Some Non-Isomorphic Kirkman Triple Systems of Order 39 and 51

D.R. Stinson and S.A. Vanstone

ABSTRACT

Kirkman triple systems of all orders $6t + 3$ exist, but very little is known regarding non-isomorphic systems. For orders 39 and 51, it was unknown if there exist non-isomorphic designs. We enumerate all Kirkman triple systems of a specific type, of orders 39 and 51, and test them for isomorphism. In this way, we find 88 non-isomorphic Kirkman triple systems of order 39, and 9419 non-isomorphic Kirkman triple systems of order 51.

1. Introduction.

Let V be a set of size $v = 6t + 3$. A *block* is a 3-subset of V, and a *resolution class* is a partition $R = \{B_1,...,B_{2t+1}\}$ of V into blocks. A *Kirkman triple system* of *order* $6t + 3$ (denoted KTS($6t+3$)) is a set $\mathbf{R} = \{R_1,...,R_{3t+1}\}$ of $3t + 1$ resolution classes, such that any two points of V are contained in precisely one block of $\bigcup_{i=1}^{3t+1} R_i$. Ray-Chaudhuri and Wilson [4] proved in 1970 that a KTS(v) exists for all positive integers $v \equiv 3$ modulo 6.

Let $\mathbf{R} = \{R_1,...,R_{3t+1}\}$ and $\mathbf{S} = \{S_1,...,S_{3t+1}\}$ be KTS(v). We say that $\mathbf{R}$ and $\mathbf{S}$ are *isomorphic* is there is a permutation ϕ of V such that $\mathbf{R}^\phi = \mathbf{S}$. (We define $\mathbf{R}^\phi = \{R_1^\phi,...,R_{3t+1}^\phi\}$, where $R_i^\phi = \{B^\phi: B \in R_i\}$ $(1 \le i \le 3t+1)$ and $B^\phi = \{x^\phi: x \in B\}$).

Let $N(v)$ denote the number of mutually non-isomorphic KTS(v). Very little is known regarding $N(v)$. The following lower bounds are taken from Mathon and Rosa [2]: $N(3) = N(9) = 1$, $N(15) = 7$, $N(21 \ge 78$, $N(27) \ge 661$, $N(33) \ge 1$, $N(39) \ge 1$, $N(45) \ge 84$, $N(51) \ge 1$, $N(57) \ge 1$, $N(63) \ge 10000$, $N(69) \ge 1$, $N(75) \ge 1$, and $N(81) \ge 10^7$.

In this paper we prove $N(39) \ge 88$ and $N(51) \ge 9419$ by enumerating all KTS(39) and KTS(51) of a particular type.

2. A Construction.

Let G be an additive abelian group of order $2n + 1$. A *strong star-*

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ter (in G) is a set $S = \{\{s_i,t_i\}: 1 \le i \le n\}$ such that: $\{\pm(s_i - t_i)\} = G\backslash\{0\}$, $\{s_i,t_i: 1 \le i \le n\} = G\backslash\{0\}$, $s_i + t_i \ne s_j + t_j$ if $i \ne j$, and $s_i + t_i \ne 0$ $(1 \le i \le n)$. Define $A(S) = \{(s_i+t_i)/2: 1 \le i \le n\}$. Now, suppose $2n + 1 = 6m + 1$. An *extension* of S is a set of m 3-subsets of G, say $T_1,...,T_m$, such that $S \cup (\bigcup_{i=1}^{m} T_i) = G\backslash\{0\}$, and $\{\pm(x-y): x,y \in T_i, 1 \le i \le m\} = G\backslash\{0\}$.

An extension of a strong starter gives rise to a KTS($12m+3$) as follows. Let $V = (G \times \{1,2\}) \cup \{\infty\}$. Define $R_0 = \{\{\infty,(0,1),(0,2)\}\} \cup \{\{(s_i,1),(t_i,1),((s_i+t_i)/2,0)\}: 1 \le i \le n\} \cup \{\{(x,0),(y,0),(z,0)\}: T_i = \{x,y,z\}, 1 \le i \le m\}$. Then define $R_x = R_0 + x$ for all $x \in G$, where addition is defined on V by $\infty + x = \infty$ for all $x \in G$ and $(a,i) + x = (a+x,i)$ $(a,x \in G,\ i = 1,2)$. Then it is easy to check that $\mathbf{R} = \{R_x: x \in G\}$ is a KTS($12m+3$). This KTS has G in its automorphism group. Also, the blocks $T_i + x$ $(1 \le i \le m,\ x \in G)$ constitute a *maximum subdesign* (see [3] for a definition and results concerning the existence of such KTS.)

3. Isomorphism.

Suppose S and S' are strong starters in $\mathbb{Z}_{2n+1}$. We say that S and S' are *equivalent* if there is an integer $c \in \mathbb{Z}_{2n+1}$ such that $S' = \{\{cs,ct\}: \{s,t\} \in s\}$. If S has an extension $T_1,...,T_m$ $(n = 3m)$, then S' has a corresponding extension $T'_1,...,T'_m$ where $T'_i = \{cx: x \in T_i\}$. The two resulting KTS($12m+3$) are easily seen to be isomorphic. Thus in our enumeration, we need only consider one strong starter from each equivalence class.

We first enumerate all inequivalent strong starters in $\mathbb{Z}_{19}$ and $\mathbb{Z}_{25}$, and then determine all possible extensions. We then test the resulting KTS for isomorphism by means of the following invariant.

Let $\mathbf{R} = \{R_1,...,R_{3t+1}\}$ be a KTS($6t+3$), and let $\mathbf{B} = \bigcup_{i=1}^{3t+1} R_i$. For $x,y \in V$, $x \ne y$, define other$(x,y) = z$ if $\{x,y,z\} \in \mathbf{B}$, and define $rc(x,y) = R_i$ if $\{x,y,$ other$(x,y)\} \in R_i$. We define a partial mapping θ from 3-subsets of V to 3-subsets of $\mathbf{R}$. If x,y,z are distinct members of V, let $z_1 =$ other(x,y), $y_1 =$ other(x,z), and $x_1 =$ other(y,z). If $\{x_1,y_1,z_1\} \notin \mathbf{B}$ then define $\theta(\{x,y,z\}) = \{rc(x_1,y_1), rc(x_1,z_1), rc(y_1,z_1)\}$. For $i \ge 0$, let f_i denote the number of 3-subsets of $\mathbf{R}$ which have an inverse image of cardinality exactly i. Finally, define Inv$(\mathbf{R}) = (f_i: 0 \le i \le v)$.

Inv$(\mathbf{R})$ is an *invariant* for KTS: if $\mathbf{R}$ and $\mathbf{R}'$ are isomorphic KTS($6t+3$), then Inv$(\mathbf{R})$ = Inv$(\mathbf{R}')$. As we shall see, this invariant is sufficiently sensitive for our purposes.

4. Computational results.

In [1], all strong starters in $\mathbb{Z}_{2t+1}$ are enumerated, for $t \leq 13$. There are precisely 52 inequivalent strong starters in $\mathbb{Z}_{19}$, and 7934 inequivalent strong starters in $\mathbb{Z}_{25}$.

In $\mathbb{Z}_{19}$, the 52 strong starters give rise to 100 KTS(39). The number of strong starters with i extensions is as follows:

i	0	1	2	3	4	5
# strong starters	12	16	8	4	4	8

Of these 100 KTS(39), some are isomorphic. These arise as follows. There exist strong starters S in $\mathbb{Z}_{19}$ for which $cS = S$ for some c (such a c is called a *multiplier*). If T_1,T_2,T_3 is an extension of S, then so is cT_1,cT_2,cT_3. These two extensions give rise to isomorphic KTS(39).

After eliminating isomorphic KTS(39) which arise in this way, 88 systems remain. These are shown to be non-isomorphic by the invariant described earlier. In Table 1 we list the 52 strong starters in $\mathbb{Z}_{19}$, and in Table 2 we list 88 non-isomorphic extensions.

Table 1

52 Strong Starters in $\mathbb{Z}_{19}$

1.	1	2	3	5	7	10	11	15	8	13	12	18	9	16	17	6	14	4
2.	1	2	3	5	7	10	12	16	8	13	9	15	11	18	17	6	14	4
3.	1	2	3	5	7	10	12	16	13	18	9	15	4	11	6	14	8	17
4.	1	2	3	5	9	12	13	17	6	11	10	16	8	15	18	7	14	4
5.	1	2	3	5	13	16	11	15	4	9	6	12	7	14	10	18	8	17
6.	1	2	3	5	15	18	10	14	8	13	6	12	4	11	9	17	7	16
7.	1	2	4	6	5	8	13	17	11	16	9	15	7	14	10	18	3	12
8.	1	2	4	6	14	17	5	9	11	16	7	13	8	15	10	18	3	12
9.	1	2	5	7	9	12	13	17	11	16	4	10	15	3	6	14	18	8
10.	1	2	5	7	10	13	14	18	3	8	11	17	9	16	4	12	6	15
11.	1	2	5	7	11	14	12	16	8	13	4	10	15	3	17	6	9	18
12.	1	2	6	8	7	10	14	18	12	17	9	15	4	11	16	5	13	3
13.	1	2	6	8	12	15	18	3	9	14	10	16	4	11	5	13	17	7
14.	1	2	6	8	13	16	7	11	4	9	12	18	10	17	14	3	15	5
15.	1	2	7	9	5	8	10	14	11	16	12	18	15	3	17	6	4	13
16.	1	2	7	9	12	15	13	17	6	11	18	5	3	10	8	16	14	4
17.	1	2	7	9	14	17	12	16	3	8	4	10	11	18	5	13	6	15
18.	1	2	8	10	4	7	14	18	11	16	9	15	5	12	17	6	13	3
19.	1	2	8	10	12	15	13	17	4	9	5	11	18	6	14	3	7	16
20.	1	2	8	10	14	17	11	15	4	9	7	13	18	6	16	5	3	12
21.	1	2	11	13	3	6	5	9	12	17	10	16	8	15	18	7	14	4
22.	1	2	11	13	7	10	12	16	4	9	18	5	15	3	6	14	8	17
23.	1	2	11	13	12	15	3	7	5	10	17	4	9	16	6	14	18	8
24.	1	2	11	13	15	18	3	7	4	9	10	16	5	12	6	14	8	17
25.	1	2	12	14	6	9	7	11	13	18	17	4	3	10	8	16	15	5
26.	1	2	12	14	6	9	18	3	11	16	4	10	8	15	5	13	17	7
27.	1	2	12	14	10	13	5	9	3	8	11	17	16	4	18	7	6	15
28.	1	2	14	16	6	9	11	15	8	13	18	5	3	10	4	12	17	7
29.	1	2	15	17	13	16	4	8	5	10	6	12	7	14	3	11	9	18

30.	1	2	16	18	5	8	6	10	12	17	9	15	7	14	3	11	4	13
31.	2	3	5	7	12	15	9	13	6	11	14	1	16	4	10	18	8	17
32.	2	3	5	7	15	18	6	10	8	13	11	17	9	16	12	1	14	4
33.	2	3	6	8	9	12	16	1	13	18	5	11	10	17	7	15	14	4
34.	2	3	7	9	5	8	12	16	10	15	14	1	11	18	17	6	4	13
35.	2	3	7	9	12	15	1	5	8	13	10	16	11	18	17	6	14	4
36.	2	3	8	10	14	17	12	16	4	9	5	11	13	1	18	7	6	15
37.	2	3	9	11	4	7	14	18	12	17	10	16	1	8	5	13	6	15
38.	2	3	10	12	5	8	7	11	13	18	14	1	16	4	9	17	6	15
39.	2	3	12	14	6	9	13	17	11	16	4	10	1	8	18	7	15	5
40.	3	4	5	7	15	18	10	14	8	13	11	17	2	9	12	1	16	6
41.	3	4	8	10	12	15	1	5	13	18	11	17	2	9	6	14	7	16
42.	3	4	8	10	12	15	16	1	13	18	5	11	2	9	6	14	17	7
43.	3	4	9	11	5	8	10	14	12	17	15	2	13	1	18	7	16	6
44.	3	4	10	12	15	18	1	5	8	13	11	17	2	9	6	14	7	16
45.	3	4	10	12	15	18	16	1	8	13	5	11	2	9	6	14	17	7
46.	3	4	11	13	7	10	12	16	15	1	18	5	2	9	6	14	8	17
47.	3	4	14	16	7	10	1	5	8	13	9	15	11	18	17	6	12	2
48.	4	5	8	10	11	14	9	13	17	3	15	2	18	6	12	1	7	16
49.	4	5	12	14	17	1	6	10	3	8	9	15	11	18	13	2	7	16
50.	7	8	3	5	14	17	6	10	16	2	9	15	11	18	12	1	4	13
51.	7	8	3	5	14	17	9	13	16	2	4	10	11	18	12	1	6	15
52.	8	9	1	3	11	14	13	17	2	7	10	16	18	6	4	12	15	5

Table 2
88 non-isomorphic KTS(39)

Strong starter	Extensions		
5	18 6 7	10 12 15	17 2 8
6	5 6 10	14 16 3	8 15 18
7	2 3 8	18 7 9	6 10 13
8	1 8 9	13 15 18	12 16 3
9	14 17 18	3 5 12	16 2 8
9	17 18 2	5 12 14	16 3 8
9	17 18 3	8 14 16	2 5 12
10	9 12 13	17 5 7	18 4 10
10	9 10 13	17 5 7	12 18 4
11	12 13 16	8 15 17	18 5 10
11	13 17 18	8 10 16	5 12 15
11	12 15 16	8 10 17	13 18 5
11	12 13 17	10 16 18	5 8 15
11	16 17 5	10 13 15	8 12 18
12	9 13 14	2 4 10	15 3 6
12	9 10 14	15 2 4	3 6 13
12	9 14 15	13 2 4	3 6 10
13	5 6 10	14 16 3	8 15 18
14	6 13 14	17 1 3	2 8 12
15	1 6 7	14 3 5	10 13 17
15	1 6 7	3 5 13	10 14 17
16	1 6 7	14 3 5	10 13 17

16	1 6 7	3 5 13	10 14 17
17	17 2 3	10 12 18	13 16 4
18	1 6 7	14 3 5	10 13 17
18	1 6 7	3 5 13	10 14 7
19	1 6 7	14 3 5	10 13 17
19	1 6 7	14 3 5	10 13 17
20	15 3 4	2 5 7	8 14 18
21	16 17 1	18 6 8	4 10 15
21	16 17 1	6 8 15	18 4 10
22	7 8 13	15 4 6	17 1 5
25	15 3 4	2 5 7	8 14 18
26	5 6 10	14 16 3	8 15 18
27	5 8 9	16 18 6	12 17 4
27	4 5 8	9 16 18	6 12 17
27	18 4 5	17 6 8	9 12 16
27	17 18 4	6 8 16	5 9 12
27	16 17 5	4 6 9	8 12 18
29	14 15 18	3 10 12	2 8 13
29	14 2 3	10 13 15	8 12 18
30	9 13 14	2 4 10	15 3 6
30	9 10 14	15 2 4	3 6 13
30	9 14 15	13 2 4	3 6 10
33	13 14 17	15 3 5	10 16 2
33	10 15 16	3 5 13	14 17 2
33	16 2 3	5 13 15	10 14 17
33	15 16 2	14 3 5	10 13 17
33	15 16 2	3 5 13	10 14 17
34	6 9 10	4 11 13	15 1 7
34	7 10 11	4 6 13	9 15 1
34	6 7 10	4 11 13	9 15 1
34	6 7 10	11 13 1	4 9 15
34	6 10 11	7 9 15	13 1 4
37	3 7 8	17 4 6	11 18 2
37	2 3 7	17 4 6	8 11 18
38	15 3 4	2 5 7	8 14 18
39	5 6 9	11 18 1	10 2 8
40	17 3 4	8 10 18	2 5 9
40	3 4 9	8 10 18	17 2 5
41	7 11 12	16 18 5	17 1 8
41	11 12 16	18 5 7	17 1 8
41	11 12 17	16 18 7	1 5 8
42	17 1 2	5 7 14	11 16 3
42	1 2 5	7 14 16	11 17 3
42	16 1 2	3 5 11	7 14 17

44	5 8 9	16 18 6	12 17 4
44	4 5 8	9 16 18	6 12 17
44	18 4 5	17 6 8	9 12 16
44	17 18 4	6 8 16	5 9 12
44	16 17 5	4 6 9	8 12 18
45	2 5 6	14 16 4	17 3 9
45	17 3 4	6 14 16	2 5 9
45	17 3 4	14 16 5	2 6 9
46	1 4 5	7 9 16	6 11 17
46	16 17 5	4 6 9	1 7 11
47	9 16 17	6 8 11	4 10 14
48	7 8 13	15 4 6	17 1 5
49	10 11 18	1 4 6	16 3 7
50	11 14 15	1 3 10	2 7 13
50	10 11 14	13 1 3	15 2 7
50	15 1 2	3 11 13	7 10 14
50	14 15 1	11 13 2	3 7 10
50	14 2 3	10 13 15	1 7 11
51	14 15 18	3 10 12	2 8 13
51	14 2 3	10 13 15	8 12 18
52	1 4 5	7 9 16	6 11 17
52	16 17 5	4 6 9	1 7 11

From the 7934 inequivalent strong starters in $\mathbb{Z}_{25}$, we obtain 9419 KTS(51). The number of strong starters with i extensions is:

i	0	1	2	3	4	5	6	7	8
# strong starters	3200	2293	1124	730	350	154	73	0	10

The 9419 KTS(51) are shown to be non-isomorphic by the invariant of Section 3.

We want to make one observation regarding extendibility of strong starters in $\mathbb{Z}_{6m+1}$. In $\mathbb{Z}_7$ and $\mathbb{Z}_{13}$, all strong starters have extensions. In $\mathbb{Z}_{19}$, 77% of the 52 inequivalent strong starters have extensions, and in $\mathbb{Z}_{25}$ the precentage is 60%. We conjecture that the probability $p(m)$ that a random strong starter in $\mathbb{Z}_{6m+1}$ has an extension approaches 0 as $m \to \infty$.

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D.R. Stinson
Department of Computer Science
University of Manitoba
Winnipeg, Manitoba
R3T 2N2 Canada

S.A. Vanstone
Department of Combinatorics and Optimization
St. Jerome's College
University of Waterloo
Waterloo, Ontario
N2L 3G1 Canada