 10)$'s while Table 11 lists, for each of the 7-regular graphs on 10 vertices, one example of a $H(7, 10)$ admitted by the graph.

One comment concerning Table 9 and the two Howell designs $H(5, 10)$ admitted by $K_{5,5}$: although we are convinced that it must be well-known that there are precisely two nonisomorphic pairs of orthogonal Latin squares of order 5, we are unable to produce a reference for this fact.

5. Concluding remarks and open problems.

Very recently, Brickell [**4**] found a Howell cube $H(6, 12)$; this is the first example of an n-regular graph on $2n$ vertices (other than $K_{n,n}$) that admits three orthogonal 1-factorizations. Many more examples should exist for $n = 6$.

One question one might ask in connection with this is, what proportion of n-regular graphs with $2n$ vertices admits a Howell design? Of course, we cannot realistically expect a complete answer.

Another question that has been asked before [**7**] is about the existence of special Howell designs $H^*(n+1, 2n)$ where n is even. The underlying graph of such a design $\bar{K}_{n-1}+Q_{n+1}$ (where + denotes the join, and Q_{n+1} is a 2-regular graph on $n+1$ vertices). The smallest known example in this class is $H^*(13, 24)$ given in [**7**].

Scott Vanstone and others conjectured that the maximum number of pairwise orthogonal 1-factorizations of any r-regular graph on $2n$ vertices does not exceed $n-1$. If true, this would generalize the upper bound on the number of pairwise orthogonal Latin squares and the conjectured bound for pairwise orthogonal (=perpendicular) symmetric Latin squares. Unfortunately, we are not aware of any progress towards settling this conjecture.

Table 1. Regular graphs on 8 vertices.

Graph No.	$\lvert\Gamma\rvert$	DPM	NPM	DOF	NOF	HD	Edges	Comment
3.1	16	5	2	2	1	-	12,13,14,23,24,35, 46,57,58,67,68,78	
3.2	4	5	3	1	1	-	12,13,14,23,25,36 45,47,58,67,68,78	
3.3	12	6	1	2	1	-	12,13,14,23,25,36 47,48,57,58,67,68	
3.4	48	9	2	4	2	-	12,13,14,25,26,35, 37,46,47,58,68,78	Q_3
3.5	16	7	3	3	2	-	12,13,14,25,26,35, 37,46,48,58,67,78	
3.6	1152	9	1	6	1	-	K_4 K_4	
4.1	16	16	4	8	3	0	co(3.1)	
4.2	4	14	5	6	3	0	co(3.2)	
4.3	12	14	2	4	1	0	co(3.3)	
4.4	48	16	4	12	5	0	co(3.4)	
4.5	16	14	3	6	2	0	co(3.5)	
4.6	1152	24	1	24	2	1	$K_{4,4}$	
5.1	16	31	7	38	9	0	co(C_8)	
5.2	128	33	4	56	8	0	co($C_4 \cup C_4$)	
5.3	60	30	1	36	2	0	co($C_3 \cup C_5$)	
6.1	384	60	2	416	13	3	K_8-F	
7.1	8!	105	1	6240	6	6	K_8	RS(8) [7]

$\lvert\Gamma\rvert$ = order of the automorphism group
DPM = number of distinct 1-factors
NPM = number of nonisomorphic 1-factors
DOF = number of distinct 1-factorizations
NOF = number of nomisomorphic 1-factorizations
HD = number of nonisomorphic Howell designs

Table 2. The 13 nonisomorphic 1-factorizations of $K_8 - F$.

No.	1-factors	Type 4+4	Type 8	OF(K_8) No.	Orthogonal mates
1	1,2,4,7,12,18	15	0	F_1	1
2	1,2,4,7,12,21	11	4	F_2	2
3	1,4,7,12,19,21	9	6	F_2	0
4	2,4,7,12,19,21	7	8	F_2	0
5	1,2,9,13,15,20	7	8	F_4	0
6	1,2,3,8,16,20	6	9	F_3	0
7	1,2,4,13,15,20	6	9	F_4	0
8	1,2,3,13,16,20	5	10	F_3	1
9	1,3,8,13,16,20	5	10	F_3	0
10	1,2,8,13,16,20	3	12	F_3	0
11	1,2,5,13,14,20	3	12	F_5	1
12	1,2,5,8,14,20	2	13	F_5	0
13	1,3,6,10,11,17	0	15	F_6	0

List of 1-factors:

1:	12 34 56 78	8:	15 27 36 48	15:	17 25 36 48
2:	13 24 57 68	9:	15 27 38 46	16:	17 25 38 46
3:	13 25 47 68	10:	15 28 37 46	17:	17 26 38 45
4:	14 23 58 67	11:	16 23 48 57	18:	17 28 35 46
5:	14 25 38 67	12:	16 25 38 47	19:	17 28 36 45
6:	14 27 36 58	13:	16 28 37 45	20:	18 26 35 47
7:	15 26 37 48	14:	17 23 46 58	21:	18 27 36 45

Table 3. The 3 nonisomorphic Howell designs $H(6,8)$.

13	68		24		57
67	14		58		23
48		15	37	26	
25		47	16	38	
	35	28		17	46
	27	36		45	18

13	57	68	24		
67	14	23	58		
48		15		36	27
		47	16	28	35
25	38			17	46
	26		37	45	18

13	68		24		57
67	14		58		23
48	27	15		36	
	35	47	16	28	
25		38		17	46
		26	37	45	18

Table 4. Regular graphs of degree 3 on 10 vertices.

Graph No.	$\|\Gamma\|$	DPM	NPM	DOF	NOF	Comment
1	32	4	1	0	0	
2	16	8	2	4	1	
3	4	7	3	2	1	
4	8	6	3	2	1	
5	16	10	2	4	1	
6	2	6	4	1	1	
7	4	7	5	1	1	
8	4	8	3	2	1	
9	8	7	3	2	1	
10	12	6	1	2	1	
11	6	9	2	4	2	
12	6	6	2	1	1	
13	2	8	5	3	2	
14	48	12	1	4	1	
15	20	11	3	5	1	
16	4	9	6	3	2	
17	20	13	4	6	2	
18	8	8	3	2	1	
19	120	6	1	0	0	
20	288	12	2	6	1	$K_4 \cup \bar{C}_6$
21	1728	18	1	12	1	$K_4 \cup K_{3,3}$

$|\Gamma|$ = order of the automorphism group

DPM = number of distinct 1-factors

NPM = number of nonisomorphic 1-factors

DOF = number of distinct 1-factorizations

NOF = number of nonisomorphic 1-factorizations

Table 4a.
Connected regular graphs of degree 3 on 10 vertices — list of edges.

Graph No.	Edges
1	1-2,3,4;2-3,4;3-5;4-5;5-6;6-7,8;7-9,10;8-9,10;9-10
2	1-2,3,4;2-3,4;3-5;4-6;5-6,7;6-8;7-9,10;8-9,10;9-10
3	1-2,3,4;2-3,4;3-5;4-6;5-7,8;6-7,9;7-10;8-9,10;9-10
4	1-2,3,4;2-3,4;3-5;4-6;5-7,8;6-9,10;7-8,9;8-10,9-10
5	1-2,3,4;2-3,4;3-5;4-6;5-7,8;6-9,10;7-9,10;8-9,10
6	1-2,3,4;2-3,5;3-6;4-5,7;5-8;6-7,9;7-10;8-9,10;9-10
7	1-2,3,4;2-3,5;3-6;4-5,7;5-8;6-9,10;7-8,9;8-10;9-10
8	1-2,3,4;2-3,5;3-6;4-5,7;5-8;6-9,10;7-9,10;8-9,10
9	1-2,3,4;2-3,5;3-6;4-7,8;5-7,8;6-9,10;7-9;8-10;9-10
10	1-2,3,4;2-3,5;3-6;4-7,8;5-7,9;6-7,10;8-9,10;9-10
11	1-2,3,4;2-3,5;3-6;4-7,8;5-7,9;6-8,9;7-10;8-10;9-10
12	1-2,3,4;2-3,5;3-6;4-7,8;5-7,9;6-8,10;7-9,8-10;9-10
13	1-2,3,4;2-3;5;3-6;4-7,8;5-7,9;6-8,10;7-10;8-9;9-10
14	1-2,3,4;2-5,6;3-5,6;4-7,8;5-9;6-10;7-9,10;8-9,10
15	1-2,3,4;2-5,6;3-5,7;4-6,8;5-9;6-10;7-8,9;8-10;9-10
16	1-2,3,4;2-5,6;3-5,7;4-6,8;5-9;6-10;7-8,10;8-9;9-10
17	1-2,3,4;2-5,6;3-5,7;4-6,8;5-9;6-10;7-9,10;8-9,10
18	1-2,3,4;2-5,6;3-5,7;4-8,9;5-8;6-9,10;7-9,10;8-10
19	1-2,3,4;2-5,6;3-7,8;4-9,10;5-7,9;6-8,10;7-10;8-9

Table 5. Regular graphs of degree 4 on 10 vertices.

Graph No.	|Γ|	DPM	NPM	DOF	NOF
1	144	18	1	0	0
2	64	24	2	16	1
3	8	26	7	16	3
4	4	22	7	8	2
5	8	22	4	8	1
6	8	24	4	8	1
7	12	28	5	24	5
8	48	26	2	8	1
9	4	24	7	12	3
10	4	24	7	12	3
11	16	24	2	8	1
12	2	24	14	12	6
13	8	26	5	16	3
14	2	26	20	20	14
15	16	24	4	16	2
16	2	22	11	8	4
17	16	20	3	4	2
18	8	20	4	4	1
19	20	22	3	10	1
20	2	22	11	10	5
21	4	24	9	14	6
22	1	24	24	14	14
23	4	24	8	12	4
24	4	22	7	10	3
25	2	24	15	14	9
26	2	24	12	14	9
27	2	22	11	10	5
28	2	24	12	12	6
29	4	26	11	20	8
30	1	24	24	14	14
31	1	24	24	14	14
32	2	22	11	12	6
33	2	26	13	16	8
34	32	24	2	8	1
35	2	26	16	18	10
36	2	24	13	12	6
37	2	24	16	14	9
38	4	24	12	12	6
39	1	24	24	10	10

Table 5. (Continued)

Graph No.	|Γ|	DPM	NPM	DOF	NOF
40	4	28	12	16	6
41	2	26	13	18	9
42	8	32	8	32	6
43	4	24	6	12	3
44	4	22	8	12	5
45	4	26	10	16	6
46	4	26	11	16	7
47	4	30	8	24	6
48	2	24	12	12	6
49	2	26	17	14	9
50	16	24	3	12	2
51	2	24	12	14	9
52	16	24	2	20	5
53	16	24	3	8	1
54	2	26	13	14	7
55	10	22	3	10	1
56	4	30	12	20	7
57	8	28	5	20	7
58	320	32	1	40	1
59	240	44	2	56	3

Table 5a.
Connected regular graphs of degree 4 on 10 vertices — list of edges.

Graph No.	Edges
1	1-2,3,4,5;2-3,4,5;3-4,5;4-6;5-7;6-8,9,10;7-8,9,10;8-9,10;9-10
2	1-2,3,4,5;2-3,4,5;3-4,6;4-6;5-7,8;6-9,10;7-8;9,10;8-9,10;9-10
3	1-2,3,4,5;2-3,4,5;3-4,6;4-7;5-6,8;6-9,10;7-8,9,10;8-9,10;9-10
4	1-2,3,4,5;2-3,4,5;3-4,6;4-7;5-8,9;6-7,8,10;7-9,10;8-9,10;9-10
5	1-2,3,4,5;2-3,4,5;3-4,6;4-7;5-8,9;6-8,9,10;7-8,9,10;8-10;9-10
6	1-2,3,4,5;2-3,4,5;3-6,7;4-6,7;5-8,9;6-8,10;7-9,10;8-9,10;9-10
7	1-2,3,4,5;2-3,4,5;3-6,7;4-6,8;5-6,9;6-10;7-8,9,10;8-9,10;9-10
8	1-2,3,4,5;2-3,4,5;3-6,7;4-6,8;5-7,8;6-9,10;7-9,10;8-9,10;9-10
9	1-2,3,4,5;2-3,4,5;3-6,7;4-6,8;5-7,9;6-8,10;7-9,10;8-9.10;9-10
10	1-2,3,4,5;2-3,4,5;3-6,7;4-6,8;5-7,9;6-9,10;7-8,10;8-9,10;9-10
11	1-2,3,4,5;2-3,4,5;3-6,7;4-6,8;5-9,10;6-7,8;7-9,10;8-9,10;9-10
12	1-2,3,4,5;2-3,4,5;3-6,7;4-6,8;5-9,10;6-7,9;7-8,10;8-9,10;9-10
13	1-2,3,4,5;2-3,4,5;3-6,7;4-6,8;5-9,10;6-9,10;7-8,9,10;8-9,10
14	1-2,3,4,5;2-3,4,6;3-4,7;4-8;5-6,7,9;6-8,10;7-9,10;8-9,10;9-10
15	1-2,3,4,5;2-3,4,6;3-4,7;4-8;5-6,9,10;6-9,10;7-8,9,10;8-9,10
16	1-2,3,4,5;2-3,4,6;3-5,6;4-7,8;5-7,9;6-8,10;7-9,10;8-9,10;9-10
17	1-2,3,4,5;2-3,4,6;3-5,7;4-5,8;5-9;6-7,8,10;7-9,10;8-9,10;9-10
18	1-2,3,4,5;2-3,4,6;3-5,7;4-5,8;5-9;6-7,9,10;7-8,10;8-9,10;9-10
19	1-2,3,4,5;2-3,4,6;3-5,7;4-6,8;5-7,9;6-8,10;7-9,10;8-9,10;9-10
20	1-2,3,4,5;2-3,4,6;3-5,7;4-6,8;5-7,9;6-9,10;7-8,10;8-9;10;9-10
21	1-2,3,4,5;2-3,4,6;3-5,7;4-6,8;5-8,9;6-7,10;7-9,10;8-9,10;9-10
22	1-2,3,4,5;2-3,4,6;3-5,7;4-6,8;5-8,9;6-9,10;7-8,9,10;8-10;9-10
23	1-2,3,4,5;2-3,4,6;3-5,7;4-6,8;5-9,10;6-9,10;7-8,9,10;8-9,10
24	1-2,3,4,5;2-3,4,6;3-5,7;4-7,8;5-6,9;6-8,10;7-9,10;8-9,10;9-10
25	1-2,3,4,5;2-3,4,6;3-5,7;4-7,8;5-8,9;6-8,9,10;7-9,10;8-10;9-10
26	1-2,3,4,5;2-3,4,6;3-5,7;4-7,8;5-9,10;6-7,8,9;7-10;8-9,10;9-10
27	1-2,3,4,5;2-3,4,6;3-5,7;4-7,8;5-9,10;6-8,9,10;7-8,9;8-10;9-10
28	1-2,3,4,5;2-3,4,6;3-5,7;4-7,8;5-9,10;6-8,9,10;7-9,10;8-9,10
29	1-2,3,4,5;2-3,4,6;3-5,7;4-8,9;5-8,9;6-7,8,10;7-9,10;8-10;9-10
30	1-2,3,4,5;2-3,4,6;3-5,7;4-8,9;5-8,10;6-7,8,9;7-9,10;8-10;9-10
31	1-2,3,4,5;2-3,4,6;3-5,7;4-8,9;5-8,10;6-7,8,10;7-9,10;8-9;9-10
32	1-2,3,4,5;2-3,4,6;3-5,7;4-8,9;5-8,10;6-7,9,10;7-9,10;8-9,10
33	1-2,3,4,5;2-3,4,6;3-5,7;4-8,9;5-8,10;6-8,9,10;7-8,9,10;9-10
34	1-2,3,4,5;2-3,4,6;3-7,8;4-7,8;5-6,9,10;6-9,10;7-9,10;8-9,10
35	1-2,3,4,5;2-3,4,6;3-7,8;4-7,9;5-6,7,8;6-9,10;7-10;8-9,10;9-10

Table 5a. (Continued)

Graph No.	Edges
36	1-2,3,4,5;2-3,4,6;3-7,8;4-7,9;5-6,7,10;6-8,9;7-10;8-9,10;9-10
37	1-2,3,4,5;2-3,4,6;3-7,8;4-7,9;5-6;8,9;6-8,10;7-9,10;8-10;9-10
38	1-2,3,4,5;2-3,4,6;3-7,8;4-7,9;5-6,8,10;6-8,10;7-9,10;8-9;9-10
39	1-2,3,4,5;2-3,4,6;3-7,8;4-7,9;5-6,8,10;6-9,10;7-8,10;8-9;9-10
40	1-2,3,4,5;2-3,4,6;3-7,8;4-7,9;5-7,8,9;6-8,9,10;7-10;8-10;9-10
41	1-2,3,4,5;2-3,4,6;3-7,8;4-9,10;5-6,7,8;6-7,9;7-10;8-9,10;9-10
42	1-2,3,4,5;2-3,4,6;3-7,8;4-9,10;5-6,7,8;6-9,10;7-9,10;8-9,10
43	1-2,3,4,5;2-3,4,6;3-7,8;4-9,10;5-6,7,9;6-7,10;7-8;8-9,10;9-10
44	1-2,3,4,5;2-3,4,6;3-7,8;4-9,10;5-6,7,9;6-8,10;7-8,9;8-10;9-10
45	1-2,3,4,5;2-3,4,6;3-7,8;4-9,10;5-6,7,9;6-8,10;7-8,10;8-9;9-10
46	1-2,3,4,5;2-3,4,6;3-7,8;4-9,10;5-6,7,9;6-8,10;7-9,10;8-9,10
47	1-2,3,4,5;2-3,4,6;3-7,8;4-9,10;5-7,8,9;6-7,8,9;7-10;8-10;9-10
48	1-2,3,4,5;2-3,4,6;3-7,8;4-9,10;5-7,8,9;6-7,8,10;7-9;8-10;9-10
49	1-2,3,4,5;2-3,4,6;3-7,8;4-9,10;5-7,8,9;6-7,9,10;7-10;8-9,10
50	1-2,3,4,5;2-3,6,7;3-8,9;4-5,6,7;5-8,9;6-8,10;7-9,10;8-10;9-10
51	1-2,3,4,5;2-3,6,7;3-8,9;4-5,6,7;5-8,10;6-8,9;7-9,10;8-10;9-10
52	1-2,3,4,5;2-3,6,7;3-8,9;4-5,6,7;5-8,10;6-9,10;7-9,10;8-9,10
53	1-2,3,4,5;2-3,6,7;3-8,9;4-5,6,8;5-7,9;6-9,10;7-8,10;8-10;9-10
54	1-2,3,4,5;2-3,6,7;3-8,9;4-5,6,8;5-7,10;6-9,10;7-8,9;8-10;9-10
55	1-2,3,4,5;2-3,6,7;3-8,9;4-5,6,8;5-7,10;6-9,10;7-8,10;8-9;9-10
56	1-2,3,4,5;2-3,6,7;3-8,9;4-6,7,8;5-6,8,9,6-10;7-9,10;8-10;9-10
57	1-2,3,4,5;2-3,6,7;3-8,9;4-6,7,8;5-8,9,10;6-9,10;7-9,10;8-10
58	1-2,3,4,5;2-6,7,8;3-6,7,8;4-6,9,10;5-6,9,10;7-9,10;8-9,10
59	1-2,3,4,5;2-6,7,8;3-6,7,9;4-6,8,9;5-7,8,9;6-10;7-10;8-10;9-10

Table 6. Regular graphs of degree 5 on 10 vertices.

Graph No.	$\|\Gamma\|$	DPM	NPM	DOF	NOF	HD
1	144	84	4	336	6	0
2	64	72	5	224	7	1
3	8	68	13	208	29	0
4	4	70	19	224	62	0
5	8	70	12	224	28	0
6	8	66	11	216	28	0
7	12	64	7	192	24	0
8	48	68	6	216	8	0
9	4	66	18	188	49	0
10	4	66	18	196	51	0
11	16	64	7	136	10	0
12	2	64	32	164	82	0
13	8	64	12	184	27	0
14	2	64	35	168	88	0.
15	16	65	9	176	20	0
16	2	66	36	204	106	0
17	16	72	15	264	32	1
18	8	70	15	224	37	0
19	20	68	11	226	20	0
20	2	66	36	190	99	0
21	4	64	26	154	47	0
22	1	64	64	174	174	0
23	4	65	24	196	57	0
24	4	68	29	234	76	0
25	2	64	39	170	91	0
26	2	66	38	194	99	0
27	2	65	35	186	95	0
28	2	66	38	204	102	0
29	4	62	22	144	41	0
30	1	64	64	170	170	0
31	1	63	63	166	166	0
32	2	64	35	176	90	0
33	2	63	35	160	84	0
34	32	67	9	216	14	0
35	2	64	39	166	91	0
36	2	64	33	180	90	0
37	2	64	39	166	90	0
38	4	63	21	148	42	0

Table 6. (Continued)

Graph No.	|Γ|	DPM	NPM	DOF	NOF	HD
39	1	64	64	166	166	0
40	4	64	23	156	45	0
41	2	61	33	142	73	0
42	8	61	12	112	15	0
43	4	62	17	148	39	0
44	4	63	21	180	58	0
45	4	63	19	176	54	0
46	4	61	18	152	44	0
47	4	62	19	144	36	0
48	2	63	34	168	88	0
49	2	62	38	154	84	0
50	16	64	12	180	21	1
51	2	64	37	182	93	0
52	16	65	12	204	23	0
53	16	62	8	176	18	1
54	2	63	34	162	83	0
55	10	63	10	186	21	0
56	4	62	22	140	42	0
57	8	63	14	156	26	0
58	320	63	4	152	3	0
59	240	56	4	36	2	0
60	28800	120	1	1344	6	2

Table 7. Regular graphs of degree 6 on 10 vertices.

Graph No.	$\|\Gamma\|$	DPM	NPM	DOF	NOF	HD	Comment
1	32	152	12	4864	168	0	
2	16	144	14	4032	269	0	
3	4	146	38	4352	1100	2	
4	8	146	24	4288	560	0	
5	16	146	16	4224	276	0	
6	2	146	87	4400	2260	6	
7	4	146	47	4336	1146	0	
8	4	146	42	4256	1064	1	
9	8	144	22	4064	520	0	
10	12	144	12	4160	348	0	
11	6	144	27	3968	690	0	
12	6	144	24	4016	674	0	
13	2	144	72	4112	2056	1	
14	48	144	10	3648	89	0	
15	20	144	15	3856	222	1	
16	4	144	43	3984	1022	1	
17	20	142	11	3552	192	1	
18	8	144	28	4064	549	1	
19	120	144	2	4064	38	1	
20	288	144	1	4032	23	1	$co(K_4 \cup \bar{C}_6)$
21	1728	144	1	4608	11	2	$co(K_4 \cup K_{3,3})$

Table 8. Regular graphs of degree 7 on 10 vertices.

Graph No.	Graph	$\|\Gamma\|$	DPM	NPM	DOF	NOF	HD
1	$co(C_{10})$	20	293	29	173008	8844	539
2	$co(C_3 \cup C_7)$	84	294	6	179232	2175	138
3	$co(C_4 \cup C_6)$	96	292	11	168384	1865	98
4	$co(C_5 \cup C_5)$	200	295	7	180000	988	57
5	$co(C_3 \cup C_3 \cup C_4)$	576	294	4	178560	369	69

Table 9. The 6 nonisomorphic $H(5,10)$'s.

Underlying graph No.	Howell design				
	1 6	5 10	4 9	2 7	3 8
	4 10	1 7	3 5	6 8	2 9
2	5 9	2 6	1 8	3 10	4 7
	3 7	4 8	2 10	1 9	5 6
	2 8	3 9	6 7	4 5	1 10
	1 6	4 10	2 5	7 8	3 9
	5 10	1 7	6 9	3 4	2 8
17	4 9	3 6	1 8	2 10	5 7
	2 7	5 8	3 10	1 9	4 6
	3 8	2 9	4 7	5 6	1 10
	1 6	8 9	2 4	5 10	3 7
	4 9	1 7	3 10	2 8	5 6
50	3 5	4 10	1 8	6 7	2 9
	2 10	3 6	5 7	1 9	4 8
	7 8	2 5	6 9	3 4	1 10
	1 6	8 9	4 10	2 5	3 7
	2 10	1 7	3 5	6 8	4 9
53	7 9	2 4	1 8	3 10	5 6
	3 4	5 10	6 7	1 9	2 8
	5 8	3 6	2 9	4 7	1 10
	1 6	4 9	2 7	5 10	3 8
	3 9	1 7	4 10	2 8	5 6
60	5 7	3 10	1 8	4 6	2 9
	2 10	5 8	3 6	1 9	4 7
	4 8	2 6	5 9	3 7	1 10
	1 6	5 10	4 9	3 8	2 7
	2 8	1 7	5 6	4 10	3 9
60	3 10	2 9	1 8	5 7	4 6
	4 7	3 6	2 10	1 9	5 8
	5 9	4 8	3 7	2 6	1 10

Table 10. The 18 nonisomorphic Howell designs $H(6,10)$.

Underlying graph No.	Howell design					
	1 5	7 8	3 9		4 10	2 6
		1 6	4 8	2 10	3 7	5 9
3	4 9	5 10	1 7	3 6	2 8	
	3 10	2 9		1 8	5 6	4 7
	2 7		6 10	4 5	1 9	3 8
	6 8	3 4	2 5	7 9		1 10

Underlying graph No.	Howell design					
	1 5		2 6	4 10	7 8	3 9
	3 8	1 6		7 9	2 10	4 5
3	2 9	5 10	1 7		3 4	6 8
		4 9	3 10	1 8	5 6	2 7
	6 10	3 7	4 8	2 5	1 9	
	4 7	2 8	5 9	3 6		1 10

Underlying graph No.	Howell design					
	1 5	3 8	6 10		2 4	7 9
	3 9	1 6		2 7	5 10	4 8
6	2 10		1 7	4 9	6 8	3 5
		4 10	5 9	1 8	3 7	2 6
	4 6	5 7	2 8	3 10	1 9	
	7 8	2 9	3 4	5 6		1 10

Underlying graph No.	Howell design					
	1 5	7 9	3 8		6 10	2 4
	7 8	1 6	2 9	5 10	3 4	
6	3 10		1 7	4 6	2 8	5 9
	4 9	2 10	5 6	1 8		3 7
		3 5	4 10	2 7	1 9	6 8
	2 6	4 8		3 9	5 7	1 10

Underlying graph No.	Howell design					
	1 5	7 8		3 9	6 10	2 4
	4 8	1 6	2 9	5 10		3 7
6	3 10		1 7	4 6	2 8	5 9
	7 9	2 10	5 6	1 8	3 4	
		3 5	4 10	2 7	1 9	6 8
	2 6	4 9	3 8		5 7	1 10

Underlying graph No.	Howell design					
	1 5	3 9	6 10		2 4	7 8
	3 8	1 6		4 10	5 7	2 9
6	2 10		1 7	5 9	6 8	3 4
		2 7	4 9	1 8	3 10	5 6
	4 6	5 10	2 8	3 7	1 9	
	7 9	4 8	3 5	2 6		1 10

Table 10. (Continued)

Underlying graph No.	Howell design					
	1 5	3 9	6 10		2 7	4 8
	2 10	1 6	4 9	5 7	3 8	
6	6 8	2 4	1 7	3 10		5 9
	7 9		3 5	1 8	4 10	2 6
		5 10	2 8	4 6	1 9	3 7
	3 4	7 8		2 9	5 6	1 10
	1 5	3 9	4 10	2 7		6 8
	2 8	1 6		4 9	5 10	3 7
6	3 10	4 8	1 7		2 6	5 9
	7 9	2 10	5 6	1 8	3 4	
		5 7	3 8	6 10	1 9	2 4
	4 6		2 9	3 5	7 8	1 10
	1 5	9 10		3 7	6 8	2 4
	7 8	1 6	5 10	2 9	3 4	
8	4 9		1 7	5 6	2 10	3 8
	3 10	2 7	4 6	1 8		5 9
		3 5	2 8	4 10	1 9	6 7
	2 6	4 8	3 9		5 7	1 10
	1 5	3 7	8 10	2 4		6 9
	4 10	1 6	3 5	7 9	2 8	
13	2 6	4 9	1 7		5 10	3 8
	3 9	2 10		1 8	6 7	4 5
		5 8	4 6	3 10	1 9	2 7
	7 8		2 9	5 6	3 4	1 10
	1 5	3 8	4 10	6 9		2 7
	2 10	1 6	3 9		4 7	5 8
15	8 9		1 7	2 4	5 10	3 6
		7 10	5 6	1 8	2 3	4 9
	6 7	4 5	2 8	3 10	1 9	
	3 4	2 9		5 7	6 8	1 10

Table 10. (Continued)

Underlying graph No.	Howell design				
16	1 5	8 10	6 9		4 7	2 3
	2 7	1 6		5 10	3 8	4 9
	4 10	2 9	1 7	3 6		5 8
	3 9		4 5	1 8	2 10	6 7
	6 8	5 7	3 10	2 4	1 9	
		3 4	2 8	7 9	5 6	1 10
17	1 5	9 10	6 8		2 3	4 7
	4 9	1 6	5 10	2 7		3 8
	2 10	3 4	1 7	6 9	5 8	
	3 6	5 7		1 8	4 10	2 9
	7 8		2 4	3 10	1 9	5 6
		2 8	3 9	4 5	6 7	1 10
18	1 5	2 7	6 8		4 10	3 9
	2 10	1 6	5 9	3 4	7 8	
	8 9		1 7	5 10	2 3	4 6
	3 6	9 10	2 4	1 8		5 7
		4 5	3 10	6 7	1 9	2 8
	4 7	3 8		2 9	5 6	1 10
19	1 5	7 8	9 10	4 6		2 3
	3 10	1 6	2 8	7 9	4 5	
		2 4	1 7	3 5	8 10	6 9
	2 9	5 10		1 8	3 6	4 7
	6 7		3 4		1 9	5 8
	4 8	3 9	5 6		2 7	1 10
20	1 5	3 8		2 6	4 10	7 9
		1 6	3 9	5 10	2 7	4 8
	3 10		1 7	4 9	6 8	2 5
	2 9	4 7	6 10	1 8	3 5	
	7 8	2 10	4 5		1 9	3 6
	4 6	5 9	2 8	3 7		1 10
21	1 5	9 10	2 6	3 7		4 8
	2 7	1 6	8 10	4 9	3 5	
	8 9	2 5	1 7		4 10	3 6
	3 10		4 5	1 8	6 7	2 9
	4 6	3 8		2 10	1 9	5 7
		4 7	3 9	5 6	2 8	1 10
21	1 5	9 10	2 6	3 7	4 8	
	2 7	1 6	8 10		3 5	4 9
	8 9	2 5	1 7	4 10		3 6
	4 6		3 9	1 8	2 10	5 7
	3 10	4 7		5 6	1 9	2 8
		3 8	4 5	2 9	6 7	1 10

Table 11. Some Howell designs $H(7,10)$.

Underlying graph	Howell design						
	1 3	6 9	8 10			2 4	5 7
	2 5	1 4		7 9		6 10	3 8
	7 10		1 5	4 8	3 9		2 6
$co(C_{10})$		3 10	4 7	1 6	2 8	5 9	
	6 8		2 9	3 5	1 7		4 10
	4 9	2 7	3 6		5 10	1 8	
		5 8		2 10	4 6	3 7	1 9
	1 8	5 7	4 9	3 6	2 10		
		2 8	1 6	5 9	4 7	3 10	
			3 8	2 7	6 9	1 5	4 10
$co(C_3 \cup C_7)$	5 10			4 8	1 3	7 9	2 6
	3 7	6 10			5 8	2 4	1 9
	2 9	1 4	7 10			6 8	3 5
	4 6	3 9	2 5	1 10			7 8
	1 3	6 9	5 7	8 10		2 4	
	4 9	1 5	3 8			7 10	2 6
		3 10	1 6	2 5	7 9		4 8
$co(C_4 \cup C_6)$		2 8	4 10	1 7	3 6		5 9
	6 10	4 7	2 9		1 8	3 5	
	5 8			4 6	2 10	1 9	3 7
	2 7			3 9	4 5	6 8	1 10
	1 3	7 9	8 10	2 4		5 6	
	6 9	1 4	2 5		7 10	3 8	
	4 8	5 10	1 6			2 7	3 9
$co(C_5 \cup C_5)$		3 6		1 7	2 9	4 10	5 8
	5 7		4 9	3 10	1 8		2 6
	2 10			6 8	3 5	1 9	4 7
		2 8	3 7	5 9	4 6		1 10
	1 3		7 10		2 4	5 8	6 9
	4 10	1 5	3 8		7 9	2 6	
		7 8	1 6	4 9		3 10	2 5
$co(C_3 \cup C_3 \cup C_4)$	5 9	2 10		1 7	3 6		4 8
	2 7	3 9	4 5	6 10	1 8		
		4 6		2 8	5 10	1 9	3 7
	6 8		2 9	3 5		4 7	1 10

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