ON λ-PACKINGS WITH BLOCK SIZE FOUR ($v \not\equiv 0 \bmod 3$)

E. J. Billington, R. G. Stanton, and D. R. Stinson

ABSTRACT. We determine the packing number $D_\lambda(2,4,v)$, where $v \not\equiv 0 \pmod 3$, for all λ.

1. *Introduction.*

Let $v > k$. A (v,k,λ)-*packing* is a collection of k-subsets *(blocks)* chosen from a set of v *points* so that no pair of points occurs in more than λ blocks. A (v,k,λ)-packing which is not also a $(v,k,\lambda-1)$-packing is said to have *index* λ.

The number $D_\lambda(2,k,v)$ is used to denote the maximum number of blocks in any (v,k,λ)-packing (the "2" indicates that pairs do not occur more than λ times). $D_\lambda(2,k,v)$ has been determined, for all v, for $k = 3$ and all λ ([3]), and for $k = 4$ and $\lambda = 1$ ([1]). In this paper we investigate the case $k = 4$ for $\lambda > 1$. We obtain a complete solution for $v \not\equiv 0 \bmod 3$. (The case $v \equiv 0 \bmod 3$ is more difficult and will be presented in a later paper.)

For a point x in a (v,k,λ)-packing, let r_x denote the number of blocks in which x occurs. Clearly $r_x \le \left\lfloor \frac{\lambda(v-1)}{k-1} \right\rfloor$. Hence the number of blocks $b \le \left\lfloor \frac{v}{k} r_x \right\rfloor \le \left\lfloor \frac{v}{k} \left\lfloor \frac{\lambda(v-1)}{k-1} \right\rfloor \right\rfloor$. In the case $k = 4$, we obtain

LEMMA 1.1. $D_\lambda(2,4,v) \le \left\lfloor \frac{v}{4} \left\lfloor \frac{\lambda(v-1)}{3} \right\rfloor \right\rfloor$.

Henceforth, we write $B_\lambda(2,4,v) = \left\lfloor \frac{v}{4} \left\lfloor \frac{\lambda(v-1)}{3} \right\rfloor \right\rfloor$. The values of the function B naturally partition into 12 classes (mod 12) for v, and 6 classes (mod 6) for λ.

It is often convenient to consider what is "left over" in a packing. The *deficiency* of a packing is the quantity $\sum_{x \neq y} (\lambda - \lambda_{xy})$, where

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λ_{xy} denotes the number of blocks containing the pair xy. Clearly, the deficiency of a packing with b blocks is $\lambda\binom{v}{2} - 6b$. The *defect graph* is the graph in which any two vertices x and y are joined by $\lambda-\lambda_{xy}$ edges. The following properties of defect graphs can be easily verified.

LEMMA 1.2. *1) The number of edges in the defect graph of a packing is equal to the deficiency of the packing.*

2) No edge has multiplicity greater than λ.

3) The degree of any vertex is congruent (mod 3) to $\lambda(v-1)$.

In most situations we can construct packings with $B_\lambda(2,4,v)$ blocks. Such a packing is said to be *good*. In Table 1, we tabulate the deficiencies of good packings, where $v = 12a + r$ $(0 \le r \le 11)$, and $\lambda = 6c + u$ $(0 \le u \le 5)$. Clearly, if we construct a $(v,4,\lambda)$-packing whose deficiency is the relevant entry of Table 1, then $D_\lambda(2,4,v) = B_\lambda(2,4,v)$ for that λ and v. We record the deficiencies of good packings in Table 1.

Table 1: Deficiencies of good packings ($v = 12a+r$, $\lambda = 6c+u$)

r \ u	0	1	2	3	4	5
0	0	12a	6a	0	12a	6a
1	0	0	0	0	0	0
2	0	6a+1	12a+2	3	6a+4	12a+5
3	0	12a+3	6a+6	3	12a+6	6a+3
4	0	0	0	0	0	0
5	0	6a+4	12a+8	0	6a+4	12a+8
6	0	12a+9	6a+6	3	12a+6	6a+3
7	0	3	0	3	0	3
8	0	6a+4	12a+8	0	6a+4	12a+8
9	0	12a+12	6a+6	0	12a+12	6a+6
10	0	3	0	3	0	3
11	0	6a+7	12a+14	3	6a+10	12a+11

A (v,k,λ)-packing with deficiency 0 is called a *balanced incomplete block design* (a (v,k,λ)-BIBD). For $k = 4$, the existence question for BIBDs is solved (cf. [2]). The result is

THEOREM 1.3. *A* $(v,4,\lambda)$-BIBD *exists if and only if one of the following conditions holds*

1) $v \equiv 1$ *or* $4 \pmod{12}$

2) $v \equiv 1 \pmod 3$ *and* $\lambda \equiv 0 \pmod 2$

3) $v \equiv 0$ *or* $1 \pmod 4$ *and* $\lambda \equiv 0 \pmod 3$

4) $\lambda \equiv 0 \pmod 6$.

Notice that the above conditions correspond precisely to the zero entries of Table 1.

In later sections, we prove

THEOREM 1.4. *If* $\lambda > 1$ *and* $v \not\equiv 0 \bmod 3$, *then* $D_\lambda(2,4,v) = B_\lambda(2,4,v)$.

2. *The Case* $\lambda = 1$.

Brouwer completed the determination of $D_1(2,4,v)$ in [1]; the result is

THEOREM 2.1. *1)* $D_1(2,4,v) = B_1(2,4,v)$ *if* $v \not\equiv 7$ *or* $10 \pmod{12}$
2) $D_1(2,4,v) = B_1(2,4,v) - 1$ *if* $v \equiv 7$ *or* $10 \pmod{12}$ *with the following small exceptions:* $D_1 = B_1 - 1$ *for* $v = 9, 17$; $D_1 = B_1 - 2$ *for* $v = 8, 10, 11$; $D_1 = B_1 - 3$ *for* $v = 19$.

3. *The Case* $\lambda = 2$.

For $v \equiv 1,4,7,$ or $10 \pmod{12}$, BIBDs exist; so $D_2(2,4,v) = B_2(2,4,v)$ in these cases.

For $v \equiv 2,5,8,$ or $11 \pmod{12}$, two copies of a good packing of index 1 will form a good packing of index 2. This handles all cases except $v = 8,11,$ and 17, for which good packings of index one do not exist.

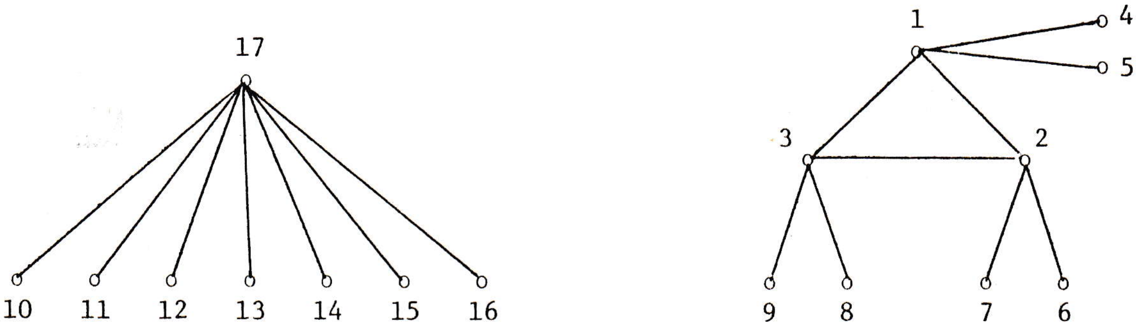
For $v = 8$, we have $D_2(2,4,8) = B_2(2,4,8) = 8$, taking the blocks 1 2 3 5, 1 2 4 6, 1 3 4 7, 1 5 6 7, 2 3 4 8, 2 5 6 8, 3 6 7 8, 4 5 7 8.

For $v = 11$, we have $D_2(2,4,11) = B_2(2,4,11) = 16$, taking the blocks 1 2 3 4, 1 2 5 6, 1 3 5 10, 1 4 6 11, 1 8 9 10, 1 7 8 11, 2 3 6 10, 2 4 5 11, 2 7 8 10, 2 7 9 11, 3 4 7 9, 3 5 9 11, 3 6 8 11, 4 5 7 10, 4 6 9 10, 5 6 8 9.

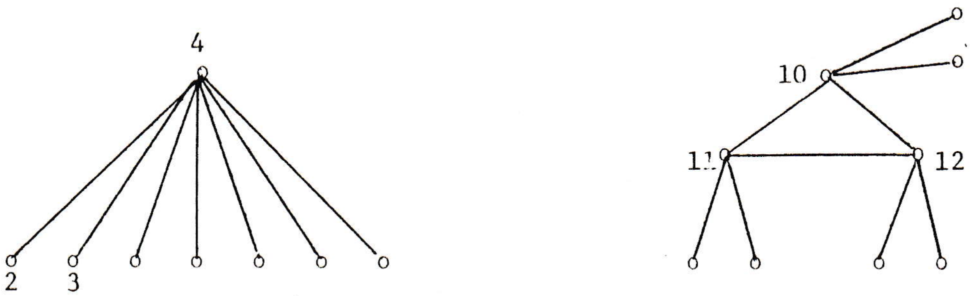
For $v = 17$ we have $D_1(2,4,17) = B_1(2,4,17) - 1 = 20$, and $B_1(2,4,17) = 42$. We construct a good packing of index 2 by taking two packings of index 1 with 20 blocks each, and adjoining two new blocks. We start with an affine plane of order 4 on points $1,\ldots,16$. Pick 3 non-collinear points, say 1,2,3. We may suppose that we have three blocks

$$\begin{aligned} B_1 &: 1\ 2\ 4\ 5 \\ B_2 &: 2\ 3\ 6\ 7 \\ B_3 &: 3\ 1\ 8\ 9. \end{aligned}$$

In B_i $(i = 1,2,3)$, replace i by a new point 17. The defect graph of the resultant packing is



Now construct an isomorphic packing in which the points have been relabelled so that the defect graph is



(The unlabelled points are irrelevant.) We can now adjoin two new blocks 1 2 3 4 and 10 11 12 17, to get a good packing of index 2.

Summarizing, we have

THEOREM 3.1. *If* $v \not\equiv 0 \pmod 3$, *then* $D_2(2,4,v) = B_2(2,4,v)$.

4. The Case $\lambda = 3$.

Here, BIBDs exist for $v \equiv 1,4,5$, and 8 (mod 12); so we need only consider $v \equiv 2,7,10$, and 11 (mod 12).

In [1], Brouwer constructs PBDs (pairwise balanced designs) in which there is a unique block of size 7 and all other blocks have size 4 for $v \equiv 7$ or 10 (mod 12), $v \neq 10,19$. Take three copies of such a PBD, thus producing three blocks 1 2 3 4 5 6 7. Replace these three blocks by the ten blocks: 1 2 3 6, 1 2 4 6, 1 2 5 7, 1 3 4 7, 1 3 5 6, 1 4 5 7, 2 3 4 7, 2 3 5 7, 2 4 5 6, and 3 4 5 6. The resulting packing has deficiency 3 (the pair 67 occurs zero times). Hence the packing is good (Table 1), and $D_3(2,4,v) = B_3(2,4,v)$ for these v.

We must handle $v = 10,19$, as special cases. For $v = 10$, we obtain $D_3(2,4,10) = B_3(2,4,10) = 22$. We use points a_i, ∞_i ($a \in \mathbb{Z}_4$, $i = 1,2$) and blocks

$$\left.\begin{array}{l} \infty_1\ 0_1\ 2_1\ 0_2 \\ \infty_1\ 0_2\ 2_2\ 1_1 \\ \infty_2\ 0_1\ 1_1\ 2_2 \\ \infty_2\ 0_2\ 1_2\ 2_1 \\ 0_1\ 1_1\ 0_2\ 1_2 \end{array}\right\} \bmod 4$$

and

$$0_1 1_1 2_1 3_1 \qquad 0_2 1_2 2_2 3_2 .$$

For $v = 19$, we construct a good packing (85 blocks) on points from $\mathbb{Z}_{17} \cup \{\infty_1, \infty_2\}$:

$$\left.\begin{array}{llll} 0 & 1 & 4 & 10 \\ 0 & 2 & 4 & 7 \\ 0 & 2 & 3 & 8 \\ \infty_1 & 0 & 1 & 7 \\ \infty_2 & 0 & 4 & 9 \end{array}\right\} \bmod 17 .$$

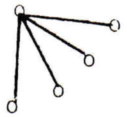
We note that both these packings have deficiency three, the pair $(\infty_1\ \infty_2)$ occurring zero times.

For v $\equiv$ 2 (mod 12), we combine three good packings of index 1 (which exist by Theorem 2.1) and adjoin some additional blocks. A good packing of index 1 has a defect graph which consists of 6a + 1 disjoint edges, where v = 12a + 2. We take three packings, with defect graphs:

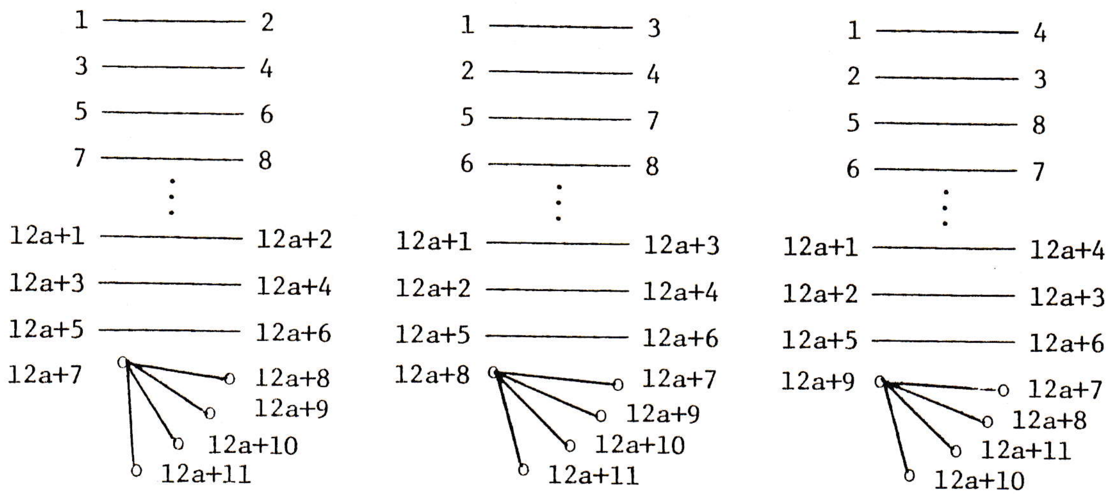
12,34,56,78, ..., 12a-3 12a-2, 12a-1 12a , 12a+1 12a+2;
13,24,57,68, ..., 12a-3 12a-1, 12a-2 12a , 12a+1 12a+2;
and 14,23,58,67, ..., 12a-3 12a , 12a-1 12a-2, 12a+1 12a+2.

We can now add 3a blocks : 1 2 3 4, 5 6 7 8, ..., 12a-3 12a-2 12a-1 12a. This forms a packing of index 3 with deficiency 3 (the pair 12a+1 12a+2 does not occur at all), which is good, by Table 1.

The case v $\equiv$ 11 (mod 12) is similar. A good packing of index 1 exists if v $\neq$ 11, and has a defect graph consisting of 6a + 3 disjoint pairs and a 4-star :



We label points in three such packings, so that the three defect graphs are:



We can now adjoin blocks 1 2 3 4, 5 6 7 8, ..., 12a+1 12a+2 12a+3 12a+4, 12a+7 12a+8 12a+9 12a+10, and 12a+7 12a+8 12a+9 12a+11. The resulting packing of index 3 is good, with 12a+5 12a+6 occurring zero times.

We have to handle v = 11 as a special case. We construct 27 blocks using points from $\mathbb{Z}_9 \cup \{\infty_1, \infty_2\}$:

$$\left.\begin{array}{llll} \infty_1 & 0 & 1 & 3 \\ \infty_2 & 0 & 1 & 4 \\ 0 & 1 & 3 & 5 \end{array}\right\} \bmod 9.$$

This packing is good, having deficiency 3 (the pair $\infty_1\ \infty_2$ occurs zero times).

Thus we have

THEOREM 4.1. $D_3(2,4,v) = B_3(2,4,v)$, *for* $v \not\equiv 0 \pmod 3$.

5. *The Case* $\lambda = 4$.

For $v \equiv 1,4,7,$ or $10 \pmod{12}$, BIBDs exist. For $v \equiv 2,5,8,11 \pmod{12}$, we can combine good packings of indices 1 and 3 to obtain a good packing of index 4 (see Table 1). We have to handle $v = 8,11,$ and 17 as special cases, since good packings of index 1 do not exist for these orders.

For $v = 8$, a good packing has 18 blocks and deficiency 4. We construct such a packing on the points of $\mathbb{Z}_4 \times \{1,2\}$:

$$\left.\begin{array}{llll} 0_1 & 1_1 & 1_2 & 2_2 \\ 0_1 & 1_1 & 3_2 & 0_2 \\ 0_1 & 1_1 & 0_2 & 2_2 \\ 0_1 & 2_1 & 0_2 & 1_2 \end{array}\right\} \bmod 4, \text{ and } \quad \begin{array}{llll} 0_1 & 1_1 & 2_1 & 3_1 \\ 0_2 & 1_2 & 2_2 & 3_2 \end{array}$$

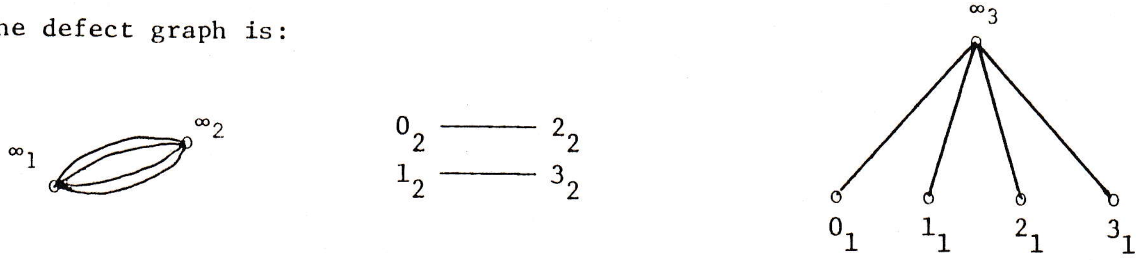
The graph is

$$\begin{array}{lcl} 0_1 & \text{———} & 2_1 \\ 1_1 & \text{———} & 3_1 \\ 0_2 & \text{———} & 2_2 \\ 1_2 & \text{———} & 3_2\ . \end{array}$$

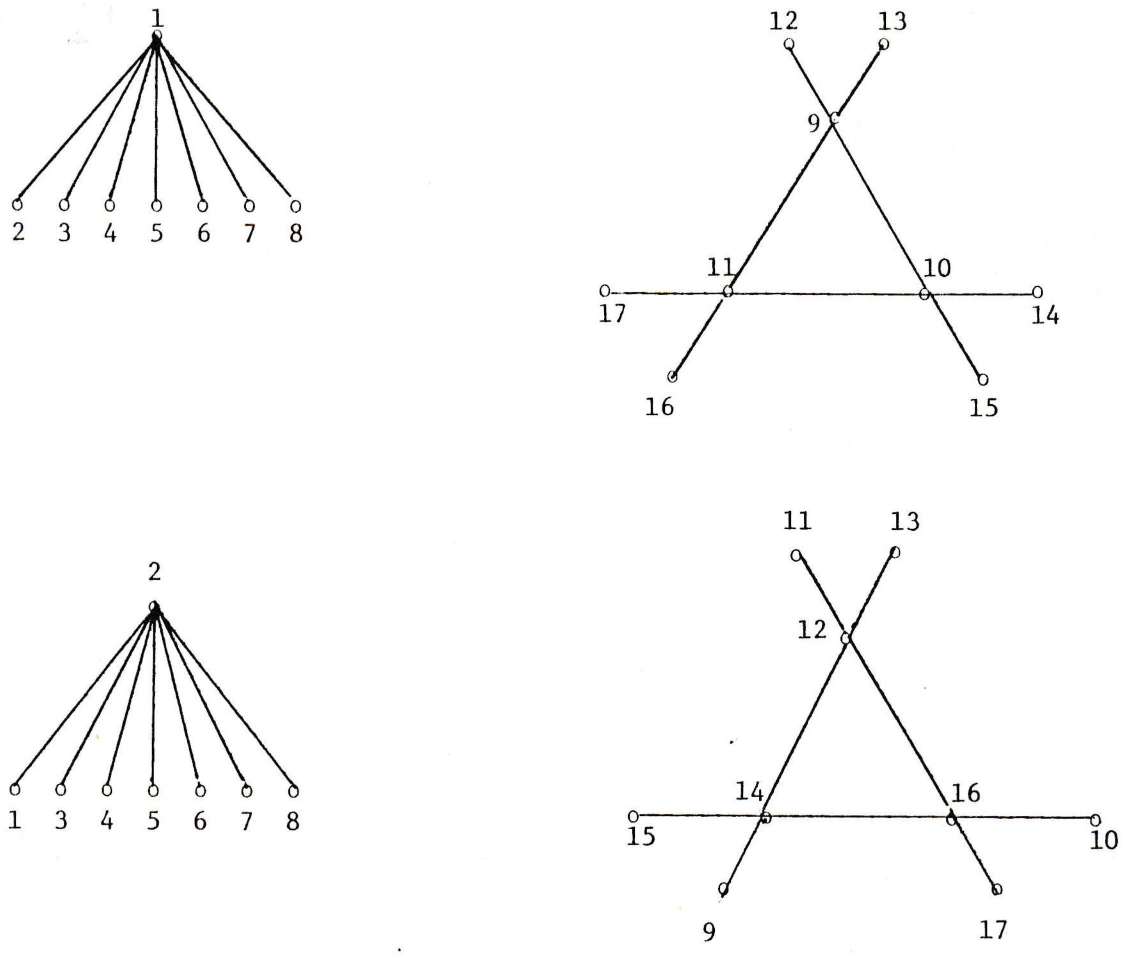
For $v = 11$, a good packing has 35 blocks and deficiency 10. Use points from a_i, ∞_i ($a \in \mathbb{Z}_4$, $i = 1,2,3$), and blocks

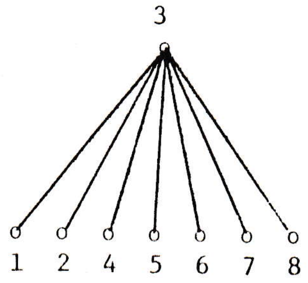
$$\left.\begin{array}{llll} \infty_1 & \infty_3 & 0_1 & 2_2 \\ \infty_2 & \infty_3 & 0_1 & 3_2 \\ \infty_1 & 0_1 & 1_1 & 2_2 \\ \infty_1 & 0_2 & 1_2 & 2_1 \\ \infty_2 & 0_1 & 2_1 & 0_2 \\ \infty_2 & 0_2 & 2_2 & 1_1 \\ \infty_3 & 0_1 & 0_2 & 1_2 \\ 0_1 & 1_1 & 0_2 & 1_2 \end{array}\right\} \bmod 4, \text{ and } \quad \begin{array}{llll} 0_1 & 1_1 & 2_1 & 3_1 \\ 0_1 & 1_1 & 2_1 & 3_1 \\ 0_2 & 1_2 & 2_2 & 3_2\ . \end{array}$$

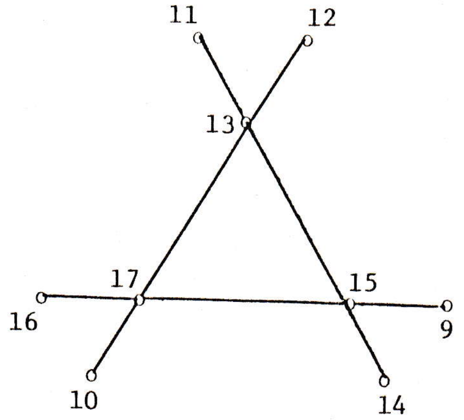
The defect graph is:

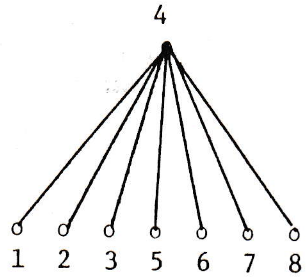


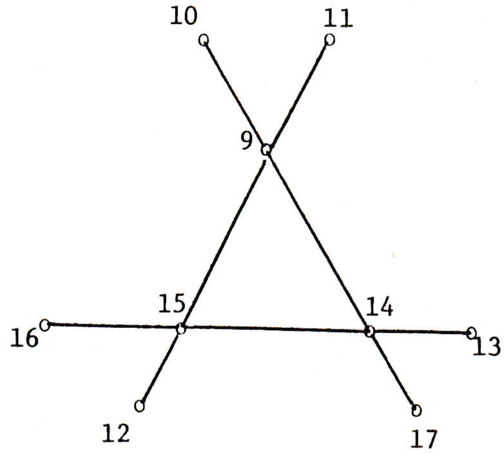
For v = 17, we require 89 blocks (the deficiency is 10). We proceed as for v = 17, λ = 2. We take four copies of the packing of index 1, with defect graphs as follows:



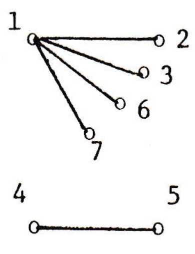








We can add 9 more blocks, for a total of 89, as desired: 1 2 3 5, 1 3 4 6, 1 2 4 7, 2 3 4 8, 12 14 15 16, 13 14 15 17, 9 10 14 15, 10 11 16 17, and 9 11 12 13. The defect graph of the resulting packing of index 4 is



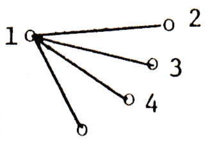
Thus we have

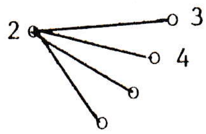
THEOREM 5.1. *If* $v \not\equiv 0 \pmod 3$, *then* $D_4(2,4,v) = B_4(2,4,v)$.

6. *The Case* $\lambda = 5$.

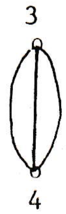
From Table 1, we see that the union of good packings of indices 2 and 3 yields a good packing of index 5, except for $v \equiv 11 \pmod{12}$.

For $v \equiv 11 \pmod{12}$, we combine three good packings of indices 1, 1, and 3, and then add one additional block to obtain a good packing of index 5. A good packing of index 1 has a defect graph consisting of a set of disjoint edges and a 4-star. The defect graph of a good packing of index 3 is a triple bond. We may label points so that these defect graphs contain the edges:





, and



This enables us to add 1 2 3 4 as a new block.

The above construction works for all $v \equiv 11 \pmod{12}$, except $v = 11$, where a good packing of index 1 does not exist. A good packing of index 5 has 44 blocks and deficiency 11, and can be constructed by taking

$$\left.\begin{matrix} 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 3 & 6 \\ 0 & 2 & 4 & 7 \end{matrix}\right\} \bmod 11.$$

The defect graph is an 11-cycle. Thus we have

THEOREM 6.1. *For* $v \not\equiv 0 \pmod 3$, $D_5(2,4,v) = B_5(2,4,v)$.

7. *The Case* $\lambda \geq 6$.

There exists a $(v,4,6)$-BIBD for all v, by Theorem 1.3. Hence, we can construct a good packing of any index $\lambda_0 + 6t$ by taking a good packing of index λ_0 and combining it with t copies of a $(v,4,6)$-BIBD.

Using the results of the previous sections, we will be finished, provided we can construct good packings of index 7 for $v \equiv 7,10 \pmod{12}$ and for $v = 8,9,11,17$. Combining good packings of indices 3 and 4 yields a good packing of index 7 for all cases except $v = 11$.

Thus we need only to construct a good packing of index 7 for $v = 11$. Such a packing has 63 blocks and defect graph

We proceed as follows. The following set of 28 blocks contains the pair (∞_1,∞_2) 6 times, and all other pairs 3 times:

∞_1 ∞_2 1 2	∞_1 2 4 6	∞_2 2 4 6
∞_1 ∞_2 3 4	∞_1 7 8 9	∞_2 7 8 9
∞_1 ∞_2 5 6	∞_1 1 4 9	∞_2 1 6 8
∞_1 ∞_2 1 7	∞_1 3 6 7	∞_2 3 2 9
∞_1 ∞_2 3 8	∞_1 5 2 8	∞_2 5 4 7
∞_1 ∞_2 5 9		

1 3 2 7	3 5 2 8
1 3 4 8	3 5 4 9
1 3 6 9	3 5 6 7
1 5 2 9	4 6 8 9
1 5 4 7	6 2 9 7
1 5 6 8	2 4 7 8

Now take a copy of the good packing of index four, and label points so that $(\infty_1\ \infty_2)$ is the edge of multiplicity four in the defect graph. These 63 blocks form a good packing of index 7.

8. *Summary.*

In the previous sections, we have shown that good packings exist whenever $\lambda > 1$ and $v \not\equiv 0 \pmod 3$, that is,

$$D_\lambda(2,4,v) = \left\lfloor \frac{v}{4}\left\lfloor \frac{\lambda(v-1)}{3} \right\rfloor \right\rfloor = B_\lambda(2,4,v) \text{ in these cases.}$$

The case $v \equiv 0 \pmod 3$ requires recursive constructions and special constructions for a number of small cases. We suspect that good packings exist, with the single exception that, for $v = 9$, $\lambda = 2$, $D_2(2,4,9) = B_2(2,4,9) - 1 = 10$.

Packings for $v \equiv 0 \pmod 3$ will be discussed in a later paper.

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Department of Mathematics
University of Queensland
St. Lucia, Australia 4067

Department of Computer Science
University of Manitoba
Winnipeg, Canada R3T 2N2