Some Improved Results Concerning the Cordes Problem

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1. Introduction.

Let X be a set of size $2m$. A *round* is a partition $R = \{B, \bar{B}\}$ of X into two subsets *(blocks)* of size m. If $R_1 = \{B_1, \bar{B}_1\}$ and $R_2 = \{B_2, \bar{B}_2\}$ are two rounds, then the *common pairs* between rounds are those pairs of elements that are in B_1 or $\bar{B}_1$, and also are in B_2 or $\bar{B}_2$. The number of common pairs between rounds is $2\binom{x}{2} + 2\binom{m-x}{2}$, where $x = |B_1 \cap B_2|$. This quantity is minimized when $x = \lfloor \frac{m}{2} \rfloor$ where the $\lfloor\ \rfloor$ signifies the greatest integer not exceeding the given expression. Thus we have

Lemma 1.1: *The minimum number of common pairs between two rounds is*

$$\begin{cases} \dfrac{(m-1)^2}{2} & \text{if } m \text{ is odd} \\ \dfrac{m(m-2)}{2} & \text{if } m \text{ is even} \end{cases}$$

We denote the above quantity by $\sigma(m)$.

Cordes [1] was the first to consider the problem of finding the maximum number of rounds, any two of which contain $\sigma(m)$ common pairs. Let this maximum be denoted by $R(m)$. Cordes proved

Theorem 1.2: *If m is even, then $R(m) \leq 2m-1$, with equality occurring if and only if there exists a Hadamard matrix of order $2m$.*

Hadamard matrices are conjectured to exist for all orders divisible by four. If this is true, then $R(m)$ would be determined for all even m. For odd m, much less is known about $R(m)$. The following results are the best bounds previously known.

Theorem 1.3: *[2] For $k \geq 2$, $R(2k+1) \leq 4k+4$.*

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Theorem 1.4: *[2] If there exists a Hadamard matrix of order* $4k$, *then* $R(4k+1) \geq 8k-1$.

Theorem 1.5: *[2] If* $4k+1$ *is a prime power, then* $R(2k+1) \geq 4k+1$.

Theorem 1.6: *[3] If* $4k-1$ *is a prime power, then* $R(4k-1) \geq 8k-1$.

Theorem 1.7: *[1]* $R(3) = 10$.

[2] $R(5) = 12$, $R(9) \geq 19$ *and* $R(13) \geq 26$.

In Section 2 we study results on the structure of a $C(2k+1,4k+4)$. We extend these results in Section 3 to show the uniqueness of $C(5,12)$. We also show that $C(7,16)$ and $C(9,20)$ do not exist. Finally in Section 4, we improve Theorem 1.6 to show that $R(4k-1) \geq 8k-1$ whenever there exists a Hadamard matrix of order $4k$.

2. Configurations with $4k+4$ rounds.

Consider a set X of $2m$ elements with rounds as defined previously. Let $C(m,r)$ denote a set of r rounds, any two of which contain $\sigma(m)$ common pairs. We will study the structure of a $C(2k+1,4k+4)$.

Suppose there are x_i pairs which occur i times in the blocks of C, a $C(2k+1,4k+4)$. Counting pairs, we have

$$\sum x_i = \binom{4k+2}{2} = 8k^2+6k+1.$$

The total number of airs, counted with respect to multiplicity is

$$\sum ix_i = 2\binom{2k+1}{2}(4k+4)$$
$$= 8(2k^3+3k^2+k).$$

Also, any two rounds contain $\sigma(2k+1) = 2k^2$ common pairs, so

$$\sum \binom{i}{2}x_i = 2k^2\binom{4k+4}{2}$$
$$= 4(4k^4+7k^3+3k^2).$$

Using the previous three equations we calculate

$$0 \leq \sum x_i(i-(2k+1))^2 = 4k^2+2k+1. \qquad (*)$$

Thus we have

Lemma 2.1: $x_{2k+1} \geq 4k^2+4k$.

Proof. Any pair which does not occur precisely $2k+1$ times yields a positive contribution of at least one in the sum in (*). Hence, at most $4k^2+2k+1$ such pairs occur. Thus $x_{2k+1} \geq \binom{4k+2}{2} - (4k^2+2k+1) = 4k^2+4k$. □

We now construct a graph **G** on vertex set X (the symbol set of **C**) The edges of **G** will be precisely those pairs that occur an odd number of times in **C**. We shall show that **G** is a complete bipartite graph.

For a subset $Y \subset X$, let $N(Y)$ denote the number of blocks $B' \in$ **C** with $Y \subset B'$. For disjoint subsets Y, $Z \subset X$, let $N(Y,Z)$ denote the number of rounds $\{B, \bar{B}\}$ with $Y \subset B$ and $Z \subset \bar{B}$, or $Y \subset \bar{B}$ and $Z \subset B$. Then for any distinct elements a, b, $c \in X$, we have

$$N(abc) + N(ab,c) = N(ab).$$

Similarly,

$$N(abc) + N(bc,a) = N(bc),$$

and

$$N(abc) + N(ac,b) = N(ac).$$

Also

$$N(abc) + N(ab,c) + N(ac,b) + N(bc,a) = 4k+4.$$

Thus

$$N(abc) = \frac{N(ab)+N(ac)+N(bc)-(4k+4)}{2} \qquad (^{**})$$

Since $N(abc)$ is an integer, we have

Lemma 2.2: *For any distinct elements* a, b, c *of* X, $N(ab) + N(ac) + N(bc)$ *is even.*

Thus, in **G**, given any three distinct vertices a, b, c, either zero or two of ab, ac, bc are edges of **G**. Such a graph can easily be shown to be a complete bipartite graph (see [4, Lemma 2.5]).

The number of edges in **G** is an integer e of the form $z(4k+2-z)$, where we may assume $0 \leq z \leq 2k+1$. But $e \geq 4k^2+4k$ from Lemma 2.1. Thus we have $z = 2k$ or $2k+1$ and $4k^2+4k \leq x_{2k+1} \leq e \leq 4k^2+4k+1$.

Let us first assume that $x_{2k+1} = 4k^2+4k$ and $e = 4k^2+4k+1$. Then some $x_{2j+1} = 1$ for $j \neq k$ and hence $(2j+1-(2k+1))^2 x_{2j+1} \geq 4$. Also, $\sum x_{2i} = 4k^2+4k$ which implies $\sum (2i-(2k+1))^2 x_{2i} \geq 4k^2+2k$. Thus $\sum (i-(2k+1))^2 x_i \geq 4k^2+2k+4$ which is a contradiction.

Hence we have $e = x_{2k+1}$. If some $x_{2j} \geq 1$ for $j \neq k, k+1$, then $(2j-(2k+1))^2 x_{2j} \geq 9$ and $\sum (i-(2k+1))^2 x_i \geq 9+4k^2+2k-1$, which is also a contradiction. Thus $x_i = 0$ if $i \notin \{2k, 2k+1, 2k+2\}$. Using this, the three equations preceding Lemma 2.1 have a unique solution, which we record as

Lemma 2.3: *In a $C(2k+1,4k+4)$ there are $k+1$ pairs that occur $2k$ times, $4k^2+4k$ pairs that occur $2k+1$ times and $4k+4$ pairs that occur $2k+2$ times.*

Now we can suppose that our complete bipartite graph G has bipartition $(Y,\bar{Y})$ where $Y = \{1,2,\ldots,2k\}$ and $\bar{Y} = \{2k+1,2k+2,\ldots,4k+2\}$. Any element occurs in $4k+4$ blocks with $2k$ other elements. Hence, for any i, we have $\sum_{i \neq j} N(ij) = 2k(4k+4)$. For $i \in \bar{Y}$ and $j \in Y$, $N(ij) = 2k+1$. Hence $\sum_{j \in Y} N(ij) = 2k(2k+1)$, for $i \in \bar{Y}$. If $i \in \bar{Y}$ then

$$\sum_{\substack{i \neq j \\ j \in \bar{Y}}} N(ij) = \sum_{i \neq j} N(ij) - \sum_{j \in Y} N(ij).$$

Hence

$$\sum_{\substack{i \neq j \\ j \in \bar{Y}}} N(ij) = 4k^2+6k.$$

Now $N(ij) = 2k$ or $2k+2$ if $i,j \in \bar{Y}$. Since $|\bar{Y}| = 2k+2$, $N(ij) = 2k+2$ for $2k$ j's and $N(ij) = 2k$ for one j where both $i, j \in \bar{Y}$.

Thus there is a unique $j = j(i) \in \bar{Y}$ with $N(ij) = 2k$ and $N(ij) = 2k+2$ if $j \in \bar{Y} \setminus \{i,j\}$. Hence the $k+1$ pairs that occur $2k$ times form a 1-factor (perfect matching) of $\bar{Y}$. Suppose, without loss of generality, that these pairs are precisely those in $F = \{\{2i+1,2i+2\}: k \leq i \leq 2k\}$. Thus we have

Lemma 2.4: *If $1 \leq i < j \leq 4k+2$, then*

$$N(ij) = \begin{cases} 2k, \text{ if } \{i,j\} \in F \\ 2k+1, \text{ if } i \in Y, j \in \bar{Y} \\ 2k+2, \text{ otherwise.} \end{cases}$$

Now, let (X,R) be any collection of $4k+4$ rounds that has the above pair-distribution (each round consists of two blocks of size $2k+1$). We can develop necessary and sufficient conditions for (X,R) to be a $C(2k+1,4k+4)$.

Let B_0 be a block in an (X,R) and define $C = C(B_0) = \sum(N(ab)-1)$ where the sum is over all pairs in B_0. Summing over all blocks other than B_0 or $\bar{B}_0$ we have

$$\sum 1 = 8k+6,$$

$$\sum |B \cap B_0| = (4k+3)(2k+1)$$

and

$$\sum \binom{|B \cap B_0|}{2} = C.$$

Thus $\sum |B \cap B_0|^2 = 2C+8k^2+10k+3$, and

$$\sum[(|B \cap B_0|-k)(|B \cap B_0|-(k+1))] = 2C-(8k^3+6k^2).$$

Since the blocks must intersect in either k or $k+1$ elements in a $C(2k+1,4k+4)$, we have that an (X,R) is a $C(2k+1,4k+4)$ if and only if the above sum is zero for all choices of B_0. This can be restated as

Lemma 2.5: *(X,R) is a $C(2k+1,4k+4)$ if and only if $\sum_{\{a,b\} \subset B} (N(ab)-1) = 4k^3+3k^2$ for every block B.*

Pick a block B. Then, for $2k \le i \le 2k+2$, let y_i denote the number of pairs $\{a,b\} \subset B$ with $N(ab) = i$. Then

$$y_{2k}+y_{2k+1}+y_{2k+2} = 2k^2+k,$$

and $$(2k-1)y_{2k}+2ky_{2k+1}+(2k+1)y_{2k+2} = C = 4k^3+3k^2.$$

Hence $y_{2k+2} = y_{2k}+k^2$ and $y_{2k+1} = k^2+k-2y_{2k}$. Let $z = |B \cap Y|$. Then $y_{2k+1} = z(2k+1-z)$, so $y_{2k} = \frac{(k-z)(k-z+1)}{2}$ is a triangular number. If we let $\theta = k-z$, then $y_{2k} = \frac{\theta^2+\theta}{2}$, $y_{2k+1} = k^2+k-(\theta^2+\theta)$ and $y_{2k+2} = k^2 + \frac{\theta^2+\theta}{2}$. We may assume that $z \le k$, by switching the roles of B and $\bar{B}$, if necessary. Thus $\theta \ge 0$. We must have $2y_{2k} \le |B \cap \bar{Y}| = 2k+1-z$ which reduces to $\theta^2 \le k+1$; hence $0 \le \lfloor\sqrt{k+1}\rfloor$.

A round $\{B,\bar{B}\}$, where $y_{2k}(B) = \frac{\theta^2+\theta}{2}$ and where $|B \cap Y| \le k$, will be called a *θ-round*, and r_θ will denote the number of θ-rounds. Clearly, $\sum r_\theta = 4k+4$. Now it is easy to show, if $y_{2k}(B) = \frac{\theta^2+\theta}{2}$, that $y_{2k}(\bar{B}) = \frac{\theta^2-\theta}{2}$. Hence we obtain

$$\sum_{\theta}\left(\frac{\theta^2+\theta}{2}+\frac{\theta^2-\theta}{2}\right)r_\theta = 2kx_{2k}$$

which simplifies to

$$\sum_{\theta}\theta^2 r_\theta = 2k^2+2k.$$

We have been unable to prove any more general results concerning the structure of a $C(2k+1,4k+4)$. However, for specific values of k, this structural information can often be applied to show existence or non-existence of configurations.

3. Specific results.

The first case which we consider is the existence of a $C(5,12)$. Such a configuration was found by computer in [2]. Thus $R(5) = 12$. We can show that this configuration is unique, up to isomorphism.

Theorem 3.1: *A $C(5,12)$ exists and is unique up to isomorphism.*

Proof. Consider a $C(5,12)$. By Lemma 2.5, it will have θ-rounds as described in the previous section. That is, $r_0+r_1 = 12$ and $r_1 = 12$. So $r_0 = 0$. Thus all rounds $\{B,\bar{B}\}$ have $y_4(B) = 1$, $y_5(B) = 4$ and $y_6(B) = 5$. $Y = \{1,2,3,4\}$ and $\bar{Y} = \{5,6,\ldots,10\}$ with $|B_i \cap Y| = 1$ and $|B_i \cap \bar{Y}| = 4$ for $1 \le i \le 12$. The pairs 5 6, 7 8, and 9 10 each occur four times in the 12 B's and since $y_4(B) = 1$, exactly one of these pairs occurs in each B. Thus let B_1,B_2,B_3 and B_4 contain the pair 5 6. Let B_5,B_6,B_7 and B_8 contain the pair 7 8, and let B_9,B_{10},B_{11} and B_{12} contain 9 10.

Consider B_1,B_2,B_3 and B_4. No two of these blocks contain the same pair $\{i,j\} \subset \{7,8,9,10\}$, for then the intersection between blocks would not be two or three, as required in a $C(5,12)$. Thus the pairs 7 9, 7 10, 8 9, 8 10, each occur in one of these blocks. A similar argument applies to the second and third groups of four blocks. Hence, we have, without loss of generality:

B_1:	5 6 7 9
B_2:	5 6 7 10
B_3:	5 6 8 9
B_4:	5 6 8 10
B_5:	7 8 5 9
B_6:	7 8 5 10
B_7:	7 8 6 9
B_8:	7 8 6 10

B_9:	9 10 5 7
B_{10}:	9 10 5 8
B_{11}:	9 10 6 7
B_{12}:	9 10 6 8

Next, we observe that the triples j 5 6 ($1 \le j \le 4$) each must occur once, by equation (**) preceding Lemma 2.2. No such triple can occur in any B_i, so each j ($1 \le j \le 4$) occurs in one of B_1, B_2, B_3 and B_4. Similarly, each such j occurs in one of B_5, B_6, B_7 and B_8, and in one of B_9, B_{10}, B_{11} and B_{12}. Without loss of generality, we can assume $j \in B_j$ for $1 \le j \le 4$. Thus we have

B_1:	1 5 6 7 9
B_2:	2 5 6 7 10
B_3:	3 5 6 8 9
B_4:	4 5 6 8 10

We now consider B_5, B_6, B_7 and B_8. By considering block intersections with B_1, B_2, B_3 and B_4, we see that neither 1 nor 3 can occur in B_5 or B_7; and neither 2 nor 4 can occur in B_6 or B_8. Thus 2 occurs in B_5 and 4 in B_7, or vice versa. However, in view of the automorphism (1 3) (2 4) (5 6) (7 8) of our partial design, we may assume that 2 occurs in B_5 and 4 occurs in B_7.

We now turn our attention to B_9, B_{10}, B_{11} and B_{12}. Calculating intersections, we see that $4 \in B_9$ and $3 \in B_{11}$. Then $2 \in B_{12}$ and $1 \in B_{10}$. Finally (again, by calculating intersections), we have $3 \in B_6$ and $1 \in B_8$. Thus the following $C(5,12)$ is unique up to isomorphism.

R_1:	1 5 6 7 9	2 3 4 8 10
R_2:	2 5 6 7 10	1 3 4 8 9
R_3:	3 5 6 8 9	1 2 4 7 10
R_4:	4 5 6 8 10	1 2 3 7 9
R_5:	2 5 7 8 9	1 3 4 6 10
R_6:	3 5 7 8 10	1 2 4 6 9
R_7:	4 6 7 8 9	1 2 3 5 10
R_8:	1 6 7 8 10	2 3 4 5 9
R_9:	4 5 7 9 10	1 2 3 6 8
R_{10}:	1 5 8 9 10	2 3 4 6 7
R_{11}:	3 6 7 9 10	1 2 4 5 8
R_{12}:	2 6 8 9 10	1 3 4 5 7

This completes the proof. □

Next we show that no $C(7,16)$ exists. We have $r_0+r_1+r_2 = 16$ and $r+4r_2 = 24$. This system has four solutions in non-negative integers.

	r_0	r_1	r_2
case (1)	10	0	6
case (2)	7	4	5
case (3)	4	8	4
case (4)	1	12	3

Our point set is $\{1,2,\ldots,14\}$ and the pairs which occur six times are those in $F = \{7\ 8, 9\ 10, 11\ 12, 13\ 14\}$. The three types of rounds are:

θ	Y_6	Y_6	Y_8
0	0	12	9
1	1	10	10
2	3	6	12

The intersection of two blocks from different rounds is 3 or 4. Case (1) and Case (2) can be quickly eliminated.

Lemma 3.2: *Cases (1) and (2) are impossible.*

Proof. A 2-round contains three of the four pairs in F. If $r_2 > 4$, then two blocks must contain three common pairs from F for an intersection of at least six. Since the intersections are supposed to be three or four, $r_2 \leq 4$. □

Lemma 3.3: *Case (3) is impossible.*

Proof. There are four 2-rounds. These rounds have blocks B which contain three of the four pairs in F and blocks $\bar{B}$ which contain no pairs of F. Now the blocks B intersect in two of these pairs giving a maximum intersection of four. Thus we may assume:

B_1:	1 7 8 9 10 11 12	$\bar{B}_1$:	2 3 4 5 6 13 14
B_2:	2 7 8 9 10 13 14	$\bar{B}_2$:	1 3 4 5 6 11 12
B_3:	3 7 8 11 12 13 14	$\bar{B}_3$:	1 2 4 5 6 9 10
B_4:	4 9 10 11 12 13 14	$\bar{B}_4$:	1 2 3 5 6 7 8

There are eight 1-rounds whose B blocks contain one pair from F and whose $\bar{B}$ blocks contain no pairs from F. Since each pair in F must occur twice more and since the 0-rounds contain no F pairs, these pairs must occur in Blocks B_5-B_{12}. We may assume without loss of generality that 7 8 is in B_5 and B_6, 9 10 is in B_7 and B_8, 11 12 is in B_9 and B_{10} and finally 13 14 is in B_{11} and B_{12}. Thus we have accounted for all occurrences of pairs from F.

Using equation (**), we know that each triple $i\ x\ x+1$, where $\{x,x+1\} \in F$ and $1 \leq i \leq 6$, occurs twice. Thus we have:

B_5:	4 5 7 8	$\bar{B}_5$:	1 2 3 6
B_6:	4 6 7 8	$\bar{B}_6$:	1 2 3 5
B_7:	3 5 9 10	$\bar{B}_7$:	1 2 4 6
B_8:	3 6 9 10	$\bar{B}_8$:	1 2 4 5
B_9:	2 5 11 12	$\bar{B}_9$:	1 3 4 6
B_{10}:	2 6 11 12	$\bar{B}_{10}$:	1 3 4 5
B_{11}:	1 5 13 14	$\bar{B}_{11}$:	2 3 4 6
B_{12}:	1 6 13 14	$\bar{B}_{12}$:	2 3 4 5

There are four 0-rounds. Using equation (**) we know that each triple i 5 6, $1 \le i \le 4$, must occur four times in the design. Thus each such triple must occur once in the remaining blocks, giving:

B_{13}:	4 5 6	$\bar{B}_{13}$:	1 2 3
B_{14}:	3 5 6	$\bar{B}_{14}$:	1 2 4
B_{15}:	2 5 6	$\bar{B}_{15}$:	1 3 4
B_{16}:	1 5 6	$\bar{B}_{16}$:	2 3 4

A triple 5 6 j, $7 \le j \le 12$ occurs three times in the design (using equation (**)). Thus each j, $7 \le j \le 12$, occurs twice in the blocks B_{13}-B_{16}. Now consider the occurrences of triples i 6 7, $1 \le i \le 4$. Each of these four triples occurs three times in the design, for a total of 12. We have three such triples in the first four founds and two in the last four rounds. So there are seven occurrences in the middle eight rounds.

Suppose the pair 6 7 occurs a times in B_5-B_{12} and b times in $\bar{B}_5$-$\bar{B}_{12}$. Then $a+3b = 7$, by the discussion above. But we can also count occurrences of the pair 6 7 in the design. There is one occurrence in the first four rounds, $a+b$ occurrences in the middle eight rounds and two occurrences in the last four rounds which should total to seven. Thus $a+b = 4$. Solving, we have $a = 5/2$ and $b = 3/2$. But a and b must be integral, so we have a contradiction. □

Lemma 3.4: *Case (4) is impossible.*

Proof. There are three 2-rounds and thus without loss of generality we have:

B_1:	7 8 9 10 11 12	$\bar{B}_1$:	13 14
B_2:	7 8 9 10 13 14	$\bar{B}_2$:	11 12
B_3:	7 8 11 12 13 14	$\bar{B}_3$:	9 10

The pair 7 8 does not occur in any blocks of 0-rounds and in three blocks of 1-rounds, say B_4, B_5 and B_6. Each of these blocks contains two points from {1,2,...,6}. If we count occurrences of triples 1 7 8, $1 \le i \le 6$, using equation (**), we see that these six triples occur twice in each block, for a total of twelve. But these triples can occur only in B_1-B_6 and in these blocks only nine such triples are found. This contradiction proves the result. □

Thus we have

Theorem 3.5: $R(7) = 15$.

Proof. Lemmata 2.7, 2.8 and 2.9 show that $R(7) \leq 15$. But $R(7) \geq 15$ by Theorem 1.6. □

We can also show that no $C(9,20)$ exists. We have $r_0+r_1+r_2 = 20$ and $r_1+4r_2 = 40$.

There are four cases to consider:

	r_0	r_1	r_2
case (1)	1	12	7
case (2)	4	8	8
case (3)	7	4	9
case (4)	10	0	10

The three θ-rounds are:

θ	y_8	y_9	y_{10}
0	0	20	16
1	1	18	17
2	3	14	19

Here a θ-round consists of $\{B,\bar{B}\}$, where $|B \cap Y| = 4-\theta$ and $|\bar{B} \cap Y| = 4+\theta$. The pairs that occur eight times are those in $F = \{9\ 10, 11\ 12, 13\ 14, 15\ 16, 17\ 18\}$. For a given pair $\{i,i+1\}$ in F, suppose that it occurs in u_θ blocks B of θ-rounds and v_θ blocks $\bar{B}$ of θ-rounds. Then it is easy to see that $u_0 = v_0 = v_1 = 0$ and $u_1+u_2+v_2 = 8$. Also we can count triples $\{i,i+1,j\}$ where $j \in \bar{Y}$ and $\{i,i+1\}$ in F. Counting (using equation (**)) gives that the eight such triples occur four times each. Hence we obtain $4u_1+5u_2+v_2 = 32$. There are three solutions for u_1, u_2, and v_2 in non-negative integers:

u_1	u_2	v_2
8	0	0
4	3	1
0	6	2

Now a block B in a 2-round contains three of the five pairs in F. No two of these blocks can contain the same three pairs, for then they would intersect in six points which is not allowed. Since $r_2 \geq 7$ and since $u_2 = 0,3$ or 6 at least two of the pairs in F must have $u_2 = 6$. Since the sets of three pairs from F must be distinct in the B blocks of the 2-rounds, we have $r_2 \geq 9$. But this implies a third pair in F has

$u_2 = 6$ which then implies $r_2 \geq 10$. So cases (1), (2) and (3) are impossible. In the remaining case, all five pairs of F correspond to the solution

$$u_1 = 0, \quad u_2 = 6, \quad v_2 = 2.$$

Now we can count triples $1\ i\ i+1$ where $\{i,i+1\}$ is in F. Each such triple occurs three times, for a total of 15. Suppose 1 occurs c times in blocks B of the 2-rounds and hence $10-c$ times in blocks $\bar{B}$ of the 2-rounds. Then $3c+10-c = 15$ or $c = \frac{5}{2}$ which is an absurdity. Thus we have

Theorem 3.6: $R(9) = 19$.

Proof. We have shown that $R(9) \leq 19$; and $R(9) \geq 19$ by Theorem 1.7. □

4. Configurations with $4k+3$ rounds, k odd.

In this section we give a construction for a $C(4k-1,8k-1)$ whenever a Hadamard matrix of order $4k$ exists. This improves Theorem 1.6.

Theorem 4.1: *If there is a Hadamard matrix of order* $4k$, *then* $R(4k-1) \geq 8k-1$.

Remark. It is well-known that a Hadamard matrix of order $4k$ exists whenever $4k-1$ is a prime power.

Proof. A $C(2k,4k-1)$ exists by Theorem 1.2. Let S denote the symbol set of this configuration and label the rounds $R_1, \ldots, R_{4k-1}$. Let x be any element of S. Each round $R_i = \{B_i, \bar{B}_i\}$ where we may assume $x \in B_i$, $1 \leq i \leq 4k-1$. We take two copies of S, say $S_1 = S \times \{1\}$ and $S_2 = S \times \{2\}$. For any $T \subset S$, let $T^i = \{t_i : t \in T\}$, $i = 1,2$. (We write t_i for (t,i) for simplicity.) Consider the following set of blocks:

$$C = \{C_i = (B_i - x)^1 \cup (B_i - x)^2 : 1 \leq i \leq 4k-1\}$$

$$D = \{D_i = (B_i - x)^1 \cup \bar{B}_i^2 : 1 \leq i \leq 4k-1\}.$$

We know that $|B_i \cap B_j| = k$ if $1 \leq i < j \leq 4k-1$. Thus, if $1 \leq i < j \leq 4k-1$, then $|D_i \cap D_j| = |(B_i - x) \cap (B_j - x)| + |\bar{B}_i \cap \bar{B}_j| = k-1+k = 2k-1$. Also, if $1 \leq i < j \leq 4k-1$, then $|C_i \cap C_j| = 2k-2$. If $1 \leq i \neq j \leq 4k-1$, we have $|C_i \cap D_j| = 2k-1$. As well, $|C_i \cap D_i| = 2k-1$ for $1 \leq i \leq 4k-1$. Finally

$|(S-x)^1 \cap C_i| = |(S-x)^1 \cap D_i| = 2k-1$ for $1 \le i \le 4k-1$. We will form a $C(4k-1, 8k-1)$ on symbol set $(S-x)^1 \cup (S-x)^2$. The blocks are supposed to be of length $4k-1$ so that the blocks C_i of C need to be lengthened by one element $c_i \notin C_i$. It is sufficient to choose these c_i's so that, for all $1 \le i \le j \le 4k-1$ either $c_i \in C_j$ or $c_j \in C_i$ (or both). Note that this increases their intersection to $2k-1$ or $2k$. Then we will let the rounds for the configuration be the following $8k-1$ blocks and their complements:

$$\{D_i\} \cup \{C_i \cup \{c_i\}\} \cup \{(S-x)^1\}.$$

It remains to determine the c_i's. This can be done by the following algorithm:

While not all c_i's are defined do

(1) Choose any i such that c_i is not yet defined.

(2) Define c_i to be any element of $(S-x)^1 - C_i$.

(3) For all c_j's such that c_j is not defined and $c_i \notin C_j$, define $c_j = c_i$.

Note that the algorithm must stop after $2k$ iterations and thus c_i can always be chosen. This completes the proof. □

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