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V-Squares

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ABSTRACT

We define a generalization of a Howell design called a V-square. The admissible parameter sets for V-squares are computed and we prove that for a special class of V-squares (crowded V-squares), the necessary conditions for existence are sufficient. Also mentioned is the connection between V-squares and graph colourings.

1. Introduction.

We begin with the definitions.

Definition 1.1. A V-square $V(k,r;v)$ is a $k \times k$ array with cells either empty or containing unordered pairs from a set S, $|S| = v$, satisfying:

i) each symbol $s \in S$ occurs in at most one cell in any row or column;

ii) each symbol occurs exactly r times in the array;

iii) Any pair of elements from S occurs in at most one cell of the array.

Notice that if $r = k$, then each symbol must occur exactly once in every row and column and thus we have a Howell Design. (See [5] or [7] for information on Howell designs.) Thus a Howell design $H(s,2n)$ is a $V(s,s;2n)$. Also, note that a Room square of side n is a $V(n,n;n+1)$. If each cell is filled, then in the notation of Kramer et al., [6], a $V(k,r;v)$ is a crowded $RR(k,k;1-([v,2k,2k],2,[r,1,1]))$. In Figure 1, we give an example of a V-square which is not a Howell design.

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1 17	2 6	7 18	4 8	9 15	12 16
2 8	1 18	6 17	3 5	10 16	9 13
3 18	4 5	1 8	7 17	11 13	10 14
4 7	3 17	2 5	6 18	12 14	11 15
9 13	10 14	11 15	12 16	1 4	5 7
12 14	9 15	10 16	11 13	2 3	6 8

Figure 1
$V(6,4;18)$

In a $V(k,r;v)$, the pairs in the cells can be thought of as edges in a graph G, called the *underlying graph* of the $V(k,r;v)$. The array V induces two proper edge k-colourings of G determined by the rows and columns of V. That is, if $\{x,y\}$ is in cell (i,j), then edge $\{x,y\}$ receives colour i in the row-induced edge k-colouring, and colour j in the column-induced edge k-colouring of G. Note that if two edges receive the same colour in the row-induced edge colouring of G then they receive different colours in the column-induced colouring and vice versa. This motivates the following.

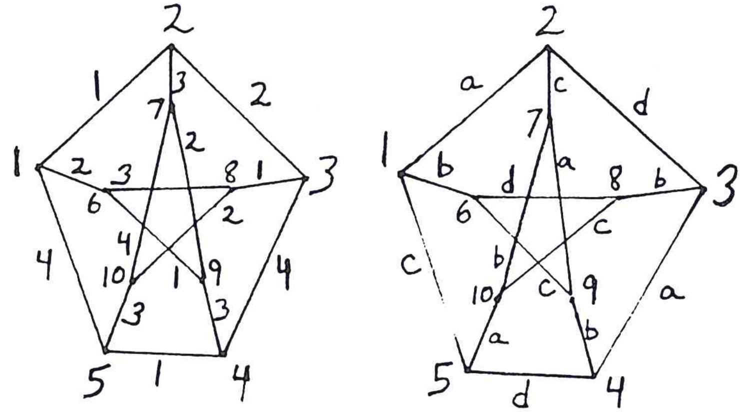
Definition 1.2. Two edge colourings C_1 and C_2 of a graph G are said to be *orthogonal* if any two edges that receive the same colour in one colouring receive different colours in the other colouring.

For example, if $G = K_{n,n}$, the complete bipartite graph with bipartitions of size n, then a pair of orthogonal Latin Squares of side n induce two orthogonal edge colourings of G.

The following proposition follows from Definition 1.2 and the discussion preceeding it.

Proposition 1.3. The existence of an r-regular graph G on v vertices having two orthogonal edge k-colourings is equivalent to the existence of a $V(k,r;v)$ with underlying graph G.

In Figure 2 we give two orthogonal 4-colourings of the Peterson graph and the $V(4,3;10)$ equivalent to it.



	a	b	c	d
1	1 2	3 8	6 9	4 5
2	7 9	1 6	8 10	2 3
3	5 10	4 9	2 7	6 8
4	3 4	7 10	1 5	

Figure 2
$V(4,3;10)$

More results concerning orthogonal edge colourings are given in [1]. Here we will restrict our attention to the construction of V-squares without necessarily considering the underlying graph. We consider a special class of V-squares, which we define. A $V(k,r;v)$ is *crowded* if there are no empty cells, i.e. if $vr = 2k^2$.

Since $v = 2k^2/r$ is a function of k and r, we denote a crowded $V(k,r;v)$ by $V(k,r)$. For the remainder of this paper we will be concerned only with crowded V-squares, and we will determine the spectrum of $V(k,r)$.

In Section 2 we determine the admissible parameters for $V(k,r)$. Section 3 supplies various constructions for $V(k,r)$ to be used in Section 4, where we determine the spectrum.

2. Admissible Parameters.

For the remainder of this paper we assume all numbers to be nonnegative integers. In this section we derive necessary conditions for the parameters k and r. We will say that $(k,r) \in V$ if there exists a $V(k,r)$.

Proposition 2.1. $(k,r) \in V$ implies $r \mid 2k^2$.

Proof. Count the number of occurrences of all the symbols in two ways to get $vr = 2k^2$. □

Let $r = 2^{e_0} \Pi p_i^{e_i}$ be the prime factorization of r. Define $\rho = 2^{\left\lceil \frac{e_0 - 1}{2} \right\rceil} \Pi p_i^{\lceil e_i/2 \rceil}$, where $\lceil \; \rceil$ denotes the ceiling function. Since necessarily $k \geq r$, we can write $k = r + x$; $x \geq 0$.

Proposition 2.2. The existence of a $V(r+x,r)$ implies $r \mid 2x^2$ and $\rho \mid x$.

Proof. By Proposition 2.1, $r \mid 2(r+x)^2$. Thus $r \mid 2x^2$. Since $r \mid 2x^2$, then $2^{e_0} \mid 2x^2$ and $p_i^{e_i} \mid 2x^2$ for $p_i \neq 2$. So $2^{\left\lceil \frac{e_0 - 1}{2} \right\rceil} \mid x$ and $p_i^{\lceil e_i/2 \rceil} \mid x$ for $p_i \neq 2$ and thus $\rho = 2^{\left\lceil \frac{e_0 - 1}{2} \right\rceil} \Pi p_i^{\lceil e_i/2 \rceil} \mid x$. □

Notice that $\rho \mid r$, so $V(k,r)$ exists only if $k \equiv n\rho$ (modulo r). Also, since $\rho \mid x$, we can write $V(k,r) = V(r+n\rho,r)$ for some n. Thus by Proposition 2.2, $r \mid 2(n\rho)^2$. If we let $t = \dfrac{2n^2\rho^2}{r}$, then $tr = 2n^2\rho^2$. But $r \mid 2\rho^2$; thus $n^2 \mid t$. Now $t/n^2 = (2n^2\rho^2/r) \cdot (1/n^2) = 2\rho^2/r$ so by the definition of ρ we see that $t/n^2 = 2\rho^2/r$ is square-free. There are two cases to consider: t/n^2 is even or odd.

If t/n^2 is even, then $t/2n^2$ is an odd square-free integer and $(t/2n^2) \mid \rho^2$. Let $s = t/2n^2$. Since s is square-free, $s \mid \rho$. Let $s \cdot m = \rho$; then $sm^2 = r$. Thus $V(r+n\rho,r) = V((t/2n^2) \cdot (m^2+mn), (t/2n^2)m^2) = V(s(m^2+mn),sm^2)$ for some odd, square-free s.

If t/n^2 is odd, then similar reasoning gives $V(r+n\rho,r) = V((t/n^2) \cdot (2m^2+mn),\; t/n^2(2m^2)) = V(s(2m^2+mn),s(2m^2))$ for some odd, square-free s.

The above analysis gives us our characterization of the admissible parameters of a $V(k,r)$.

Proposition 2.3. The necessary conditions for the existence of a $V(k,r)$ fall into two cases:

Case 1.

If $r = sm^2$ where s is odd and square-free, then $V(k,r) = V(s(m^2+mn),sm^2)$ for some $n \geq 0$.

Case 2.

If $r = s(2m^2)$ where s is odd and square-free, then $V(k,r) = V(s(2m^2+mn),s(2m^2))$ for some $n \geq 0$.

3. Useful Constructions.

In order to construct may $V(k,r)$ it is necessary to generalize the idea of V-square to that of a V-rectangle.

Definition 3.1. A $V(k_1 \times k_2, r; v)$ is a $k_1 \times k_2$ array which satisfies all of the conditions of Definition 1.1. A crowded $V(k_1 \times k_2, r; v)$ is termed a $V(k_1 \times k_2, r)$. We also say that a V-rectangle or V-square is *bipartite* if the underlying graph is bipartite.

Lemma 3.2. *There exists a bipartite $V(a \times b, b)$ and $V(b \times a, b)$ for every $a \geq b$, except $a = b = 2$ and $a = b = 6$. Also, there is a nonbipartite $V(6,6)$.*

Proof. Let L_1 and L_2 be two orthogonal Latin squares of side $a(a \neq 2$ or $6)$ on different symbol sets. If L_1 and L_2 are superimposed then we obtain a bipartite $V(a,a)$. Now delete $a - b$ rows or columns to obtain a bipartite $V(b \times a, b)$ or $V(a \times b, b)$.

If $a = 6$, then let L_1 and L_2 be the two 6×6 Latin squares found by Horton [4], again on different symbol sets. Say L_1 is on the symbol set $S_1 = \{1,2,...,6\}$ and L_2 is on $S_2 = \{7,...,12\}$. These squares have the property that when superimposed, every ordered pair in $S_1 \times S_2$ occurs exactly once, except that in the lower right 2×2 corner, (6,12) and (5,11) occur twice, and (6,11) and (5,12) never occur. A bipartite $V(6 \times b, b)$, $b \leq 5$, can be constructed by superimposing L_1 and L_2 and then deleting the final $6-b$ columns. The $V(b \times 6, b)$ is the transpose.

The nonbipartite $V(6,6)$ is a Howell design $H(6,12)$ and can be found in Hung and Mendelsohn [5].

Finally, if $a = 2$, a $V(2 \times 1, 1)$ is trivial and a $V(2,2)$ does not exist. □

Our next theorem is similar to the usual Kronecker product for Latin squares.

Theorem 3.3. *(First Product Construction) If there exists a $V(a \times b, r_1)$ and a bipartite $V(c \times d, r_2)$, then there exists a $V(ac \times bd, r_1 r_2)$.*

Proof. Let $V_1 = V(a \times b, r_1)$ and $V_2 = V(c \times d, r_2)$ and say the symbol set of V_1 is S_1. Since V_2 is bipartite, it is equivalent to two orthogonal Latin rectangles L_1 and L_2 on the same symbol set S_2. Say $\{x,y\}$ is in a cell of V_1. Define xL_1 to be the $c \times d$ array obtained from L_1 by replacing each symbol $s \in L_1$ by the new symbol x_s. Now, in V_1, replace the cell containing $\{x,y\}$ with the $c \times d$ array obtained by superimposing xL_1 and yL_2. It is straightforward to check that each symbol in $S_1 \times S_2$ occurs $r_1 r_2$ times and that this resulting array is indeed a $V(ac \times bd, r_1 r_2)$. □

Corollary 3.4. *If $n \neq 2$ or 6, then the existence of a $V(k,r)$ implies the existence of a $V(nk,nr)$.*

Proof. By Lemma 3.2, if $n \neq 2$ or 6 there exists a bipartite $V(n,n)$. The result follows by Theorem 3.3. □

Using Corollary 3.4, we note that, in Proposition 2.3, $(s(m^2+mn), sm^2) \in V$ if $(m^2+mn, m^2) \in V$, since $s \neq 2$ or 6. Also $(s(2m^2+mn), s \cdot 2m^2) \in V$ if $(2m^2+mn, 2m^2) \in V$. Thus we focus our attention on these orders.

We will not use the next theorem; however we include it for completeness.

Theorem 3.5. *(Second Product Construction) If $(k,r) \in V$, then $(kt,r) \in V$, for every t.*

Proof. Place t^2 copies of the $V(k,r)$, all on different symbol sets, in a $t \times t$ array. □

The remaining theorems all deal with the orders specified in Proposition 2.3.

Theorem 3.6. *If there exist 4 pairwise orthogonal Latin squares of side $2m + n$, where $0 \leq n \leq m$, and 2 orthogonal Latin squares of side m, then there exists a $V((2m+n)m, 2m^2)$.*

Proof. Let A and B be two of the orthogonal Latin squares of order $2m + n$. Use the two other orthogonal squares to find two sets of common transversals in A and B. Let one set be $T_1, T_2, \ldots, T_{2m+n}$, and from the other set choose $2m$ transversals $F_1, F_2, \ldots, F_m$, $S_1, \ldots, S_m$. Let V_1 be the $V(2m+n, 2m+n)$ formed by the superimposition of A and B.

Let V_2 be the bipartite $V(m,m)$ obtained from the two orthogonal Latin squares C and D of side m. Now use Theorem 3.3 to take the product of V_1 and V_2. Call this V_3.

Let A be on the symbol set $S_A = \{\alpha_1, \ldots, \alpha_{2m+n}\}$, B be on the symbol set $S_B = \{\beta_1,...,\beta_{2m+n}\}$, and C and D be on the set $M = \{1,2,...,m\}$. Then V_3 is on the symbols $(S_A \times M) \cup (S_B \times M)$. Thus there are $2(2m+n)m = 4m^2 = 2mn$ symbols, each occurring $2m^2 + mn$ times. We desire $4m^2 + 4mn + n^2$ symbols each occurring $2m^2$ times. Thus we must add $2mn + n^2$ new symbols, and delete each old symbol exactly mn times.

Call the new symbols ∞_i^j where $1 \le i \le 2m+n$ and $1 \le j \le n$. Let E be an $m \times n$ Latin rectangle on the symbols $1,2,...,m$.

Remember that V_3 consists of $(2mn+n)^2$ $m \times m$ blocks. Now, we replace the old symbols of an $m \times m$ block X depending on which transversals X occurs in:

1. If X is in transversals T_i and F_j, then replace each pair $\{(\alpha_k,r),(\beta_{k'},s)\}$ in X by the pair $\{\infty_i^p,(\beta_{k'},s)\}$ where $r = E(j,p)$. If for every $1 \le p \le n$, $r \ne E(j,p)$, then no replacement is made.
2. If X is in transversals T_i and S_j, then replace each pair $\{\alpha_k,r),(\beta_{k'},s)\}$ in X by the pair $\{(\alpha_k,r),\infty_i^p\}$ where $s = E(j,p)$. If for every $1 \le p \le n$, $s \ne E(j,p)$, then no replacement is made.
3. If X is not in one of the transversals $F_1,F_2,...,F_m$, $S_1,...,S_m$, then no replacements are made in X.

We check that this square is a $V(2m^2+mn,2m^2)$. First, each symbol (α_i,r) or (β_i,r) has been deleted from exactly as many $m \times m$ blocks as the number of times r occurs in the array E. Thus each symbol has been deleted from mn cells. The symbol ∞_i^j occurs m times for each transversal of the $F_1,...,F_m$, $S_1,...,S_m$. Thus it occurs exactly $2m^2$ times also.

It is relatively straightforward to verify conditions (i) and (iii) in Definition 1.1, to finish the proof. □

The next theorem is similar, but requires the existence of only 3 MOLS of order $2m+n$. However, we now require n to be even.

Theorem 3.7. *If there exist three pairwise orthogonal Latin squares of order $2m+n$ where n is even and $0 \le n \le 2m$, and 2 orthogonal Latin squares of order n, then there is a $V(2m^2+mn,2m^{2)}$.*

Proof. Let A and B be two orthogonal Latin squares of order $2m+n$, and let $F_1,...,F_n$, $S_1,...,S_n$ be $2n$ disjoint common transversals of A and B, obtained from the third orthogonal square (note $2n \le 2m + n$). Let V_1, V_2 and V_3 be as in the previous construction, on the symbol sets $S_A \cup S_B, M$ and $(S_A \cup S_B) \times M$, respectively. The new symbols will be denoted f_i^j and s_i^j, where $1 \le i \le m + n/2$ and $1 \le j \le n$. Let E denote a $(2m+n) \times m$ rectangle on the symbols $1,...,m+n/2$, in which each symbol occurs $2m$ times in total, and at most once in each row. (E

can be constructed, for example, by defining $E_{ij} = 1 + (((i-1)m+j) \bmod (m+n/2))$.

Now consider the $m \times m$ block arising from the cell in row i of F_j. In this block, replace all occurrences of (α,k) $(\alpha \epsilon S_A, 1 \leq k \leq m)$ by f_e^j where $e = E_{ik}$. Similarly, in the block arising from the cell in row i of S_j, replace all symbols (β,k) $(\beta \epsilon S_B, 1 \leq k \leq m)$ by s_e^j, where $e = E_{ik}$.

It is clear that each old symbol has been deleted mn times, and each new symbol now occurs $2m \cdot m = 2m^2$ times. No symbol occurs twice in any row or column, so we have a $V(2m^2+mn, 2m^2)$. □

A *diagonal Latin square* of order n is a Latin square each of whose main diagonal and back diagonal are transversals. It has been shown [8] that two orthogonal diagonal Latin squares exist for all orders $n \geq 7$, $n \neq 10$.

Theorem 3.8. *If there exists a Howell design $H(m,2m)$ with one transversal and two orthogonal diagonal Latin squares of order $2m+1$, then there exists a $V(2m^2+m, 2m^2)$.*

Proof. Let L_1 and L_2 be the two orthogonal diagonal Latin squares of order $2m+1$, on the same symbol set $S = \{1,2,...,2m+1\}$. Superimpose L_1 and L_2 so that in each cell is a pair of elements from S. Down the main diagonal, replace each first symbol in a cell by the new symbol ∞ and down the back diagonal replace each second symbol in a cell by ∞. Leave the cell that is in both diagonals empty. Call this square W. Note that W contains $2m+2$ symbols and each symbol in $S' = S \cup \{\infty\}$ occurs exactly $4m$ times.

Now, let H be a Howell design $H(m,2m)$ containing a transversal, on the symbol set $T = \{1,2,...,2m\}$. Let $H_{ab} (a,b,\epsilon S')$ be the square H with each pair $\{i,j\}$ in H replaced by $\{i_a, j_b\}$. Construct the array $\overline{W}$ from the array W by replacing each pair $\{a,b\}$ in W by the array H_{ab}. Also replace the empty cell in W by an empty $m \times m$ array. This array $\overline{W}$ of side $(2m+1)m$ contains the $(2m+2) \cdot 2m = 4m^2 + 4m$ symbols from $T \times S'$. To get our V-square we need exactly one more symbol, which we call Ω.

Let the transversal of H contain the symbols $1,2,...,m$ as the first elements in its cells and the symbols $m+1,...,2m$ as the second elements. In every other cell in the main diagonal of $\overline{W}$ (cells (i,i) where i is odd, $1 \leq i \leq 2m+1$), replace every occurrence of the symbol a_∞, $1 \leq a \leq m$, in the transversal of $H_{\infty b}$ by the symbol Ω. In every other cell in the back diagonal of $\overline{W}$ (cells $(i, 2m+2-i)$ where i is even, $1 \leq i \leq 2m+1$) replace each occurrence of the symbol b_∞, $m+1 \leq b \leq 2m$, in the transversal of $H_{a\infty}$ by the symbol Ω. Call this square V.

We have deleted each of the $2m$ symbols a_∞, $1 \leq a \leq 2m$, exactly

m times. In the $m \times m$ empty array in the centre of V, place a Howell design $H(m,2m)$ on the symbols a_∞, $1 \leq a \leq 2m$. This completes the construction of V. □

Corollary 3.9. *For every $m \geq 4$, $(2m^2+m,2m^2) \in V$.*

Proof. In [2] it is shown that 2 orthogonal Latin squares with a common transversal exist for al orders $n \neq 2,3$, or 6. By superimposing these Latin squares we have a Howell design $H(n,2n)$ with one transversal for all $n \neq 2,3$ or 6. When $n = 6$, the required $H(6,12)$ is given in [7, Figure 1]. By the remark preceding Theorem 3.8, the appropriate diagonal Latin squares also exist. The result then follows from Theorem 3.8. □

Our final construction is a direct construction of $V(2m^2+mn,2m^2)$ with the only constraints being arithmetic in nature.

Theorem 3.10. *Suppose that there exist distinct integers $a_1,\ldots,a_m$ such that*

1) $1 \leq a_i < (2m+n)/2$ and
2) either $2m+n)/g_i$ is even, or $2n+m)/g_i$ is odd and $g_i \leq n$, for $1 \leq i \leq m$, where g_i = g.c.d.$(a_i,2m+n)$.

Then there exists a $V(2m^2+mn,2m^2)$.

Proof. First observe that, in a $V(2m^2+mn,2m^2)$, the number of symbols is $(2m+n)^2$. We will use the symbol set $\mathbf{Z}_{2m+n} \times \mathbf{Z}_{2m+n}$.

We group the $2m^2+mn$ rows and columns into blocks of size $(2m+n)\times(2m+n)$, and note that there are m blocks across each row and down each column. Let (i,j,r,c) denote the $(2m+n)\times(2m+n)$ block formed by placing the pair $\{(x-r,y-c),(i+x-r,j+y-c)\}$ in row x and column y, $1 \leq x$, $y \leq 2m+n$ (all arithmetic is done mod $2m+n$). Note that (s_1,s_2) appears in row $s_1 + r$, column $s_2 + c$, where it is paired with (s_1+i,s_2+j); and also in row $s_1 + r - i$, column $s_2 + c - j$, where it is paired with (s_1-i,s_2-j).

Let A denote an $m \times m$ square, in which each cell contains a 4-tuple (i,j,r,c) such that:

1) There are no repeated pairs in the multiset $\{(i,j)\} \cup \{(-i,-j)\}$;
2) across a row there are no repeated elements in the multiset $\{r\} \cup \{r-i\}$; and
3) down each column there are no repeated elements in the multiset $\{c\} \cup \{c-j\}$.

We will show that A generates a $V(2m^2+mn,2m^2)$ when we replace each cell with the $(2m+n)\times(2m+n)$ square which corresponds to the 4-

tuple in that cell. First note that we do get a square of side $2m^2 + mn$, and that each symbol occurs $2m^2$ times (twice in each $(2m+n) \times (2m+n)$ block). Next note that no pair occurs twice, since each pair (s_1,s_2) occurs with (s_1+i,s_2+j) and (s_1-1,s_2-j). By Condition 1) on the square A, these symbols are all distinct. Similarly, across a row of A (after replacing the cells by the square of side $2m+n$) the symbol (s_1,s_2) appears in rows $s_1 + r$, and $s_1 + r - i$. By Condition 2) on A these are all distinct. The proof that no symbol occurs twice in a column is similar.

We now show how to use the conditions of this theorem to build the square A with the desired properties.

Let $a_1,...,a_m$ be as hypothesized. In row x, column y of A we set $i = a_x$, $j = -a_y$. Thus the i's are constant across the rows of A, and the $-j$'s are constant down the columns. Moreover, since $1 \le a_x < (2m+n)/2$, the multiset $\{(i,j)\} \cup \{(-i,-j)\}$ contains no repeated elements, therefore Condition 1) on A is satisfied.

We next show how to assign the r values across the row x and the c values down column x. Consider the graph on the vertex set $\mathbf{Z}_{2m+n}$ formed by connecting each b with $b + a_x$. This graph consists of g_x cycles, each of length $(2m+n)/g_x$, where g_x = g.c.d.$\{a_x,2m+n\}$. Thus there exists an independent set of edges of size $(2m+n)/2$ (if $(2m+n)/g_x$ is even) or of size $(2m+n-g_x)/2$ (if $(2m+n)/g_x$. By the second hypothesis on a_x, this set has at least m independent edges, say (b_k,b_k+a_x), $k = 1,...,m$.

Across row x we assign to the r-values these sums $b_k + a_x$. Since the edges are independent, the multiset $\{r\} \cup \{r-i\} = \{b_k+a_x\} \cup \{b_k\}$ has no repeated elements. Similarly, down column x we set the c-values equal to the b_k's. The multiset $\{c\} \cup \{c-j\} = \{b_k\} \cup \{b_k-(-a_x)\}$ consists of distinct elements.

Having constructed square A, the proof of the theorem is completed. □

As an example, we construct a $V(10,8)$. Here $m = 2$, $n = 1$ and we pick $a_1 = 1$, $a_2 = 2$. We use independent edges 40 and 12 for row 1, 30 and 24 for row 2, 01 and 34 for column 1, and 02 and 13 for column 2. We obtain the array

$A =$

(1,4,0,0)	(1,3,2,0)
(2,4,0,3)	2,3,4,1)

which gives rise to the $V(10,8)$ given in Figure 3. (The second coordinate is written as a subscript).

$0_0\ 1_4$	$0_1\ 1_0$	$0_2\ 1_1$	$0_3\ 1_2$	$0_4\ 1_3$	$3_0\ 4_3$	$3_1\ 4_4$	$3_2\ 4_0$	$3_3\ 4_1$	$3_4\ 4_2$
$1_0\ 2_4$	$1_1\ 2_0$	$1_2\ 2_1$	$1_3\ 2_2$	$1_4\ 2_3$	$4_0\ 0_3$	$4_1\ 0_4$	$4_2\ 0_0$	$4_3\ 0_1$	$4_4\ 0_2$
$2_0\ 3_4$	$2_1\ 3_0$	$2_2\ 3_1$	$2_3\ 3_2$	$2_4\ 3_3$	$0_0\ 1_3$	$0_1\ 1_4$	$0_2\ 1_0$	$0_3\ 1_1$	$0_4\ 1_2$
$3_0\ 4_4$	$3_1\ 4_0$	$3_2\ 4_1$	$3_3\ 4_2$	$3_4\ 4_3$	$1_0\ 2_3$	$1_1\ 2_4$	$1_2\ 2_0$	$1_3\ 2_1$	$1_4\ 2_2$
$4_0\ 0_4$	$4_1\ 0_0$	$4_2\ 0_1$	$4_3\ 0_2$	$4_4\ 0_3$	$2_0\ 3_3$	$2_1\ 3_4$	$2_2\ 3_0$	$2_3\ 3_1$	$2_4\ 3_2$
$0_2\ 2_1$	$0_3\ 2_2$	$0_4\ 2_3$	$0_0\ 2_4$	$0_1\ 2_0$	$1_4\ 3_2$	$1_0\ 3_3$	$1_1\ 3_4$	$1_2\ 3_0$	$1_3\ 3_1$
$1_2\ 3_1$	$1_3\ 3_2$	$1_4\ 3_3$	$1_0\ 3_4$	$1_1\ 3_0$	$2_4\ 4_2$	$2_0\ 4_3$	$2_1\ 4_4$	$2_2\ 4_0$	$2_3\ 4_1$
$2_2\ 4_1$	$2_3\ 4_2$	$2_4\ 4_3$	$2_0\ 4_4$	$2_1\ 4_0$	$3_4\ 0_2$	$3_0\ 0_3$	$3_1\ 0_4$	$3_2\ 0_0$	$3_3\ 0_1$
$3_2\ 0_1$	$3_3\ 0_2$	$3_4\ 0_3$	$3_0\ 0_4$	$3_1\ 0_0$	$4_4\ 1_2$	$4_0\ 1_3$	$4_1\ 1_4$	$4_2\ 1_0$	$4_3\ 1_1$
$4_2\ 1_1$	$4_3\ 1_2$	$4_4\ 1_3$	$4_0\ 1_4$	$4_1\ 1_0$	$0_4\ 2_2$	$0_0\ 2_3$	$0_1\ 2_4$	$0_2\ 2_0$	$0_3\ 2_1$

Figure 3
A $V(10,8)$

Corollary 3.11. *If $n \geq m$, then there exists a $V(2m^2+mn, 2m^2)$.*

Proof. In Theorem 3.10, we let $a_i = i$ for $1 \leq i \leq m$. Then $1 \leq a_i \leq (2m+n)/2$, and, since $g_i \leq a_i \leq m \leq n$, we get $g_i \leq n$, for each i. Thus the conclusion. □

4. The Spectrum.

By Proposition 2.3, the necessary conditions for the existence of a $V(k,r)$ fall into two cases:

Case 1. If $r = sm^2$ where s is odd and square-free, then $V(k,r) = V(s(m^2+mn), sm^2)$ for some n.

Case 2. If $r = s(2m^2)$ where s is odd and square-free, then $V(k,r) = V(s(2m^2+mn), s(2m^2))$ for some n.

In this section we will show that these necessary conditions are also sufficient.

First, we can easily dispose of Case 1.

Theorem 4.1. *There exists a $V(s(m^2+mn), sm^2)$, for all odd square-free s, and all $m,n \in \mathbf{Z}^+$.*

Proof. By Lemma 3.2, there exists a bipartite $V((m+n) \times m, m)$ and $V(m \times (m+n), m)$ (except for the two cases $m = 6$, $n = 0$ and $m = 2$, $n = 0$, which we will handle separately). Now, by Theorem 3.3 there exists a $V(m^2+mn, m^2)$. Since s is odd, then $s \neq 2$ or 6, so by Corollary 3.4 there exists a $V(s(m^2+mn), sm^2)$. If $m = 6$, $n = 0$, then there exists a $V(36,36)$ by Lemma 3.2, and thus by Corollary 3.4 there is a $V(s \cdot 36, s \cdot 36)$ for all odd s. Similarly, if $m = 2$, $n = 0$, there exists a $V(4,4)$ by Lemma 3.2 and again by Corollary 3.4 there is a $V(s{\cdot}4, s{\cdot}4)$ for all odd s. □

Case 2 is more difficult to settle. However, the constructions from Section 3 will handle this case. We will use Theorem 3.6 as our main tool. Therefore, we must consider the special cases when $m = 2$ or 6.

Lemma 4.2. *If $m = 2$ or 6 and $0 \leq n < m$, then $(2m^2+mn, 2m^2) \in V$.*

Proof. When $m = 2$ we have $V(8,8)$ by Lemma 3.2 and a $V(10,8)$ was given following the proof of Theorem 3.10.

When $m = 6$, Table 1 below shows $(72+6n, 72) \in V$, for $0 \leq n < 6$.

Parameters	Equation	Authority	Remarks
$V(72,72)$		Theorem 3.2	
$V(78,72)$		Cor. 3.9	
$V(84,72)$	$4 \times V(21,18)$	Cor. 3.4	$(21,18) \in V$ by Thm. 3.6
$V(90,72)$	$9 \times V(10,8)$	Cor. 3.4	$V(10,8)$ given above
$V(96,72)$	$4 \times V(24,18)$	Cor. 3.4	$(24,18) \in V$ by Thm. 3.6
$V(102,72)$		Theorem 3.10	$\{a_i\} = \{1,2,3,4,5,6\}$ □

Table 1.

Since we wish to employ Theorem 3.6 to construct $V((2m+n)m, 2m^2)$ we must also consider as special cases the orders $2m+n$ for which there do not exist 4 orthogonal Latin squares. Let $S = \{2,3,4,5,6,10,14,18,20,22,24,26,28,30,33,34,38,42,44,52\}$. Then, if $2m+n \notin S$, there exist 4 orthogonal Latin squares of order $2m+n$ [3].

Lemma 4.3. *If $2m+n \in S$, $0 \leq n < m$, then $((2m+n)m,2m^2) \in V$, except* $(2,2) \notin V$.

Proof. Let $T = \{4,18,20,22,24,26,28,30,34,38,42,44,52\}$. If $2m+n \in T$, then there are 3 orthogonal Latin squares of order $2m+n$ [3]. Note that every $t \in T$ is even. Thus, if $2m+n \in T$, then n is necessarily even. By Theorem 3.7, we conclude that $((2m+n)m,2m^2) \in V$ for all $2m+n \in T$ and $0 \leq n < m$.

If $2m+n \in S \backslash T$, then $2m+n$ = 2,3,6,10,14 or 33. If $2m+n = 2$, then $m = 1$, $n = 0$, and $V(2,2)$ does not exist. If $2m+n = 3$, then there are no cases to consider when $0 \leq n < m$. If $2m+n = 6$, then $m = 3$, $n = 0$, and a $V(18,8)$ exists by Lemma 3.2. The remaining cases are displayed in Table 2.

$2m+n$	$m n$	V-square	Equation	Authority	Comments
10	4 2	$V(40,32)$	$4 \times V(10,8)$	Cor. 3.4	$(10,8) \epsilon V$ by Lemma 4.2
10	5 0	$V(50,50)$		Lemma 3.2	
14	5 4	$V(70,50)$		Thm. 3.10	$\{a_i\}=\{1,2,3,4,5\}$
14	6 2	$V(84,72)$		Lemma 4.2	
14	7 0	$V(98,98)$		Lemma 3.2	
33	12 9	$V(396,288)$	$9 \times V(44,32)$	Cor. 3.4	$(44,32) \epsilon V$ by Thm 3.6
33	13 7	$V(429,338)$		Thm. 3.10	$\{a_i\}=\{1,2,...,10,12,13,14\}$
33	14 5	$V(462,392)$		Thm. 3.10	$\{a_i\}=\{1,2,...,10,12,13,14,15\}$
33	15 3	$V(495,450)$	$9 \times V(55,50)$	Cor. 3.4	$(55,50) \epsilon V$ by Cor. 3.9
33	16 1	$V(528,512)$		Cor.3.9	

□

Table 2.

Our main result is that the necessary conditions given in Proposition 2.3 are also sufficient. We state this as

Theorem 4.4. *The necessary and sufficient conditions for the existence of a $V(k,r)$ are that either*

1. $r = sm^2$ where s is odd and square-free and $V(k,r) = V(s(m^2+mn),sm^2)$ for some $n \geq 0$, or
2. $r = s(2m^2)$ where s is odd and square-free and

$V(k,r) = V(s(2m^2+mn),s(2m^2))$ for some $n \geq 0$, except that no $V(2,2)$ exists.

Proof. These conditions are necessary by Proposition 2.3.

By Theorem 4.1 a $V(s(m^2+mn),sm^2)$ exists for all odd, square-free s, and all $m,n \geq 0$. Thus all the squares in Case 1 exist.

If $n \geq m$, then by Corollary 3.11 there exists a $V(2m^2+mn,2m^2)$. Assume $0 \leq n < m$. If $2m+n \in S$ then there exists a $V(2m^2+mn,2m^2)$ by Lemma 4.3, except $V(2,2)$ does not exist. If $2m+n \notin S$, then there exists a $V(2m^2+mn,2m^2)$ for $m \neq 2$ or 6 by Theorem 3.6. Finally if $m = 2$ or 6, then there exists a $V(2m^2+mn,2m^2)$ by Lemma 4.2. Thus for every $m,n \geq 0$, $(2m^2+mn,2m^2) \in V$, except $(2,2) \notin V$. So by Corollary 3.4 and Lemma 3.2 there exists a $V(s,(2m^2+mn),s(2m^2))$ for all odd square-free s and all $m,n \geq 0$, except $V(2,2)$. □

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