A SHORT PROOF OF A THEOREM OF DE WITTE

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Abstract

de Witte has proved that a finite linear space on v points, other than a near-pencil, where $n^2 \leq v \leq n^2+n+1$, can be embedded into a projective plane of order n if and only if the number of lines $b \leq n^2+n+1$. We give a short proof of this result.

1. *Introduction.*

A *finite linear space* (or FLS) is a pair $(X,\mathcal{B})$ where X is a finite set and $\mathcal{B}$ is a set of proper subsets of X, each of size at least two, such that every unordered pair $\{x_1,x_2\} \subseteq X$ is contained in precisely one $B \in \mathcal{B}$. Elements of X are called *points* and members of $\mathcal{B}$ are called *lines*. We usually denote the number of points by v and the number of lines by b. The *length* of a line is the number of points it contains; the *degree* of a point is the number of lines on which it lies.

A *projective plane* of order n is an FLS with n^2+n+1 points and n^2+n+1 lines, in which every line has length $n+1$ and every point has degree $n+1$.

A *near-pencil* is an FLS with one line of length $v-1$ and $v-1$ lines of length two, where v is the number of points.

The following is a fundamental result of de Bruijn and Erdos [1].

THEOREM 1.1. *A finite linear space has at least as many lines as points, with equality occurring if and only if the space is either a projective plane or a near-pencil.*

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Of course, there is a near-pencil on v points for all $v \geq 3$. A near-pencil is a very uninteresting FLS, and henceforth we consider only FLS that are not near-pencils, i.e. *non-degenerate* FLS. We abbreviate the term non-degenerate FLS to NLS.

Let $F_i = (X_i, \mathcal{B}_i)$ be FLS, $i = 1,2$. We say that F_1 is *embedded* in F_2 if and only if $X_1 \subseteq X_2$ and $\mathcal{B}_1 = \{B \cap X_1 : B \in \mathcal{B}_2 \text{ and } |B \cap X_1| \geq 2\}$. A basic question concerning NLS is "when can an NLS be embedded in a projective plane of order n?" Clearly, we must have $v \leq b \leq n^2+n+1$. In an unpublished manuscript, de Witte [5] showed that for $n^2 \leq v$, the above necessary condition is also sufficient. The purpose of this paper is to give a short proof of de Witte's result.

We will require the idea of an (r,1)-design. An (r,1)-design is a pair $(X,\mathcal{B})$ where X is a finite set (of *points*) and $\mathcal{B}$ is a family of subsets of X (called *blocks*), which satisfies

(1) every point occurs in precisely r blocks

(2) every pair of points occurs in a unique block.

We use v and b to denote respectively the number of points and blocks. Note that one can obtain an FLS from an (r,1)-design by deleting all blocks of length one, and conversely, the addition of sufficiently many blocks of length one to an FLS will produce an (r,1)-design for some r. Note also that an (r,1)-design may contain repeated blocks of length one.

Vanstone [3,4] and de Witte [5] proved the following embedding result for (n+1,1)-designs.

THEOREM 1.2. *An $(n+1,1)$-design with $n^2-1 \leq v \leq n^2+n+1$ can be embedded into a projective plane of order n.*

2. *The Proof.*

In this section we prove the main result. The following simple Lemma will be used several times in the proof.

LEMMA 2.1. *Suppose* F *is an NLS which satisfies:*

(1) $b = n^2+n+1$

(2) every point of F *has degree at least* $n+1$

(3) there is a line of F *of length* $n+1$.

Then F *is an* $(n+1,1)$*-design.*

Proof. Let B be the longest line. Counting lines that meet B, we obtain

$$n^2+n+1 \geq 1+n|B|,$$

or
$$|B| \leq n+1.$$

But $|B| \geq n+1$, so $|B| = n+1$. Also, B meets all lines, and every point of B has degree precisely $n+1$. If a point x not on B has degree exceeding $n+1$, then there is a line through x disjoint from B, an impossibility. Thus every point has degree precisely $n+1$. □

LEMMA 2.2. *If an NLS has* $n^2+2 \leq v \leq n^2+n+1$ *then* $b \geq n^2+n+1$. *Equality can occur only if the longest line has length* $n+1$.

Proof. Erdős et al [2].

LEMMA 2.3. *An NLS with* $n^2+2 \leq v \leq n^2+n+1$ *can be embedded into a projective plane of order* n *if and only if* $b = n^2+n+1$.

Proof. In view of Lemma 2.1, the only if portion is clear. Thus, assume $b = n^2+n+1$.

If some point x has degree at most n, then the average length of the lines through x is at least $1 + \frac{v-1}{n} > n+1$. This contradicts Lemma 2.2.

Thus every point has degree at least $n+1$. By Lemma 2.1, this NLS is an $(n+1,1)$-design, and can be embedded in a projective plane of order n by Theorem 1.2. □

It remains to consider the cases $v = n^2$ and $v = n^2+1$.

LEMMA 2.4. *If an NLS has* $v = n^2$ *or* $n^2 + 1$, *then* $b \geq n^2+n$. *Further,* $b > n^2+n+1$ *unless the longest line has length* n *or* $n+1$.

Proof. See [2]. □

LEMMA 2.5. *Suppose* F *is an NLS with* $v = n^2$ *or* $n^2 + 1$, *in which no line has length exceeding* $n+1$. *Then every point has degree at least* n. *Further if* x *has degree* n, *then* x *is the only point of degree* n, *and all lines of length* $n+1$ *pass through* x.

Proof. The longest line has length at most $n+1$, so any point has degree at least $\frac{v-1}{n} \geq n$.

Suppose x has degree n. Then any line not containing x has length at most n, so all lines of length $n+1$ pass through x. Let y be any point different from x, and suppose y has degree r. Counting points on lines through y, we obtain $v-1 \leq n+(r-1)(n-1)$. Thus $r \geq n+1$, and x is the only point of degree n. □

LEMMA 2.6. *An FLS with* $v = n^2$ *or* $n^2 + 1$ *and* $b = n^2+n$ *can be embedded in a projective plane of order* n.

Proof. By Lemma 2.4, the longest line has length n or $n+1$.

First, suppose the longest line has length n. Since all pairs must be covered, we must have $\binom{v}{2} \leq \binom{n}{2}(n^2+n)$. Hence $v = n^2$ and all lines have length n. The FLS is thus an affine plane and can be embedded in the projective plane of order n.

Now suppose the longest line has length $n+1$, and let B be a line of length $n+1$. At most one point of B has degree n, and all other point on B have degree at least $n+1$ (Lemmata 2.4 and 2.5). We count lines that meet B: $n^2+n \geq 1+n-1 + n\cdot n = n^2+n$. Thus B contains a point x of degree n, and all other points in the FLS have degree exactly $n+1$. If we adjoin $\{x\}$ to our FLS, we obtain an $(n+1,1)$-design with $v \geq n^2$ and $b = n^2+n+1$. We can embed this $(n+1,1)$-design in a projective plane of order n; hence the FLS can be embedded also. □

LEMMA 2.7. *An NLS with* $v = n^2+1$ *and* $b = n^2+n+1$ *can be embedded in a projective plane of order* n.

Proof. First, suppose there is a point x of degree n. On deleting x, we obtain an NLS with $v = n^2$, $b = n^2+n+1$, in which the longest line has length n. We will show that no such FLS exists (Lemma 2.9).

Thus, assume that every point has degree at least $n+1$. If there is a line of length $n+1$, then we have an $(n+1,1)$-design (Lemma 2.1), which can be embedded in a projective plane of order n. Hence, we can assume that the longest line has length n. Then every point has degree at least $\left\lceil \frac{n^2}{n-1} \right\rceil = n+2$. Then $b \geq \left\lceil \frac{(n^2+1)(n+2)}{n} \right\rceil > n^2+n+1$, a contradiction. This completes the proof. □

It remains to consider NLS with $v = n^2$ and $b = n^2+n+1$. We first consider the case when there is a point of degree n.

LEMMA 2.8. *An NLS with* $v = n^2$ *and* $b = n^2+n+1$, *which has a point of degree* n, *can be embedded in a projective plane of order* n.

Proof. Suppose F is such an NLS, and ∞ has degree n. Every point other than ∞ has degree at least $n+1$ (Lemma 2.5). If no point has degree exceeding $n+1$, then delete ∞ to obtain an $(n+1,1)$-design F' with at least n^2-1 points. By Theorem 1.2, F' can be embedded in projective plane of order n, and the point ∞ will be restored in this embedding. Hence F can be so embedded.

Thus we can assume that there is a point y of degree at least $n+2$. The point ∞ occurs on $n-1$ lines of length $n+1$ and one line of length n. We have $n \geq 3$ since the only NLS on four points has six lines. Thus there is a line B_0 of length $n+1$ with $y \in B_0$. There is a line B_1 through y and disjoint from B_0. If z is any point on B_1, then there are $n+1$ lines joining z to B_0, so z has degree at least $n+2$.

Now, the number of lines meeting B_0 is at least $1+n-1+n\cdot n = n^2+n$. Thus B_1 is the only line disjoint from B_0, every point on B_1 has degree $n+2$, and every other point (except ∞) has degree $n+1$. A point on $B_0\backslash\{\infty\}$ occurs on one line of length $n+1$, $n-1$ lines of length n, and one line of length $n-1$.

Let r_x denote the degree of a point x; and let C denote the set consisting of the $n-1$ lines of length $n+1$, the n lines of length $n-1$ that meet B_0, and B_1. Thus $|C| = 2n$ and $\sum_{B\in C} |B| = 2n^2-n-1+|B_1|$. We have $\sum_x r_x = \sum_B |B| = n^2(n+1)+|B_1|-1 = n^3+n^2-1+|B_1|$. The average length of the lines not in C is at least

$$\frac{n^3+n^2-1+|B_1|-(2n^2-n-1+|B_1|)}{n^2+n+1-2n} = n.$$

Since all the lines of length $n+1$ are in C, we have $|B| = n$ for all $B \notin C$.

Now we count pairs:

$$\binom{n^2}{2} = \sum_B \binom{|B|}{2} = (n-1)\binom{n+1}{2} + (n^2-n+1)\binom{n}{2} + n\binom{n-1}{2} + \binom{|B_1|}{2}.$$

This reduces to $\binom{|B_1|}{2} = 0$ so $|B_1| = 0$ or 1. But $|B_1| \geq 2$, so we have a contradiction. □

LEMMA 2.9. *There is no NLS with* $v = n^2$ *and* $b = n^2+n+1$, *in which the longest line has length* n.

Proof. If every line is of length n, then $b = \binom{n^2}{2}/\binom{n}{2} = n^2+n$, as contradiction. Thus there is some line B_0 of length $k < n$. A point has degree $n+1$ if and only if it occurs on $n+1$ lines of length n. Thus any point on B_0 has degree at least $n+2$. Now $\sum_x r_x = \sum_B |B| \geq n^2(n+1)+k$. If we calculate the average length of the lines other than B_0, we obtain

$$n \geq \frac{n^2(n+1)+k-k}{n^2 + n} = n.$$

Thus our NLS contains n^2+n lines of length n and one line of length k. Counting pairs, we find $k = 0$ or 1, a contradiction. □

LEMMA 2.10. *An NLS with* $v = n^2$ *and* $b = n^2+n+1$ *can be embedded in a projective plane of order* n.

Proof. Lemma 2.8 handles the case when there is a point of degree n. Thus we may assume every point has degree at least $n+1$. If there is a line of length $n+1$, then we have an $(n+1,1)$-design (Lemma 2.1), which can be embedded in the projective plane by Theorem 1.2. Otherwise, the longest line has length n, and Lemma 2.9 shows that no such NLS exists. □

Summarizing, we have the main Theorem.

THEOREM 2.11. *A non-degenerate finite linear space with* v *points* $(n^2 \leq v \leq n^2+n+1)$ *can be embedded in a projective plane of order* n *if and only if there are at most* n^2+n+1 *lines.*

Proof. Lemmata 2.3, 2.4, 2.6, 2.7, and 2.10. □

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