Room squares with maximum empty subarrays

D.R. Stinson

ABSTRACT

If a Room square of side $2n+1$ contains a $t \times t$ block of empty cells, then $t \leq n$. If $t = n$ we say the empty subarray is maximum. We construct such Room squares of sides 9 and 11, and show that no such Room square exists of side 7. Recursive constructions enable one to construct an infinite number of these, but the spectrum is not determined. Also, we describe several equivalent formulations of the problem, indicating connections with other types of designs.

1. Introduction.

Let n be a non-negative integer. A *Room square* of side $2n + 1$ is a square array R of side $2n + 1$, in which each cell either is empty or contains an unordered pair of symbols chosen from a set S of size $2n + 2$, such that each symbol occurs in exactly one cell of each row and each column of R, and each pair of symbols occurs in exactly one cell of R. R. Mullin and W. Wallis [1] proved

Theorem 1.1. *There is a Room square of side* $2n + 1$ *if and only if* $2n + 1 \neq 3$ *or* 5.

Each row or column of a Room square of side $2n + 1$ contains $n + 1$ filled cells and n empty cells. A square $t \times t$ subarray of empty cells must therefore satisfy $t \leq n$. If equality occurs, we say that the given Room square contains a *maximum empty subarray*. We will denote by MESRS($2n+1$) a Room square of side $2n + 1$ which contains a maximum empty subarray.

In this paper, we investigate MESRS. First, in Section 2, we give several equivalent formulations. In Section 3, we describe direct constructions. Then, in Section 4, we give some recursive constructions and some consequences. We conjecture that there exists a MESRS($2n+1$) if and only if $2n + 1 \neq 3$, 5, or 7; however, we are far from proving this fact.

Ars Combinatoria, **20** (1985), pp.159-166

We close this section by presenting MESRS(9) and MESRS(11) in Fig. 1.

3 7					2 8	5 9	4 10	1 6
	5 6				1 10	4 7	2 9	3 8
		2 10			6 7	1 8	3 5	4 9
			4 8		3 9	2 6	1 7	5 10
				1 9	4 5	3 10	6 8	2 7
1 2	8 10	5 7	6 9	3 4				
4 6	1 3	8 9	7 10	2 5				
5 8	7 9	1 4	2 3	6 10				
9 10	2 4	3 6	1 5	7 8				

5 10						2 4	9 11	3 7	1 12	6 8
	1 6					7 9	3 5	10 11	4 8	2 12
		2 7				3 12	8 10	1 4	6 11	5 9
			3 8			1 10	4 12	6 9	2 5	7 11
				4 9		8 11	2 6	5 12	7 10	1 3
					11 12	5 6	1 7	2 8	3 9	4 10
1 2	4 11	9 12	6 7	3 10	5 8					
4 6	2 3	5 11	10 12	7 8	1 9					
8 9	5 7	3 4	1 11	6 12	2 10					
7 12	9 10	1 8	4 5	2 11	3 6					
3 11	8 12	6 10	2 9	1 5	4 7					

Figure 1.

2. Equivalent formulations.

Suppose R is a MESRS$(2n+1)$. We can permute rows and columns so that the maximum empty subarray T occurs in the lower right corner of R. Hence, we regard R as being partitioned $R = \left[\begin{array}{c|c} D & F_1 \\ \hline F_2 & T \end{array}\right]$. It is easy to see that F_1 and F_2 contain only filled cells, and D contains precisely one filled cell in each row and column. Permuting rows and columns, we can stipulate that the filled cells of D occur on the main diagonal.

Furthermore, each symbol occurs in n cells of F_1, in n cells of F_2, and hence in one cell of D. So, the filled cells of D form a perfect matching M of the symbol set S, i.e. a one-factor of the complete graph on vertex set S.

If we project the cells of D horizontally, we obtain an $(n+1) \times (n+1)$ array $H_1 = \left[M \mid F_1 \right]$. This array is a Howell design $H(n+1,2n+2)$: every symbol occurs in one cell of each row and column of H_1, and no pair is repeated. Similarly if we project the cells of M downward, we obtain $H_2 = \left[\frac{M^T}{F_2} \right]$, which is also an $H(n+1,2n+2)$. These two Howell designs contain no common pairs, except for those in M. Every pair of symbols of S not in M occurs in precisely one of H_1, H_2.

Thus we say two $H(n+1,2n+2)$, H_1 and H_2, are *almost disjoint* if there exists one row/column of H_1 which contains the same pairs as one row/column of H_2, but no other pair occurs in both H_1 and H_2.

We have constructed a pair of almost disjoint $H(n+1,2n+2)$ from a MESRS$(2n+1)$. It is easy to see that the construction is reversible, so we have

Theorem 2.1. *There exists a MESRS$(2n+1)$ if and only if there exists a pair of almost disjoint $H(n+1,2n+2)$.*

There is another way to alter our given MESRS$(2n+1)$, R. Consider the array $B = \left[F_2^T \mid M \mid F_1 \right]$. This array is of size $n \times (2n+1)$, and contains every unordered pair of the $2n+2$ symbols exactly once. Every column contains each symbol once, and each row contains each symbol once or twice. Such an array is called a *balanced tournament design*, denoted *BTD*$(n+1)$.

Not all BTD$(n+1)$ give rise to MESRS$(2n+1)$, however. The necessary property is that the BTD can be partitioned as shown above. That is, we should be able to partition the columns into 3 sets C_1,C_2,C_3 of sizes $1,n,n$ respectively, so that the columns in $C_1 \cup C_2$ form an $H(n+1,2n+2)$, as do the columns in $C_1 \cup C_3$. We will say that such a BTD is *partitionable,* and denote it PBTD$(n+1)$. This discussion establishes

Theorem 2.2. *There exists a MESRS$(2n+1)$ if and only if there exists a PBTD$(n+1)$.*

3. Direct Constructions.

The major direct construction for Room squares is the method of starters and adders; however, it seems that MESRS can not be produced by this method. The equivalent formulations of Section 2 suggest that we can produce MESRS by other means. Balanced tournament designs are studied in [3], and many constructions are given. However, none of these appear to yield partitionable BTDs.

The idea of almost disjoint Howell designs $H(n+1,2n+2)$ seems to be more promising, and it was by this method that we constructed the MESRS(9) and MESRS(11) given in the introduction. In the remainder of this section we describe possible methods of constructing almost disjoint Howell designs. It could be very nice to construct an infinite class of these designs; however, we only have the two small designs at present.

The *graph* $G(H)$ of a Howell design H is the graph on vertex set S (the symbol set of the design) whose edges are the pairs occurring in the cells of H. One approach is to first find two $(n+1)$-regular graphs G_1, G_2 on $2n + 2$ vertices, S, such that $M = G_1 \cap G_2$ is a 1-factor, and $G_1 \cup G_2$ is the complete graph on vertex set S. We want to find H_i $(i = 1,2)$, such that $G(H_i) = G_i$ $(i = 1,2)$, and such that G_1, G_2 both have M as a row/column.

Alternatively, we can start with an $H(n+1,2n+2)$, H_1. For each row/column M of H_1, construct the graph $G_2 = G(H_1)^c \cup M$. (The superscript "c" denotes the complement of the specified graph.) Then, determine if there is an $H(2n,2n+2)$, H_2 with $G(H_2) = G_2$, having M as a row/column.

This approach quickly rules out the existence of a pair of almost disjoint $H(4,8)$, and hence a MESRS(7). There are precisely six non-isomorphic 4-regular graphs on 8 vertices. It can easily be checked by hand that the only one of these that is the graph of an $H(4,8)$ is $K_{4,4}$. For any one-factor F of $K_{4,4}$, $K_{4,4}^c \cup F$ is not isomorphic to $K_{4,4}$, and hence is not the graph of an $H(4,8)$. Hence, a pair of almost disjoint $H(4,8)$ do not exist.

Let us also mention that the non-existence of a MESRS(7) can also be verified from the list of 6 non-isomorphic Room squares of side 7, enumerated in [5]. S.A. Vanstone observed that the incidence pattern of empty vs. filled cells of any of these RS(7) forms the incidence matrix of the complement of the Fano plane, which certainly does not contain a 3×3 submatrix of zeroes.

Next, we consider almost disjoint $H(5,10)$. There are 60 non-isomorphic 5-regular graph on 10 vertices, and the Howell designs having each of these as graphs were enumerated in [2]. There is a Howell design with graph $K_{5,5}$ (this is just a pair of orthogonal Latin squares of order 5) but it is not one of a pair of almost disjoint designs. Of the remaining 59 graphs, only 4 are the graphs of $H(5,10)$ designs, and only 1 non-isomorphic design is obtained in each case. Knowing this, we were pessimistic that a pair of almost disjoint designs would exist, but in fact the search was successful.

The next case is a pair of almost disjoint $H(6,12)$. There are too many 6-regular graphs on 12 vertices to conveniently enumerate all Howell designs, so we wanted to restrict our search. We picked a graph G, and a one-factor F of G, and found a permutation Π such that $F^{\Pi} = F$, and $G^c \cup F = G^{\Pi}$. If there is a Howell design H with graph G having F as a row/column, then H and H^{Π} are almost disjoint Howell designs.

We obtained the MESRS(11) Fig. 1 as a result. The relevant permutation Π is (1 3 2 4)(5)(6 8 7 9)(10)(11 12). Also note that both Howell designs have as an automorphism the cyclic group of order 5 generated by (1 2 3 4 5)(6 7 8 9 10)(11)(12).

Summarizing the results of this section, we have

Theorem 3.1. *There exists a MESRS(9) and MESRS(11), whereas there does not exist a MESRS(7).*

4. Recursive constructions.

We have two main recursive constructions, which will give rise to an infinite number of MESRS.

Construction 4.1. (Direct Product).

Suppose there exist MESRS$(2m-1)$ and MESRS$(2n-1)$, and a pair of orthogonal Latin squares of order m (i.e. $m \neq 2$ or 6). Then there exists a MESRS$(2mn-1)$.

Proof. Let R be a MESRS$(2n-1)$, partitioned $R = \left[\begin{array}{c|c} D & F_1 \\ \hline F_2 & T \end{array}\right]$, as in Section 2. Inflate R by a factor of m as follows. First, replace every empty cell by an $m \times m$ array of empty cells, then replace a cell of $F_1 \cup F_2$, containing $\{x,y\}$, say, by the superposition of a pair of orthogonal Latin squares on symbol set $(\{x\} \times \{1,\ldots,m\}) \times (\{y\} \times \{1,\ldots,m\})$. Finally, for each $\{x,y\} \in D$, let $S(x,y) = \left[\begin{array}{c|c} D'_{xy} & F'_{xy,1} \\ \hline F'_{xy,2} & T'_{xy} \end{array}\right]$ be a MESRS$(2m-1)$ on $\{x,y\} \times \{1,\ldots,m\}$. Replace each cell $\{x,y\}$ of D by D'_{xy}. Now adjoin $m-1$ new rows and columns to this $(2n-1)m \times (2n-1)m$ array R'. The last $m-1$ columns contain the $F'_{xy,1}$ arrays; the last $m-1$ rows contain the $F'_{xy,2}$ arrays; and the rest is empty. This modified array is a $RS(2mn-1)$, and the lower right corner is an empty square subarray of side $m(n-1) + m - 1 = mn - 1$, as required. Hence, we have a MESRS$(2mn-1)$. □

The second construction is generally weaker than the direct product, but can sometimes be applied where the direct product cannot.

Construction 4.2. (PBD construction).

Suppose $(X, \mathbf{A})$ is a pairwise balanced design, and the blocks in $\mathbf{A}$ can be partitioned as $\mathbf{A} = \bigcup_{i=1}^{r} \mathbf{R}_i$, where each $\mathbf{R}_i$ is a partition of X into blocks of size n_i. If there exist MESRS$(2n_i - 1)$, for $1 \leq i \leq r$, then there exists a MESRS$(2|X| - 1)$.

Proof. This is a consequence of [4, Construction 2]. In the notation of [4], a MESRS$(2n+1)$ is an $H^{**}(2n-1, 2n)$. For each $\mathbf{R}_i$, define $k_i = 2n_i - 1$ and apply the above-mentioned construction. □

One application of this construction is to start with a resolvable (126,6,1)-BIBD. A MESRS(251) is obtained.

Finally, we mention how Construction 4.1 can be altered, with the aid of the array in Fig. 2. We obtain the following.

Construction 4.3.

Suppose there exists a MESRS$(4m - 1)$, and a pair of orthogonal Latin squares of side m. Then there exists a MESRS$(12m - 1)$.

Proof. Apply the same operations as in Construction 4.1, starting with the array of Fig. 2 in place of the MESRS$(2n - 1)$. □

The only MESRS of side less than 100 produced by these constructions are: MESRS(49) from Const. 4.1 with $m = n = 5$; MESRS(59) from Const. 4.1 with $m = 5$, $n = 6$; and MESRS(35) from Construction 4.3 with $m = 3$.

0 3						1 11	5 10	4 8	2 7
6 9						5 7	1 8	2 10	4 11
		1 4				0 8	2 9	3 11	5 6
		7 10				2 6	0 11	5 9	3 8
				2 5		3 10	4 6	0 7	1 9
				8 11		4 9	3 7	1 6	0 10
4 2	10 8	5 0	11 6	3 1	9 7				
10 11	4 5	2 3	8 9	6 7	0 1				
1 5	7 11	8 6	2 0	9 10	3 4				
7 8	1 2	11 9	5 3	0 4	6 10				

Figure 2

There are many other recursive constructions for Room squares, but none of them seem to be applicable to this problem. However, we conjecture the spectrum for MESRS$(2n+1)$ is the set of all odd integers exceeding 7.

Acknowledgement. We would like to thank Dan Archdeacon and Scott Vanstone for their helpful comments.

References.

[1] R.C. Mullin and W.D. Wallis, *The existence of Room squares*, Aequationes Math. 13 (1975), 1-7.

[2] A. Rosa and D.R. Stinson. *One-factorizations of regular graphs and Howell designs of small order*, preprint.

[3] P.J. Schellenberg, G.H.J. van Rees, and S.A. Vanstone, *The existence of balanced tournament designs*, Ars Combinatoria 3 (1977), 303-318.

[4] D.R. Stinson, *The existence of Howell designs of odd side*, J. Comb. Theory A 32 (1982), 53-65.

[5] W.D. Wallis, A.P. Street and J.S. Wallis, *Combinatorics: Room Squares, Sum-free sets, Hadamard matrices*, Lecture Notes in Math. 292, Springer-Verlag, Berlin, 1972.

Department of Computer Science
University of Manitoba
Winnipeg, Manitoba
R3T 2N2 Canada