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Perfect Pair-Coverings with Block Sizes Two, Three, and Four

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We investigate the minimum number of blocks in a perfect pair-covering (pairwise balanced design) which contains only blocks of sizes 2, 3, and 4. This number is denoted by $g^{(4)}(v)$, where v is the number of points. We determine $g^{(4)}(v)$ for all positive integers v, with three exceptions:

$v = 17$, 18, and 19.

1. INTRODUCTION

Let X be a finite set. A *perfect covering* of X is a set $\mathcal{B}$ of subsets of X such that every pair of points $\{x_1, x_2\} \subseteq X$ occurs in a unique $B \in \mathcal{B}$. We refer to the members of X as *points* and the members of $\mathcal{B}$ as *blocks*. (A perfect covering is also referred to in the literature as a pairwise balanced design or a finite linear space.)

Stanton *et al* introduced the covering number $g^{(k)}(v)$, which denotes the minimum number of blocks in a perfect covering of a v-set, where k is the size of the longest block. In [4], $g^{(k)}(v)$ is determined for $k = 2$ and 3 and all v. Also, $g^{(k)}(v)$ is determined for all $k \geqslant v/2$ in [4]; and $g^{(k)}(2k + 1)$ was found in [3] and [4].

In this paper, we study $g^{(4)}(v)$. Complete results are given for all v with the exception of $v = 17$, 18, and 19.

2. LOWER BOUNDS FOR $g^{(4)}(v)$

Let $B_1, B_2, \ldots, B_g$ form a perfect covering of the v-set $\{1, 2, \ldots, v\}$, where each B_i has cardinality at most four. We will let $k_i = |B_i|$, $1 \leqslant i \leqslant g$. For $1 \leqslant i \leqslant v$, let r_i denote the number of blocks containing the point i. Since all blocks have size at most four, it is clear that $r_i \geqslant \left\lceil \frac{v-1}{3} \right\rceil$. Now

$$\sum_{i=1}^{g} k_i = \sum_{i=1}^{v} r_i = v\left\lceil \frac{v-1}{3} \right\rceil + \epsilon \text{ for some } \epsilon \geqslant 0$$

Also,

$$\sum_{i=1}^{g} 1 = g,$$

and

$$\sum_{i=1}^{g} k_i(k_i - 1) = v(v - 1).$$

Denote the number of blocks of size i by $g_i (i = 2, 3, 4)$. Then we obtain

$$2g_2 = \sum_{i=1}^{g} (k_i - 3)(k_i - 4)$$

$$= v(v-1) - 6\left(v\left\lceil \frac{v-1}{3} \right\rceil + \epsilon\right) + 12g.$$

Solving for g, we obtain

LEMMA 2.1. *In any perfect covering of a v-set in which the largest block has size four, we have*

$$g = \frac{v\left(6\left\lceil \frac{v-1}{3}\right\rceil - v + 1\right) + 6\epsilon + 2g_2}{12}$$

COROLLARY 2.2.

$$g^{(4)}(v) \geqslant \left\lceil \frac{v\left(6\left\lceil \frac{v-1}{3}\right\rceil - v + 1\right)}{12} \right\rceil$$

Proof. ϵ and g_2 are non-negative, and $g^{(4)}(v)$ is an integer. ∎

Any positive integer v can be written in the form $12t + \delta$, where $-1 \leqslant \delta \leqslant 10$ and δ and t are integers. We record the bound of Corollary 2.2, for v written in the above form, in Table 1. We denote this bound by $g_0(v)$.

TABLE 1

v	$g_0(v)$
$12t - 1$	$12t^2 + t$
$12t$	$12t^2 + t$
$12t + 1$	$12t^2 + t$
$12t + 2$	$12t^2 + 7t + 1$
$12t + 3$	$12t^2 + 7t + 1$
$12t + 4$	$12t^2 + 7t + 1$
$12t + 5$	$12t^2 + 13t + 4$
$12t + 6$	$12t^2 + 13t + 4$
$12t + 7$	$12t^2 + 13t + 4$
$12t + 8$	$12t^2 + 19t + 8$
$12t + 9$	$12t^2 + 19t + 8$
$12t + 10$	$12t^2 + 19t + 8$

For $v \equiv 7$ or 10 (mod 12), we are able to improve the bound of Table 1. Our proof is based on the packing number $D(2, 4, v)$, which denotes the maximum number of blocks of size four, chosen from a v-set so that no pair of points occurs in more than one block. It is well-known, for $v \equiv 7$ or 10 (mod 12), that

$$D(2, 4, v) \leqslant \left\lfloor \frac{v}{4} \left\lfloor \frac{v-1}{3} \right\rfloor \right\rfloor - 1$$

(see, for example, [1]).

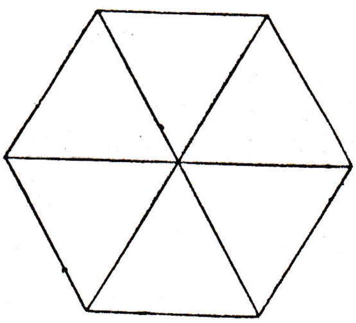
LEMMA 2.3. *For* $v \equiv 7$ *or* 10 (mod 12), $g^{(4)}(v) \geqslant g_0(v) + 3$.

Proof. We describe the proof for $v \equiv 7$ (mod 12); it is similar for $v \equiv 10$ (mod 12). Notation is as before. We have

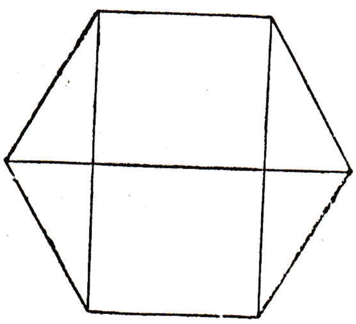
$$g_4 \leqslant D(2, 4, v) \leqslant \left\lfloor \frac{v}{4} \left\lfloor \frac{v-1}{3} \right\rfloor \right\rfloor - 1 = 12t^2 + 13t + 2.$$

The number of pairs not covered in the blocks of size four is $\binom{v}{2} - 6g_4$. In the graph formed by these pairs, every vertex has valence divisible by three.

If $g_4 = 12t^2 + 13t + 2$, we obtain a cubic graph on six vertices. There are two such graphs:



and



G_1 G_2

To cover these pairs would require nine blocks, for G_1, or five blocks, for G_2. Hence $g \geqslant 12t^2 + 13t + 7 = g_0(v) + 3$.

If $g_4 = 12t^2 + 13t + 1$, we have fifteen pairs not covered by the blocks of size four. If $g < 12t^2 + 13t + 7$, then we must be able to cover these fifteen pairs by at most five blocks. Thus we must have precisely five blocks of size three. But there is no way that five edge-disjoint triangles can be combined to form a graph in which every vertex has valence divisible by three. Thus $g \geqslant 12t^2 + 13t + 7$ in this case, as well.

Finally, assume $g_4 \leqslant 12t^2 + 13t$. We have at least 21 pairs not yet covered, which requires at least seven more blocks. Hence $g \geqslant 12t^2 + 13t + 7$.

Since we have covered all cases, we have

$$g^{(4)}(12t + 7) \geqslant 12t^2 + 13t + 7 = g_0(v) + 3. \blacksquare$$

3. Upper Bounds

Lemma 3.1. *For* $v \equiv -1, 0, 1, 2, 3,$ *or* $4 \pmod{12}$ $g^{(4)}(v) = g_0(v)$.

Proof. Delete zero, one, or two points from a *BIBD* $(12t + 4, 4, 1)$ or a *BIBD* $(12t + 1, 4, 1)$. ∎

Lemma 3.2. *For* $v \equiv 7$ *or* $10 \pmod{12}$, $v \neq 7, 10, 19$, $g^{(4)}(v) = g_0(v) + 3$.

Proof. For $v \equiv 7$ or $10 \pmod{12}$, $v \neq 10, 19$, Brouwer [1] has constructed a perfect covering with one block of seven and $g_0(v) - 4$ blocks of size four. Replace the block of size seven by a Fano geometry (seven blocks of size three), thus constructing a perfect covering with $g_0(v) + 3$ blocks. (We cannot do this for $v = 7$, for the resulting covering would contain no block of size four.) Lemma 2.2 proves that $g^{(4)}(v) \geqslant g_0(v) + 3$; hence, we have equality. ∎

Lemma 3.3. *For* $v \equiv 5, 6, 8,$ *or* $9 \pmod{12}$, $v \geqslant 20$, $g^{(4)}(v) = g_0(v)$.

Proof. For all $v \equiv 7$ or $10 \pmod{12}$, $v \geqslant 22$, Mills [2] has shown that there exists a collection C of $g_0(w)$ blocks of size 4 which contain one pair four times, and all other pairs once. Let xy be the repeated pair. If x is deleted from all blocks of C containing it, then we obtain $g_0(w - 1) = g_0(w)$ blocks which form a perfect covering of the $w - 1$ points other than x. If any other point z is deleted from all blocks containing it in this resulting configuration, we have $g_0(w - 2) = g_0(w)$ blocks which form a perfect covering of the $w - 2$ points other than x and z.

Hence $g^{(4)}(v) \leqslant g_0(v)$ for the stated v; of course $g^{(4)}(v) \geqslant g_0(v)$. ∎

There are nine small values of v not covered by the above results: $v = 5$, 6, 7, 8, 9, 10, 17, 18, and 19. In [4], $g^{(4)}(v)$ is found for all $v \leqslant 12$. The results can be summarized as

Lemma 3.4. $g^{(4)}(5) = 5$, $g^{(4)}(6) = 8$, $g^{(4)}(7) = 10$, $g^{(4)}(8) = 11$, $g^{(4)}(9) = 12$, and $g^{(4)}(10) = 12$.

Thus we have determined $g^{(4)}(v)$ for all $v \neq 17, 18, 19$. These last three values are currently under investigation; we know that $g^{(4)}(17) \geqslant 30 = g_0(17) + 1$.

We summarize our results in tabular form.

TABLE 2

V	$g^{(4)}(v)$	exceptions
$12t-1$	$12t^2+t$	
$12t$	$12t^2+t$	
$12t+1$	$12t^2+t$	
$12t+2$	$12t^2+7t+1$	
$12t+3$	$12t^2+7t+1$	
$12t+4$	$12t^2+7t+1$	
$12t+5$	$12t^2+13t+4$	$g^{(4)}(5)=5$, $g^{(4)}(17)$ not known
$12t+6$	$12t^2+13t+4$	$g^{(4)}(6)=8$, $g^{(4)}(18)$ not known
$12t+7$	$12t^2+13t+7$	$g^{(4)}(7)=10$, $g^{(4)}(19)$ not known
$12t+8$	$12t^2+19t+8$	$g^{(4)}(8)=11$
$12t+9$	$12t^2+19t+8$	$g^{(4)}(9)=12$
$12t+10$	$12t^2+19t+11$	$g^{(4)}(10)=12$

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