On Scheduling Perfect Competitions

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The following problem was posed by M.S. Brandly in [2; p. 759]. Let \( v \geq 4 \) be an integer. A perfect competition is the set of all possible games, where in each game two of \( v \) players play against two other players. Clearly, there are \( \frac{1}{2} \binom{v}{2} \binom{v-2}{2} = 3 \binom{v}{4} \) games. It is desired to schedule these games into rounds, so that any player plays in at most one game in each round. Denote by \( R(v) \) the minimum number of rounds necessary to schedule a perfect competition, subject to the above constraint. The problem posed by Brandly is to determine \( R(v) \).

In this paper we accomplish this: for any integer \( v \geq 4 \), \( R(v) = 3 \binom{v}{4} \left\lfloor \frac{v}{4} \right\rfloor \). (Note that this quantity is always an integer.) First, we observe that at most \( \left\lfloor \frac{v}{4} \right\rfloor \) games can be played in a given round, so clearly \( R(v) \geq 3 \binom{v}{4} \left\lfloor \frac{v}{4} \right\rfloor \). The remainder of this paper describes the construction of schedules with the desired number of rounds.

Our proof follows easily from a result of Baranyai. It is necessary first to give some terminology. Let \( X \) be a finite set, and denote \( v = |X| \). If \( 1 \leq k \leq v \), then \( \binom{X}{k} \) denotes the set of all \( k \)-subsets (called edges) of \( X \). If \( k_1, \ldots, k_m \) is a list of integers (not necessarily distinct), with \( 1 \leq k_i \leq v \) for \( 1 \leq i \leq m \), then \( K(v; k_1, \ldots, k_m) \) (based on \( X \)) denotes the multiset union of the \( \binom{X}{k_i} \), \( 1 \leq i \leq m \). Thus, edges will be repeated if \( k_i = k_j \) for some \( i \neq j \).

A scheme based on \( K(v; k_1, \ldots, k_m) \) is an \( m \times n \) matrix \( A = (a_{ij}) \) of non-negative integers, which satisfies

\[
\sum_{j=1}^{n} a_{ij} = \binom{v}{k_i}, \quad 1 \leq i \leq m
\]  

(i)

and

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\[ \sum_{i=1}^{m} k_i a_{ij} = v, \ 1 \leq j \leq n. \]  

(ii)

A resolution of \( K(v; k_1, \ldots, k_m) \) according to \( A \) is a partition \( P = \{P_1, \ldots, P_n\} \) of \( K(v; k_1, \ldots, k_m) \), where each \( P_j = \bigcup_{i=1}^{m} X_{ij} \), which satisfies

\[ |X_{ij}| = a_{ij}, \ 1 \leq i \leq m, \ 1 \leq j \leq n \]  

(i)

\[ \bigcup_{j=1}^{n} X_{ij} = \binom{X}{k_i}, \ 1 \leq i \leq m \]  

(ii)

\[ \bigcup_{i=1}^{m} \bigcup_{s \in X_{ij}} s = X, \ 1 \leq j \leq n. \]  

(iii)

We remark that each \( X_{ij} \) consists of disjoint \( k_i \)-subsets of \( \binom{X}{k_i} \).

Baranyai proved the following Theorem in [1].

**Theorem 1.** If \( A \) is any scheme based on \( K(v; k_1, \ldots, k_m) \), then there exists a resolution of \( K(v; k_1, \ldots, k_m) \) according to \( A \).

Baranyai's theorem is related to scheduling perfect competitions as follows.

**Lemma 2.** Let \( v \geq 4 \) be a positive integer. Suppose \( k_1 = k_2 = k_3 = 4 \), and \( 1 \leq k_i \leq 3 \) for \( 4 \leq i \leq m \). Suppose \( \bigcup_{j=1}^{n} P_j \) is a resolution of \( K(v; k_1, \ldots, k_m) \) according to some scheme \( A \). Then a perfect competition for \( v \) players can be scheduled, using \( n \) rounds.

**Proof.** For each \( P_j \), delete any edges of size less than 4. Then replace an edge \( abcd \) (where \( a < b < c < d \)) by:

\[
\begin{align*}
ab \text{ vs. } cd, & \text{ if } abcd \in \binom{X}{k_1} \\
ac \text{ vs. } bd, & \text{ if } abcd \in \binom{X}{k_2} \\
ad \text{ vs. } bc, & \text{ if } abcd \in \binom{X}{k_3}.
\end{align*}
\]

Thus it is necessary only to construct suitable schemes.

For \( v \equiv 0 \mod 4 \), this is easy. We take \( m = 3 \), and our scheme is
\[
A = \begin{pmatrix}
\frac{v}{4} & \cdots & \frac{v}{4} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & \frac{v}{4} & \cdots & \frac{v}{4} & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & \frac{v}{4} & \cdots & \frac{v}{4}
\end{pmatrix}
\]

Here \( n = 3(\frac{v-1}{3}) = 3(\frac{v}{4})^{\frac{v}{4}} \).

The case \( v = 3 \mod 4 \) is almost as simple. Here we set \( m = 6 \), \( k_1 = k_2 = k_3 = 3 \), and let
\[
A = \begin{pmatrix}
\frac{v-3}{4} & \cdots & \frac{v-3}{4} & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & \frac{v-3}{4} & \cdots & \frac{v-3}{4} & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & \frac{v-3}{4} & \cdots & \frac{v-3}{4}
\end{pmatrix}
\]

Here, \( n = 3(v) = 3(\frac{v}{4})^{\frac{v-3}{4}} \).

For \( v = 1 \mod 4 \), we desire \( n = \frac{v(v-2)(v-3)}{2} = 3(\frac{v}{4})^{\frac{v-1}{4}} \). We set \( m = 3 + \frac{(v-2)(v-3)}{2} \), and \( k_i = 1 \) for \( 4 \leq i \leq m \). We have two subcases: \( v = 5 \) or \( 9 \mod 12 \); and \( v = 1 \mod 12 \).

If \( v = 5 \) or \( 9 \mod 12 \), then \( m, n = 0 \mod 3 \). For \( 1 \leq i_1 \leq 3 \), \( 4 \leq i_2 \leq m \), and \( i_1 = i_2 \mod 3 \), construct \( v \) columns of \( A \), with \( a_{i_1j} = \frac{v-1}{4} \), \( a_{i_2j} = 1 \), and \( a_{ij} = 0 \) if \( i \neq i_1, i_2 \).

If \( v = 1 \mod 12 \), then \( m, n = 1 \mod 3 \). First, we describe the first three rows of \( A \). Construct \( \frac{n-1}{3} \) columns each of the form
\[
\begin{pmatrix}
(v-1)/4 & 0 & 0 \\
0 & (v-1)/4 & 0 \\
0 & 0 & (v-1)/4
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}; \quad \text{and one column} \quad \begin{pmatrix}
(v-1)/12 \\
(v-1)/12 \\
(v-1)/12
\end{pmatrix}.
\]

We now add \( m - 3 \) rows, where each row contains \( v \) 1's and \( n - v \) 0's, and each column contains one 1. This forms the desired matrix \( A \).

Lastly, we consider \( v = 2 \) mod 4. Here, we desire \( n = \frac{v(v-1)(v-3)}{2} = 3 \left( \frac{v-2}{4} \right) \). We set \( m = 3 + (v-1)(v-3) \), and \( k_i = 1 \) for \( 4 \leq i \leq m \). We have two subcases: \( v = 6 \) or 10 mod 12; and \( v = 2 \) mod 12.

If \( v = 6 \) or 10 mod 12, then \( m, n = 0 \) mod 3. First, construct the first three rows of \( A \): \( \frac{n}{3} \) columns each of the form

\[
\begin{pmatrix}
(v-2)/4 \\
0 \\
0
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}.
\]

Call this matrix \( A_1 \). Now, let \( A_2 \) be any \( m - 3 \) by \( n \) \( 0 - 1 \) matrix, with all column sums 2 and all row sums \( v \). Then \( A = \left( \frac{A_1}{A_2} \right) \).

Finally, we consider \( v = 2 \) mod 12. Note that \( m, n = 2 \) mod 3. Construct \( A_1 \), the first 3 rows of \( A \): \( \frac{n-2}{3} \) columns each of the form

\[
\begin{pmatrix}
(v-2)/4 \\
0 \\
0
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}; \quad \text{and two columns} \quad \begin{pmatrix}
(v-2)/12 \\
(v-2)/12 \\
(v-2)/12
\end{pmatrix}.
\]

Let \( A_2 \) be any \( m - 3 \) by \( n \) \( 0 - 1 \) matrix, with all row sums \( v \) and all column sums 2. Then \( A = \left( \frac{A_1}{A_2} \right) \).

As a consequence of Theorem 1, Lemma 2, and the matrices \( A \) we have constructed, we have

**Theorem 3.** For any \( v \geq 4 \), the minimum number of rounds required to schedule a perfect competition with \( v \) players is \( 3 \left( \frac{v}{4} \right) \left\lceil \frac{v}{4} \right\rceil \).
References.


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