On Scheduling Perfect Competitions

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The following problem was posed by M.S. Brandly in [2; p. 759]. Let $v \geq 4$ be an integer. A *perfect competition* is the set of all possible games, where in each game two of v players play against two other players. Clearly, there are $\frac{1}{2}\binom{v}{2}\binom{v-2}{2} = 3\binom{v}{4}$ games. It is desired to schedule these games into rounds, so that any player plays in at most one game in each round. Denote by $R(v)$ the minimum number of rounds necessary to schedule a perfect competition, subject to the above constraint. The problem posed by Brandly is to determine $R(v)$.

In this paper we accomplish this: for any integer $v \geq 4$, $R(v) = 3\binom{v}{4}/\lfloor\frac{v}{4}\rfloor$. (Note that this quantity is always an integer.) First, we observe that at most $\lfloor\frac{v}{4}\rfloor$ games can be played in a given round, so clearly $R(v) \geq 3\binom{v}{4}/\lfloor\frac{v}{4}\rfloor$. The remainder of this paper describes the construction of schedules with the desired number of rounds.

Our proof follows easily from a result of Baranyai. It is necessary first to give some terminology. Let X be a finite set, and denote $v = |X|$. If $1 \leq k \leq v$, then $\binom{X}{k}$ denotes the set of all k-subsets (called *edges*) of X. If $k_1,\ldots,k_m$ is a list of integers (not necessarily distinct), with $1 \leq k_i \leq v$ for $1 \leq i \leq m$, then $K(v;k_1,\ldots,k_m)$ (based on X) denotes the multiset union of the $\binom{X}{k_i}$, $(1 \leq i \leq m)$. Thus, edges will be repeated if $k_i = k_j$ for some $i \neq j$.

A *scheme* based on $K(v;k_1,\ldots,k_m)$ is an $m \times n$ matrix $A = (a_{ij})$ of non-negative integers, which satisfies

$$\sum_{j=1}^{n} a_{ij} = \binom{v}{k_i}, \quad 1 \leq i \leq m \qquad \text{(i)}$$

and

$$\sum_{i=1}^{m} k_i a_{ij} = v, \quad 1 \le j \le n. \qquad \text{(ii)}$$

A *resolution* of $K(v;k_1,\ldots,k_m)$ *according to* A is a partition $\mathbf{P} = \{P_1,\ldots,P_n\}$ of $K(v;k_1,\ldots,k_m)$, where each $P_j = \bigcup_{i=1}^{m} X_{ij}$, which satisfies

$$|X_{ij}| = a_{ij}, \quad 1 \le i \le m, \quad 1 \le j \le n \qquad \text{(i)}$$

$$\bigcup_{j=1}^{n} X_{ij} = \binom{X}{k_i}, \quad 1 \le i \le m \qquad \text{(ii)}$$

$$\bigcup_{i=1}^{m} \bigcup_{s \in X_{ij}} s = X, \quad 1 \le j \le n. \qquad \text{(iii)}$$

We remark that each X_{ij} consists of disjoint k_i-subsets of $\binom{X}{k_i}$.

Baranyai proved the following Theorem in [1].

Theorem 1. *If A is any scheme based on $K(v;k_1,\ldots,k_m)$, then there exists a resolution of $K(v;k_1,\ldots,k_m)$ according to A.*

Baranyai's theorem is related to scheduling perfect competitions as follows.

Lemma 2. *Let $v \ge 4$ be a positive integer. Suppose $k_1 = k_2 = k_3 = 4$, and $1 \le k_i \le 3$ for $4 \le i \le m$. Suppose $\bigcup_{j=1}^{n} P_j$ is a resolution of $K(v;k_1,\ldots,k_m)$ according to some scheme A. Then a perfect competition for v players can be scheduled, using n rounds.*

Proof. For each P_j, delete any edges of size less than 4. Then replace an edge $abcd$ (where $a < b < c < d$) by:

$$\begin{cases} ab \text{ vs. } cd, \text{ if } abcd \in \binom{X}{k_1} \\ ac \text{ vs. } bd, \text{ if } abcd \in \binom{X}{k_2} \\ ad \text{ vs. } bc, \text{ if } abcd \in \binom{X}{k_3}. \end{cases} \quad \square$$

Thus it is necessary only to construct suitable schemes.

For $v \equiv 0 \bmod 4$, this is easy. We take $m = 3$, and our scheme is

$$A = \begin{pmatrix} \frac{v}{4} \cdots \frac{v}{4} & 0 \cdots 0 & 0 \cdots 0 \\ 0 \cdots 0 & \frac{v}{4} \cdots \frac{v}{4} & 0 \cdots 0 \\ 0 \cdots 0 & 0 \cdots 0 & \frac{v}{4} \cdots \frac{v}{4} \\ \underbrace{\qquad\qquad}_{\binom{v-1}{3}} & \underbrace{\qquad\qquad}_{\binom{v-1}{3}} & \underbrace{\qquad\qquad}_{\binom{v-1}{3}} \end{pmatrix}$$

Here $n = 3\binom{v-1}{3} = 3\binom{v}{4}/\frac{v}{4}$.

The case $v \equiv 3 \bmod 4$ is almost as simple. Here we set $m = 6$, $k_4 = k_5 = k_6 = 3$, and let

$$A = \begin{pmatrix} \frac{v-3}{4} \cdots \frac{v-3}{4} & 0 \cdots 0 & 0 \cdots 0 \\ 0 \cdots 0 & \frac{v-3}{4} \cdots \frac{v-3}{4} & 0 \cdots 0 \\ 0 \cdots 0 & 0 \cdots 0 & \frac{v-3}{4} \cdots \frac{v-3}{4} \\ 1 \cdots 1 & 0 \cdots 0 & 0 \cdots 0 \\ 0 \cdots 0 & 1 \cdots 1 & 0 \cdots 0 \\ 0 \cdots 0 & 0 \cdots 0 & 1 \cdots 1 \\ \underbrace{\qquad\qquad}_{\binom{v}{3}} & \underbrace{\qquad\qquad}_{\binom{v}{3}} & \underbrace{\qquad\qquad}_{\binom{v}{3}} \end{pmatrix}$$

Here, $n = 3\binom{v}{3} = 3\binom{v}{4}/\frac{v-3}{4}$.

For $v \equiv 1 \bmod 4$, we desire $n = \frac{v(v-2)(v-3)}{2} = 3\binom{v}{4}/\frac{v-1}{4}$. We set $m = 3 + \frac{(v-2)(v-3)}{2}$, and $k_i = 1$ for $4 \leq i \leq m$. We have two subcases: $v \equiv 5$ or $9 \bmod 12$; and $v \equiv 1 \bmod 12$.

If $v \equiv 5$ or $9 \bmod 12$, then $m,n = 0 \bmod 3$. For $1 \leq i_1 \leq 3$, $4 \leq i_2 \leq m$, and $i_1 \equiv i_2 \bmod 3$, construct v columns of A, with $a_{i_1 j} = \frac{v-1}{4}$, $a_{i_2 j} = 1$, and $a_{ij} = 0$ if $i \neq i_1, i_2$.

If $v \equiv 1 \bmod 12$, then $m,n \equiv 1 \bmod 3$. First, we describe the first three rows of A. Construct $\frac{n-1}{3}$ columns each of the form

$$\begin{pmatrix}(v-1)/4\\0\\0\end{pmatrix},\quad \begin{pmatrix}0\\(v-1)/4\\0\end{pmatrix}\quad\text{and}\quad \begin{pmatrix}0\\0\\(v-1)/4\end{pmatrix};\quad\text{and one column}\quad \begin{pmatrix}(v-1)/12\\(v-1)/12\\(v-1)/12\end{pmatrix}.$$

We now add $m-3$ rows, where each row contains v 1's and $n-v$ 0's, and each column contains one 1. This forms the desired matrix A.

Lastly, we consider $v \equiv 2 \mod 4$. Here, we desire $n = \frac{v(v-1)(v-3)}{2} = 3\binom{v}{4}/\frac{v-2}{4}$. We set $m = 3 + (v-1)(v-3)$, and $k_i = 1$ for $4 \leq i \leq m$. We have two subcases: $v \equiv 6$ or $10 \mod 12$; and $v \equiv 2 \mod 12$.

If $v \equiv 6$ or $10 \mod 12$, then $m,n \equiv 0 \mod 3$. First, construct the first three rows of A: $\frac{n}{3}$ columns each of the form

$$\begin{pmatrix}(v-2)/4\\0\\0\end{pmatrix},\quad \begin{pmatrix}0\\(v-2)/4\\0\end{pmatrix},\quad\text{and}\quad \begin{pmatrix}0\\0\\(v-2)/4\end{pmatrix}.$$

Call this matrix A_1. Now, let A_2 be any $m-3$ by n $0-1$ matrix, with all column sums 2 and all row sums v. Then $A = \left(\frac{A_1}{A_2}\right)$.

Finally, we consider $v \equiv 2 \mod 12$. Note that $m,n \equiv 2 \mod 3$. Construct A_1, the first 3 rows of A: $\frac{n-2}{3}$ columns each of the form

$$\begin{pmatrix}(v-2)/4\\0\\0\end{pmatrix},\quad \begin{pmatrix}0\\(v-2)/4\\0\end{pmatrix},\quad\text{and}\quad \begin{pmatrix}0\\0\\(v-2)/4\end{pmatrix};\quad\text{and two columns}\quad \begin{pmatrix}(v-2)/12\\(v-2)/12\\(v-2)/12\end{pmatrix}.$$

Let A_2 be any $m-3$ by n $0-1$ matrix, with all row sums v and all column sums 2. Then $A = \left(\frac{A_1}{A_2}\right)$.

As a consequence of Theorem 1, Lemma 2, and the matrices A we have constructed, we have

Theorem 3. *For any $v \geq 4$, the minimum number of rounds required to schedule a perfect competition with v players is* $3\binom{v}{4}/\lfloor\frac{v}{4}\rfloor$.

References.

[1] Z. Baranyai, ***On the factorization of complete uniform hypergraphs***, Proc. Colloq. on Infinite and Finite Sets (ed. Hajnal et al) (1973), 91-108.

[2] M.S. Brandly, ***Query 293***, Notices of Amer. Math. Soc., Vol. 30, No. 7 (1983), 759.

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