ON THE EXISTENCE OF 30 MUTUALLY ORTHOGONAL LATIN SQUARES

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Abstract

If $N(v)$ denotes the maximum number of mutually orthogonal Latin squares of order v, then $N(v) \geq 30$ if $v > 121605$.

1. *Introduction.*

A *Latin square* L of order v is a v by v array of elements of a v-set S, such that each element of S occurs exactly once in each row and each column of S. Two Latin squares, L_1 and L_2, on v-sets S_1 and S_2 respectively, are said to be *orthogonal* if, for any ordered pair $(s_1,s_2) \in S_1 \times S_2$, there exists a unique cell (i,j) such that $s_1 \in L_1(i,j)$ and $s_2 \in L_2(i,j)$. Several Latin squares are said to be *mutually orthogonal* if each pair of them is orthogonal. Henceforth we refer to mutually orthogonal Latin squares as MOLS.

Let $N(v)$, $v \geq 0$, denote the maximum number of MOLS of order v. Let n_r denote the least integer such that $N(v) \geq r$ if $v > n_r$. We note that $N(0)$ and $N(1)$ are each greater than any finite number.

Bounds on n_r have been established for several small values of r. The following table lists many of the known results.

Table 1

Bounds	References
$n_2 = 6$	Bose, Shrikhande and Parker [1]
$n_3 \leq 14$	Hanani [3], Wang and Wilson [9]
$n_4 \leq 52$	Hanani [3], Wilson [10], Van Lint [8]
$n_5 \leq 62$	Hanani [3]
$n_6 \leq 90$	Wilson [10]
$n_7 \leq 2862$	Van Lint [8], Mullin et al [5], Stinson [7]

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$n_8 \le 7768$	Mullin et al [5], Stinson [7], Brouwer [2]
$n_{29} \le 34115553$	Hanani [3]

In this paper we will establish that $n_{30} \le 121605$. As we will show in the next section, the basis for establishing a bound for n_{30} is that $N(31), N(32) \ge 30$. We apply a construction of Wilson to obtain our result.

When Hanani established the above bound for n_{29} he used older constructions which are not as powerful as Wilson's constructions; hence he constructed 29 MOLS rather than 30.

2. *Theorems concerning the construction of MOLS.*

We will use the following three theorems in order to establish our bound for n_{30}. Theorem 2.1 is due to MacNeish [4], Theorem 2.2 is due to Wilson [10], and Theorem 2.4 is due to Wotjas [12].

THEOREM 2.1. *If* $v = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ *is the factorization of the integer* $v \ge 2$, *into powers of distinct primes* p_i, *then* $N(v) \ge \min\{p_i^{a_i}-1, 1 \le i \le k\}$.

THEOREM 2.2. *If* $0 \le u \le t$, *then*

$$N(mt+u) \ge \min\{N(m), N(m+1), N(t)-1, N(u)\}.$$

Theorem 2.1 implies that $N(31) \ge 30$, $N(32) \ge 31$. (In fact we have equality since $N(v) \le v-1$ for all integers $v \ge 2$). Thus we have the following corollary to Theorem 2.2.

COROLLARY 2.3. *If* $0 \le u \le t$, $N(t) \ge 31$, *and* $N(u) \ge 30$, *then* $N(31t+u) \ge 30$.

Proof: Set $m = 31$ in Theorem 2.2. □

THEOREM 2.4. *If* $0 \le u \le t$ *then*

$$N(mt+u) \ge \min\{N(m), N(m+1), N(m+u)-1, N(t)-u\}.$$

For p a prime integer, let $u(p)$ denote the least integer α such that, if m_0 is any positive integer, then there exists an integer m, $m_0 \le m \le m_0 + \alpha - 1$, such that $(m,q) = 1$ if q is prime and $q \le p$. As is usual, (m,q) denotes the greatest common divisor of m and q. The following is established in [6] by means of a computer search.

THEOREM 2.5. *$u(31) = 58$.*

The following theorem enables us to determine a bound for n_{30}.

THEOREM 2.6. *Suppose $0 \le u_0 < 31$, and suppose that $N(31t+u_0) \ge 30$ if $t_0 - 57 \le t \le t_0$. Then $N(31t_1+u_0) \ge 30$ if $t \ge 32t_0 + u_0$.*

Proof: Suppose $t_1 \ge 32t_0 + u_0$. By Theorem 2.5, there is an integer i such that $0 \le i \le 57$ and $t_1 - t_0 + i$ is relatively prime to all primes not exceeding 31. For such i, Theorem 2.1 guarantees that $N(t_1-t_0+i) \ge 31$. Further, $t_1-t_0+i \ge 31\ (t_0-i) + u_0$, since $i \ge 0$. Also $N(31(t_0-i)+u_0) \ge 30$ by assumption, since $0 \le i \le 57$. If we take $t = t_1 - t_0+i$ and $u = 31(t_0-i) + u_0$ in Corollary 2.3, we have the result. □

Suppose $0 \le u < 31$. Define $N_u(t) = N(31t+u)$. We may restate Corollary 2.3.

COROLLARY 2.7. *If $N_u(t) \ge 30$, $m \ge 31t+u$, and $N(m) \ge 31$, then $N_u(m+t) \ge 30$.*

Also, we may rephrase Theorem 2.6 as follows.

THEOREM 2.8. *If $N_u(t) \ge 30$ for $t_0-57 \le t \le t_0$, then $N_u(t) \ge 30$ if $t \ge 32t_0 + u$.*

3. *Description of Method.*

The purpose of this paper is two-fold: firstly, to obtain a "good" result, and secondly, to accomplish the first purpose in an efficient manner, that is, without using large amounts of computer time. We have tried to approach the problem in such a way that it can be easily and efficiently implemented on a computer. The total computer time used to establish the result of section 4 was under three minutes.

We define the set of non-negative integers congruent to a particular integer modulo 31 to be a *fibre* (modulo 31), and we identify a fibre by its least member.

In establishing our bound for n_{30}, we attack each fibre separately. In view of the form of Theorem 2.8, this seems to be a natural approach.

It is desirable to construct 30 MOLS of many consecutive orders within each fibre. In fact, we must construct 30 MOLS of 58 consecutive orders within a fibre to even apply Theorem 2.8. Then we wish to construct much longer sequences of 30 MOLS to fill in the gap from t_0 to $32t_0+u$, in Theorem 2.8.

Our approach is the following. We construct a set T_u of small values t for which $N_u(t) \geq 30$. We use Theorem 2.1 and Corollary 2.3 Further, we desire that each T_u contain exactly as many even numbers as odd (we will say more about this later).

We next obtain a list M_k of all orders t, below some positive integer k, for which Theorem 2.1 guarantees $N(t) \geq 31$. In view of Theorem 2.5, T_k contains no gap of length greater than 57. However $|M_k|/k$ is much greater than 1/58.

Let P denote the set of primes not exceeding 31, and write $p^* = \prod_{p \in P} p$. Then, in any block of p^* consecutive integers, there are exactly $\prod_{p \in P} (p-1)$ integers relatively prime to p^*. Now since $\frac{1}{p^*} \prod_{p \in P} (p-1) > .1528$, we have that $|M_k|/k > .1528$ if k is large. Notice that we consider only odd members of M_k in this estimate. The estimate is good enough for our purposes since M_k contains few even numbers.

Corollary 2.7 states that, if $m \in M_k$, $t \in T_u$, and $m \geq 31t + u$, then $N_u(m+t) \geq 30$. Given lists of integers M_k and T_u, it is a very simple matter to determine all sums $m+t$, subject to $m \geq 31t + u$. A computer can perform many such calculations in a short amount of time.

Thus we wish to write integers in the form $m+t$, with m and t above. We want to choose T_u small enough that all sums $m+t$ can be computed quickly, yet large enough so that not too many integers are missed. This is where parity is important. Since almost all the elements of M_k are odd, we wish to have equal numbers of odd and even t's so that sums $m+t$ do not favour either the odds or the evens.

We can obtain a crude estimate on how big T_u should be. Suppose ℓ is an integer large enough that $32t_{max} + u \leq \ell$, where $t_{max} = \max\{t, t \in T_u\}$. Further, suppose $k \geq \ell$. Then we can be certain that $N_u(\ell) \geq 30$ if $\ell - t \in M_k$ for some $t \in T_u$. The probability of this occuring can be estimated to be $1 - (1-.3056)^{1/2|T_u|}$. We consider only those $t \in T_u$ which make $\ell - t$ odd, then estimate the probability that $\ell - t \in M_n$ by $2(.1528)$. If $|T_u| = 50$, then the estimate obtained is $1 - (.6944)^{25} <$ $.0001097$, which is very close to 10^{-4}. Thus it appears that $|T_u| = 50$ is sufficient to ensure that very few orders are missed.

We emphasize that the above discussion proves nothing. However, it serves as an indicator of what will happen when the problem is programmed for the computer. Indeed, without some estimate like the one obtained above, it would not be possible to establish a bound for n_{30} in an efficient manner.

4. *A Bound for* n_{30}.

In the appendix, we list orders of squares corresponding to the sets T_u which we used. As stated in section 3, $t \in T_u$ only if it can be established that $N_u(t) \geq 30$ by Theorem 2.1 or Corollary 2.3. Each T_u consists of the 25 smallest even, and then 25 smallest odd t's described above.

The results of the computer run are contained in Table 2. For each fibre u, and each t such that $t_1(u) \leq t \leq 32(t_1(u)+57)+u$, we have $N_u(t) \geq 30$ except for the possible exceptions $t \in E_u$.

Table 2

Fibre	$t_1(u)$	Possible exceptions (t-values)
0	2009	
1	1959	
2	6004	
3	5757	150866
4	3888	63612, 11742
5	3187	136313, 3856
6	2650	85592, 83492, 80972, 69422, 57032, 55562
7	5431	29631, 6176
8	4416	145088, 110558, 85418, 52268, 47138, 33548, 4979
9	3941	
10	2568	
11	3607	39147, 3940
12	2368	
13	5577	
14	5430	81055
15	6129	
16	2639	
17	3941	
18	2949	
19	3941	4978

Fibre	$t_1(u)$	Possible exceptions (t-values)
20	5430	
21	4482	154541, 143441, 138401, 126191, 106061, 89021, 86195, 50621, 45341, 28961, 15311, 11721
22	3817	
23	10315	332346, 331026, 286212, 275616, 244296, 226476, 215196, 210906, 201546, 171876, 168336, 90786, 79866, 63066, 61913, 60756, 40836, 39306, 11520, 10944, 10374, 10314, 10266, 10026, 10014, 9996, 9948, 9936, 9900, 9870, 9858, 9846, 9816, 9696,
24	7138	195254
25	6009	
26	7834	
27	4415	66457, 38401, 29917, 4979,
28	2650	2843
29	10315	246597, 158607, 90786, 54447, 54234, 11232, 10974, 10944, 10314, 10200, 9956, 9948, 9900, 9896, 9870, 9816, 9790
30	3528	

LEMMA 4.1. *Suppose for each* $t \in E_u$ *that* $N_u(t) \geq 30$. *Then* $N_u(t) \geq 30$ *if* $t \geq t_1(u)$.

Proof: Theorem 2.8 implies the result. □

In Table 3, we list constructions which eliminate the possible exceptions E_u. All constructions are applications of Theorems 2.2 and 2.4.

Table 3

fibre	t	31t+u	Construction	Reference
3	150866	4676849	256.18217+13297	Theorem 2.2
4	63612	1971976	31.61643+61043	Theorem 2.2
4	11742	364006	31.11387+11009	Theorem 2.2
5	136313	4225708	31.132071+131507	Theorem 2.2
5	3856	119541	31.3851+160	Theorem 2.4
6	85592	2653358	31.82939+82249	Theorem 2.2
6	83492	2588258	31.80887+80761	Theorem 2.2
6	80972	2510138	31.78457+77971	Theorem 2.2

fibre	t	31t+u	Construction	Reference
6	69422	2152088	31.67267+66811	Theorem 2.2
6	57032	1767998	31.55261.54907	Theorem 2.2
6	55562	1722428	31.53851+53047	Theorem 2.2
7	29631	918568	31.28757+27101	Theorem 2.2
7	6176	191463	31.6173+100	Theorem 2.4
8	145088	4497736	31.140579+139787	Theorem 2.2
8	110558	3427306	31.107123+106493	Theorem 2.2
8	85418	2647966	31.82757+82499	Theorem 2.2
8	52268	1620316	31.50657+49949	Theorem 2.2
8	47138	1461286	31.45677+45299	Theorem 2.2
8	33548	1039996	31.32507+32279	Theorem 2.2
8	4989	154357	31.4969+318	Theorem 2.4
11	39147	1213568	31.37961+37087	Theorem 2.2
11	3940	122151	31.3929+352	Theorem 2.4
14	81055	2512719	256.9781+8783	Theorem 2.2
19	4978	154337	31.4957+670	Theorem 2.4
21	154541	4790792	31.149713+149689	Theorem 2.2
21	143441	4446692	31.138979+138343	Theorem 2.2
21	138401	4290452	31.134119+132763	Theorem 2.2
21	126191	3911942	31.122263+121789	Theorem 2.2
21	106061	3287912	31.102763+102259	Theorem 2.2
21	89021	2759672	31.86269+85333	Theorem 2.2
21	86195	2672066	31.83507+83349	Theorem 2.2
21	50621	1569272	31.49069+48133	Theorem 2.2
21	45341	1405592	31.43933+43669	Theorem 2.2
21	28961	897812	31.28069+27673	Theorem 2.2
21	15311	474662	31.14851+14281	Theorem 2.2
21	11721	364922	31.11413+11119	Theorem 2.2
23	332346	10302749	256.40123+31261	Theorem 2.2
23	331026	10261829	256.39931+39493	Theorem 2.2
23	286212	8872595	256.34537+31123	Theorem 2.2
23	275616	8544119	256.33247+32887	Theorem 2.2
23	244296	7573199	256.29473+28111	Theorem 2.2
23	226476	7020779	256.27331+24043	Theorem 2.2
23	215196	6671099	256.25981+19963	Theorem 2.2

fibre	t	31t+u	Construction	Reference
23	210906	6538109	256.25447+23667	Theorem 2.2
23	201546	6247949	256.24373+8461	Theorem 2.2
23	171876	5328179	256.20737+19507	Theorem 2.2
23	168336	5218439	256.20331+13703	Theorem 2.2
23	90786	2814389	256.10957+9397	Theorem 2.2
23	79866	2475869	256.9643+7261.	Theorem 2.2
23	63066	1955069	256.7621+4093	Theorem 2.2
23	61913	1919326	31.59987+59729	Theorem 2.2
23	60756	1883459	256.7331+6723	Theorem 2.2
23	40836	1265939	31.40829+240	Theorem 2.4
23	39306	1218509	256.4753+1741	Theorem 2.2
23	11520	357143	31.11491+922	Theorem 2.4
23	10944	339287	31.10937+240	Theorem 2.4
23	10374	321617	31.10337+1170	Theorem 2.4
23	10314	319757	31.10301+426	Theorem 2.4
23	10266	318269	31.10259+240	Theorem 2.4
23	10026	310829	31.10007+612	Theorem 2.4
23	10014	310457	31.10007+240	Theorem 2.4
23	9996	309899	31.9967+922	Theorem 2.4
23	9948	308411	31.9941+240	Theorem 2.4
23	9936	308039	31.9929+240	Theorem 2.4
23	9900	306923	31.9887+426	Theorem 2.4
23	9870	305993	31.9857+426	Theorem 2.4
23	9858	305621	31.9851+240	Theorem 2.4
23	9846	305249	31.9839+240	Theorem 2.4
23	9816	304319	31.9803+426	Theorem 2.4
23	9696	300599	31.9689+240	Theorem 2.4
24	195254	6052898	31.189169+188659	Theorem 2.2
27	66457	2060194	31.64433+62771	Theorem 2.2
27	38401	1190458	31.37243+35923	Theorem 2.2
27	29917	927454	31.28991+28733	Theorem 2.2
27	4979	154345	31.4969+306	Theorem 2.4
28	2844	88161	31.2843+28	Theorem 2.4
29	246597	7644536	31.238897+238729	Theorem 2.2

fibre	t	31t+u	Construction	Reference
29	158607	4916846	31.153679+152797	Theorem 2.2
29	90786	2814395	256.10951+10939	Theorem 2.2
29	54447	1687886	31.52747+52729	Theorem 2.2
29	54234	1681283	256.6547+5251	Theorem 2.2
29	11232	348221	31.11173+1858	Theorem 2.4
29	10974	340223	31.10957+556	Theorem 2.4
29	10944	339293	31.10807+2346	Theorem 2.4
29	10314	319763	31.10303+370	Theorem 2.4
29	10200	316229	31.10193+246	Theorem 2.4
29	9956	308665	31.9949+246	Theorem 2.4
29	9948	308417	31.9941+246	Theorem 2.4
29	9900	306929	31.9887+432	Theorem 2.4
29	9896	306805	31.9883+432	Theorem 2.4
29	9870	305999	31.9859+370	Theorem 2.4
29	9816	304325	31.9803+432	Theorem 2.4
29	9790	303519	31.9707+742	Theorem 2.4.

The results in Table 3, together with Lemma 4.1, proves the following.

THEOREM 4.2. *If* $t \geq t_1(u)$, *then* $N_u(t) \geq 30$.

From Theorem 4.2, we obtain Table 4, a list of the highest order of each fibre for which 30 MOLS are not known. Note that Table 4 lists orders of Latin squares, not t-values.

Table 4

Fibre	Highest Possible Exception	Fibre	Highest Possible Exception
0	62217	4	120501
1	60699	5	98771
2	186095	6	82125
3	178439	7	168337

Fibre	Highest Possible Exception	Fibre	Highest Possible Exception
8	136873	20	168319
9	122149	21	138932
10	79587	22	118318
11	111797	23	299297
12	73389	24	221271
13	172869	25	186273
14	168313	26	242849
15	189983	27	136861
16	81794	28	82147
17	122157	29	299303
18	91406	30	109367
19	122159		

Thus we have a preliminary bound.

THEOREM 4.3. $n_{30} \leq 299303$.

5. *An Improved Bound.*

A computer programme was run which attempted to construct 30 MOLS of orders between 100000 and 300000, where 30 MOLS were not constructed by the method of Section 4. The programme used Theorems 2.2 and 2.4 with m = 31. In applying Theorem 2.2, we use odd values of t and u, for which theorem guarantees that $N(t) \geq 31$, $N(u) \geq 30$. In applying Theorem 2.4, we use odd t and even u, and verify that $N(31+u) \geq 31$ by Theorem 2.1, and that t is divisible by no prime not exceeding $u + 30$. Thus we do not use these theorems in their most general form, but in an easily programmed way which proves very effective. Notice that Theorem 2.2 is used for odd orders and Theorem 2.4 is used for even orders.

The results of this programme are as follows. There were 21 possible exceptions. The three highest of these possible exceptions are eliminated in Lemma 5.2. The remaining 18 possible exceptions are listed in Table 5.

THEOREM 5.1. *If* $0 \le u, v \le t$, *then*

$$N(mt+u+v) \ge \min\{N(m), N(m+1), N(m+2), N(t)-2, M(u), N(v)\}.$$

Proof: See Wilson [10].

LEMMA 5.2. *There exist 30 MOLS of orders 185905, 143607, and 138932.*

Proof: Since 138932 = 127.1093+121, Theorem 2.2 implies N(138932) ≥ 30.

Since 143607 = 31.4489+4448, Theorem 2.2 implies N(143607) ≥ 30 (note that 4448 = 32.139, 4489 = 67^2).

We use Theorem 5.1 to construct 30 MOLS of order 185905. First we have 2294 = 31.73+31, 2295 = 31.73+32, 2296 = 31.73+33. Thus we have 30 MOLS of these orders by Theorems 2.2, 2.2 and 2.4 respectively. Then 185905 = 2294.81+59+32, and Theorem 5.1 implies the result. □

Table 5 -- Possible Exceptions over 100000.

Order	Fibre	Order	Fibre
121605	23	107206	8
121515	26	106975	25
121076	21	102927	7
119317	29	101878	24
118318	22	101625	7
114766	4	100827	15
109246	2	100682	25
109215	2	100515	13
108823	13	100029	23

Thus we have our main result.

THEOREM 5.3. $n_{30} \le 121605$.

6. *Conclusion.*

Thus we have established the new bound $n_{30} \le 121605$. Further, there are at most 18 orders over 100000 for which 30 MOLS are not known to exist.

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APPENDIX

FIBRE — ORDERS OF SQUARES USED AS VALUES FOR T LISTS

FIBRE									
0	0	31	961	992	1023	1147	1178	1271	1302
	1333	1364	1457	1488	1599	1550	1643	1674	1829
	1860	1891	1922	1984	2015	2077	2108	2201	2232
	2263	2294	2449	2480	2511	2542	2573	2604	2759
	2790	3007	3038	3131	3162	3193	3224	3317	3348
	3379	3410	3503	3534	3782				
1	1	32	125	311	373	683	993	1024	1117
	1148	1179	1272	1303	1334	1365	1427	1458	1489
	1520	1551	1613	1644	1675	1830	1861	1892	1923
	1985	2016	2078	2109	2202	2233	2264	2295	2357
	2419	2450	2481	2512	2543	2574	2760	3008	3132
2	64	157	281	343	467	529	653	839	1087
	1459	1583	1831	2017	2048	2141	2203	2265	2327
	2389	2513	2575	2637	2699	2823	3071	3195	3257
	4032	5024	5210	5334	5396	5520	5706	5768	6016
	6078	6140	6264	6326	6698	7070	7194	7256	7380
3	127	251	313	499	809	1181	1367	1429	1553
	1739	1801	1987	2111	2173	2297	2731	2917	3041
	3413	3599	3847	4033	4064	4095	4157	4188	4219
	4374	4436	4746	4808	4994	5056	5180	5304	5366
	7164	7226	7350	7536	7598				
4	97	128	283	593	841	1151	1213	1399	1523
	1709	2081	2143	2267	2701	2887	3011	3104	3228
	3259	3290	3321	3414	3476	3569	3600	3631	3848
	3972	4003	4034	4065	4097	4127	4158	4189	4344
	4375	4406	4437	4716	4778	4964	5150	5274	5336
	5460	5646	5708	6018	6080				
5	67	191	439	563	625	811	997	1307	1369
	1493	1741	1927	2113	2144	2237	2268	2330	2423
	2516	2578	2609	2640	2671	2826	2857	3074	3136
	3167	3198	3229	3260	3384	3446	3539	3570	3818
	3911	3942	4004	4035	4128	4159	4283	4314	4376
	4686	4748	4934	5120	5244				
6	37	223	347	409	719	967	1091	1153	1184
	1277	1308	1370	1494	1556	1680	1866	1928	2021
	2083	2114	2207	2238	2269	2300	2393	2486	2548
	2579	2610	2796	3044	3137	3168	3230	3323	3354
	3416	3540	3571	3788	3881	3912	3943	3974	4005
	4067	4098	4129	4253	4625				
7	131	193	317	379	503	751	937	1061	1123
	1433	1619	1681	1867	2053	2239	2549	2797	3169
	3293	3479	3541	3727	3851	4099	4192	4223	4378
	4440	4750	4812	4998	5184	5308	5370	5494	5680
	5742	6052	6114	6176	6238	6300	6362	6672	6734
	7044	7106	7168	7230	7292				

FIBRE	ORDERS OF SQUARES USED AS VALUES FOR T LISTS							
8	101	163	256	349	659	907	1031	1093	1217
	1279	1961	2209	2333	2767	2891	2953	3139	3232
	3294	3418	3449	3480	3511	3604	3697	3821	3852
	3976	4008	4038	4069	4162	4348	4410	4441	4720
	4751	4782	4813	4968	5154	5216	5278	5340	5402
	5464	5526	5650	5712	5774				
9	71	257	443	691	877	1063	1187	1249	1373
	1559	1621	1931	1993	2179	2272	2303	2334	2520
	2551	2582	2644	2830	2861	2923	3078	3109	3202
	3233	3264	3388	3450	3481	3574	3822	3853	3946
	3977	4008	4039	4132	4318	4349	4380	4690	4752
	4938	5124	5248	5310	5434				
10	41	103	227	289	599	661	971	1033	1312
	1374	1498	1560	1591	1684	1777	1870	1901	1932
	2025	2087	2118	2242	2273	2304	2459	2490	2521
	2552	2614	2707	2800	2048	3079	3172	3203	3234
	3296	3358	3389	3420	3482	3544	3606	3637	3761
	3792	3823	3854	3947	4009				
11	73	197	383	569	631	941	1499	1747	1871
	1933	2243	2336	2491	2522	2584	2646	2677	2801
	2832	3049	3080	3204	3266	3359	3390	3452	3483
	3576	3607	3793	3824	3917	3948	4010	4041	4134
	4289	4320	4382	4661	4692	4723	4754	4847	4940
	5126	5250	5312	5436	5622				
12	43	167	229	353	601	787	911	1097	1283
	1376	1500	1531	1562	1686	1872	1934	2027	2089
	2120	2213	2244	2306	2368	2399	2492	2554	2616
	2647	2802	2833	2957	3019	3050	3174	3236	3329
	3360	-391	3422	3546	3577	3701	3763	3794	3918
	3980	4011	4073	4104	4290				
13	137	199	509	571	757	881	1129	1439	1811
	1873	1997	2183	2617	2741	2803	2927	2989	3299
	3361	3392	3547	3671	3733	3919	4229	4384	4446
	4477	4756	4818	5004	5190	5314	5376	5500	5686
	5748	6058	6120	6244	6306	6368	6678	6740	7050
	7112	7174	7236	7298	7360				
14	107	169	293	479	541	727	1223	1409	1471
	1657	2029	2153	2339	2401	2711	2773	2897	3083
	3331	3424	3486	3517	3610	3827	3858	3889	3982
	4013	4044	4075	4168	4261	4354	4416	4726	4788
	4974	5160	5284	5346	5408	5470	5532	5656	5718
	5780	6028	6090	6152	6214				

FIBRE	ORDERS OF SQUARES USED AS VALUES FOR T LISTS							
15	139	263	449	821	883	1069	1193	1627	1813
	1999	2309	2371	2557	2867	3053	3239	3301	3673
	3797	4231	4293	4448	4603	4758	4789	4820	4913
	5006	5099	5192	5316	5378	5502	5688	5750	6060
	6122	6246	6308	6680	7052	7176	7238	7362	7548
	7610	7672	7920	8106	8292				
16	47	109	233	419	512	729	853	977	1039
	1163	1504	1566	1597	1690	1721	1783	1876	1907
	1938	2031	2124	2248	2279	2310	2341	2496	2558
	2620	2713	2806	2837	3023	3054	3178	3209	3240
	3271	3364	3426	3457	3488	3550	3581	3612	3643
	3767	3798	3860	3922	3984				
17	79	389	761	823	947	1009	1319	1381	1567
	1753	1877	2063	2187	2311	2528	2590	2621	2653
	2683	2838	2993	2086	3210	3272	3396	3458	3551
	3582	3613	3737	3830	3923	3954	4016	4047	4140
	4171	4326	4357	4388	4481	4698	4760	4946	5132
	5256	5318	5442	5504	5628				
18	49	173	359	421	607	1103	1289	1568	1692
	1723	1847	1878	1940	2033	2126	2250	2281	2312
	2467	2498	2560	2591	2622	2777	2808	2963	3056
	3149	3180	3242	3366	3397	3428	3552	3583	3769
	3800	3924	3986	4017	4079	4110	4141	4296	4327
	4358	4451	4513	4544	4668				
19	81	577	701	887	1259	1321	1693	1879	2003
	2251	2437	2592	2623	2654	2747	2809	2840	3088
	3119	3181	3212	3274	3398	3460	3491	3584	3677
	3739	3832	3863	3956	4018	4049	4111	4142	4297
	4328	4390	4421	4483	4700	4762	4948	5134	5258
	5320	5444	5630	5692	6002				
20	113	361	547	733	857	919	1229	1291	1601
	1663	1787	1849	1973	2221	2531	3593	2624	2903
	3089	3337	3461	3616	3709	3833	3864	3988	4019
	4050	4081	4174	4360	4391	4422	4732	4794	4980
	5166	5290	5352	5476	5226	5724	6034	6096	6220
	6282	6592	6654	7026	7150				
21	83	269	331	641	827	1013	1447	1571	2129
	2377	2501	2656	2687	2749	2842	2997	3090	3121
	3214	3276	3307	3400	3431	3462	3586	3617	3803
	3834	3958	3989	4020	4051	4144	4330	4361	4392
	4423	4547	4702	4733	4764	4950	5136	5260	5322
	5446	5632	5694	6004	6066				

FIBRE	ORDERS OF SQUARES USED AS VALUES FOR T LISTS								
22	53	239	487	673	797	859	983	1231	1696
	1789	1882	1913	1944	2037	2099	2130	2161	2254
	2316	2347	2502	2564	2626	2657	2719	2812	2843
	3060	3184	3246	3370	3432	3463	3556	3649	3804
	3928	3959	3990	4021	4114	4300	4331	4362	4517
	4672	4703	4734	4889	4920				
23	271	457	643	829	953	1201	1511	1697	1759
	2069	2131	2441	2503	2627	2689	2999	3061	3371
	3433	3557	3929	4177	4363	4549	4673	5696	8672
	8858	8982	9044	9230	9354	9664	9788	9912	9974
	10098	10532	10718	10904	11028	11090	11214	11400	11462
	11648	11834	12020	12144	12330				
24	179	241	613	1109	1171	1481	1543	1667	2039
	2287	2411	2473	2597	2659	2752	2969	3217	3403
	3527	3589	3713	4271	4457	4519	4643	4736	4891
	5728	5790	6100	6162	6286	6348	6720	7092	7216
	7278	7402	7588	7650	7712	7774	7960	8022	8146
	8208	8332	8394	8518	8580				
25	149	211	397	521	769	1327	1451	1637	1699
	1823	2009	2257	2381	2753	2939	3001	3125	3187
	3373	3559	3776	3807	3869	3931	4241	4489	4768
	4830	5016	5202	5326	5388	5512	5698	5760	6070
	6132	6256	6318	6690	6752	7062	7124	7186	7248
	7310	7372	7434	7558	7620				
26	181	243	367	491	677	739	863	1049	1297
	1483	1607	1669	1979	2351	2537	2909	2971	3343
	3467	3529	3901	4087	4211	4273	4397	5792	6102
	6164	6288	6350	6722	6784	7094	7218	7280	7404
	7590	7652	7714	7776	7962	8024	8148	8210	8334
	8396	8520	8582	8644	8768				
27	89	151	337	461	523	647	709	1019	1453
	1763	1949	2011	2197	2383	2693	2848	2879	3096
	3127	3220	3251	3282	3313	3406	3468	3499	3592
	3623	3840	3871	3964	4026	4057	4150	4181	4243
	4336	4398	4708	4770	4832	4956	5018	5142	5204
	5266	5328	5390	5452	5514				
28	59	121	307	431	617	1051	1237	1361	1423
	1609	1733	1888	1950	2043	2136	2260	2322	2477
	2508	2539	2570	2632	2663	2818	2911	3066	3190
	3221	3252	3283	3376	3407	3438	3469	3562	3593
	3779	3810	3872	3934	3996	4027	4058	4089	4120
	4182	4306	4337	4368	4430				

FIBRE	ORDERS OF SQUARES USED AS VALUES FOR T LISTS								
29	277	401	463	587	773	1021	1331	1517	1579
	1889	1951	2137	2447	2633	2819	2881	3067	3191
	3253	3811	3904	4183	4307	4493	4679	5051	8864
	8988	9050	9236	9360	9794	9918	9980	10104	10538
	10724	10910	11034	11096	11220	11406	11468	11654	11840
	12026	12150	12336	12584	12708				
30	61	433	557	619	743	929	991	1301	1487
	1549	1952	2045	2107	2138	2262	2293	2324	2417
	2479	2510	2572	2634	2789	2820	2851	3037	3068
	3192	3254	3347	3378	3440	3533	3564	3719	3812
	3936	3967	3998	4029	4091	4122	4153	4308	4370
	4680	4742	4928	5114	5238				

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