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A Note on Non-Isomorphic Kirkman Triple Systems

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We prove that the number of non-isomorphic Kirkman triple systems of order v ($v \equiv 3 \bmod 6$) is at least $\dfrac{8^{(v-1)(v-3)/48}}{v!}$.

1. INTRODUCTION

A *Steiner triple system* of order v is a pair $(X, \mathcal{B})$, where $|X| = v$, and $\mathcal{B}$ is a set of $v(v-1)/6$ 3-element subsets of X such that any distinct pair of elements (*points*) of x are contained in a unique member (*block*) of $\mathcal{B}$. A Steiner triple system of order v exists for all $v \equiv 1$ or $3 \bmod 6$. A *Kirkman triple system* of order v is a pair $(X, \mathcal{P})$ which satisfies the conditions (0) $|X| = v$; (1) $\mathcal{P} = \{P_1, \ldots, P_{(v-1)/2}\}$, where each P_i is a set of $v/3$ blocks that partitions X; (2) $\left(X, \bigcup_{i=1}^{(v-1)/2} P_i\right)$ is a Steiner triple system of order v. A Kirkman triple system of order v exists if and only if $v \equiv 3 \bmod 6$ [5]. We abbreviate the phrase Steiner (Kirkman) triple system of order v to STS(v)(KTS(v)). The STS(v) of condition (2) is called the *underlying* STS(v) of the given KTS(v). An STS(v) is said to be *resolvable* if it is the underlying STS(v) of some KTS(v).

Let $(X, \mathcal{B}_i)$ be STS(v), $i = 1, 2$. These are said to be *isomorphic* if there is a permutation ϕ of X such that, for every $B \in \mathcal{B}_1$, $B^\phi \in \mathcal{B}_2$, where $B^\phi = (x^\phi : x \in B\}$. We denote by $N(v)$ the number of pair-wise non-isomorphic STS(v) (on a specified point set X). In [6], Wilson proved that $(e^{-5}v)^{v^2/6} \leqslant N(v) \leqslant (e^{-1/2}v)^{v^2/6}$.

Isomorphism of KTS is defined as follows. Two KTS(v), say $(X, \mathcal{P}_i)$, $i = 1, 2$, are said to be *isomorphic* if there is a permutation ϕ of X such that $\mathcal{P}_1^\phi = \mathcal{P}_2$ (where $\mathcal{P}^\phi = \{\{B^\phi : B \in P\} : P \in \mathcal{P}\}$). Note that non-isomorphic KTS(v) can have underlying STS(v) which are isomorphic. This happens, for example, at order 15. There exist precisely 80 non-isomorphic STS(15). Four of these are resolvable, but three of these four give rise to two non-isomorphic KTS(15). The number of non-isomorphic KTS(15) is therefore 7. (These results can be found in Mathon, Phelps, and Rosa [3]). We denote by NK(v) the number of non-isomorphic KTS(v), and by NR(v) the number of non-isomorphic resolvable STS(v). Hence NK(15) = 7 and NR(15) = 4. Of course NK(v) $\geqslant$ NR(v) for all v.

In this paper we prove that, for any $v \equiv 3 \bmod 6$, NR(v) $\geqslant \dfrac{8^{(v-1)(v-3)/48}}{v!}$. Hence NR($v$) $\to \infty$ as $v \to \infty$. We observe that it was previously unknown if NK(v) > 1 for all but finitely many v. The best previous result is found in Lenz [2]; he proved that, for all $v \equiv 9 \bmod 18$, NR(v) $\geqslant \dfrac{6^{v(v-3)/54}}{v!}$.

2. A Recursive Construction

We use a well-known recursive construction for KTS. (See, for example, Wilson [7].) We briefly review this construction.

A PBD (pairwise balanced design) is a pair $(X, \mathcal{B})$, where $\mathcal{B}$ is a set of subsets of X (blocks), each of size at least 2, such that every pair of points in X is contained in a unique block of $\mathcal{B}$. For each block $B \in \mathcal{B}$, suppose that there is a KTS($2|B| + 1$) (this requires that $|B| \equiv 1 \bmod 3$). We construct a KTS($2|B| + 1$), $((B \times \{1, 2\}) \cup \{\infty\}, \mathcal{P}_B)$, where $\mathcal{P}_B = \{P_{B,x} : x \in B\}$ and $\{\infty, x_1, x_2\} \in P_{B,x}$ for all $x \in B$. This is done for each block B. We then observe that $((X \times \{1, 2\}) \cup \{\infty\}, \mathcal{P})$ is a KTS($2|X| + 1$), where we define $\mathcal{P} = \{P_x : x \in X\}$ and $P_x = \bigcup_{x \in B} P_{B,x}$, for each $x \in X$.

3. A Bound

This recursive construction can be used to construct large numbers of *distinct* KTS(v) (on a specified symbol set), by using different component KTS. For example, suppose we have a block $B \in \mathcal{B}$ of size 4, say $B = \{a, b, c, d\}$. It is not difficult to see that there are precisely 8 distinct resolvable STS(9) on point-set $\{\infty, a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2\}$ in which the underlying STS(9) contains the blocks $\{\infty, a_1, a_2\}$, $\{\infty, b_1, b_2\}$, $\{\infty, c_1, c_2\}$, and $\{\infty, d_1, d_2\}$. Suppose we take $(X, \mathcal{B})$ to be a $(w, 4, 1)$-BIBD (which exists for any $w \equiv 1$ or $4 \bmod 12$). $\mathcal{B}$ consists of $w(w-1)/12$ blocks of size 4. The recursive construction produces $8^{w(w-1)/12}$ *distinct* resolvable STS(v) ($v = 2w + 1$). No more than $v!$ of these can be mutually isomorphic, so NR(v) $\geqslant \dfrac{8^{w(w-1)/12}}{v!} = \dfrac{8^{(v-1)(v-3)/48}}{v!}$. This works for all $v \equiv 3$ or $9 \bmod 24$.

For $v \equiv 15$ or $21 \bmod 24$, we use a slight variation. Start with a PBD $(X, \mathscr{B})$ on $w \equiv 7$ or $10 \bmod 12$ points, containing one block of size 7 and $(w(w-1)-42)/12$ blocks of size 4. Such a PBD is shown to exist for all $w \equiv 7$ or $10 \bmod 12$, except $w = 10$ and 19, by Brouwer [1]. Blocks of size 4 are handled as before. For the block of size 7, we need distinct resolvable STS(15). For example, suppose we begin with STS # 61 in the list of 80 (see [3]). This STS(15) is resolvable, and has an automorphism group of order 21. There are $2^7 \cdot 7!$ permutations of $\{\infty, a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, e_1, e_2, f_1, f_2, g_1, g_2\}$ which fix ∞, and also fix the set of blocks $\{\infty, a_1, a_2\}, \ldots, \{\infty, g_1, g_2\}$. We get at least $2^7 \cdot 7!/21 > 8^{7/2}$ distinct resolvable STS(15) which contain blocks $\{\infty, a_1, a_2\}, \ldots, \{\infty, g_1, g_2\}$.

So, when we fill these resolvable STS into one of Brouwer's PBDs, we get $8^{(w(w-1)-42)/12} \cdot 8^{7/2} = 8^{w(w-1)/12}$ distinct resolvable STS$(2w+1)$. Hence, as before, we obtain $\mathrm{NR}(v) \geqslant \dfrac{8^{(v-1)(v-3)/48}}{v!}$ for $v \equiv 15$ or $21 \bmod 24$, $v \neq 15$, 39. The exceptions $v \neq 15, 39$ can in fact be removed since this bound is less than 1, and resolvable STS(15) and STS(39) exist. Hence, we have our main

THEOREM. *For all $v \equiv 3 \bmod 6$, there are at least* $\dfrac{8^{(v-1)(v-3)/48}}{v!}$ *non-isomorphic resolvable Steiner triple systems of order v.*

4. Comments

Lower bounds on $\mathrm{NK}(v)$ for small v can be found in [4]. These are generally obtained by ad hoc techniques. The recursive techniques we use do not yield new bounds until v is about 75. At that point, an explosion occurs, since $v!$ is roughly $e^{v \log v}$, and the numerator is about e^{cv^2}. Hence $\mathrm{NK}(v) \geqslant c_1 e^{c_2 v^2}$ for constants c_1, c_2, and sufficiently large v.

The true value of $\mathrm{NR}(v)$ is probably $c_1 e^{c_2 v^2 \log v}$, but it does not seem possible to prove this using known constructions.

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