APPLICATIONS AND GENERALIZATIONS OF THE VARIANCE METHOD IN COMBINATORIAL DESIGNS.

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1. *Introduction - An Inequality.*

Many well-known inequalities for designs can be proved by the method of "intersection numbers". This method is based on the fact that a sum of squares is non-negative. We prove a single inequality, and derive several other inequalities as corollaries. In later sections, we consider extensions of this method.

A pairwise balanced design, or PBD, of index λ, is a pair $(X,\mathcal{B})$, where X is a finite set, and $\mathcal{B}$ is a family of subsets of X, such that, for each unordered pair $\{x_1,x_2\} \subseteq X$, there are precisely λ $B \in \mathcal{B}$ with $\{x_1,x_2\} \subseteq B$. Elements of X are called points, and elements of $\mathcal{B}$ are called blocks. We will denote the number of points by v, and the number of blocks by b. Notice that we allow both repeated blocks and blocks of size 1.

An (r,λ)-design) is a PBD of index λ in which every point occurs in precisely r blocks. A balanced incomplete block design (or BIBD) is an (r,λ)-design in which every block has size k, for some constant $k < v$. The parameters of a BIBD are usually written (v,b,r,k,λ). The relations $bk=vr$ and $\lambda(v-1) = r(k-1)$ follow easily.

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We now prove our inequality. This inequality was established for $(r,1)$-designs in [9], using the same method of proof as is used here. An alternate proof is given in [6] (for (r,λ)-designs).

THEOREM 1.1. *Suppose a PBD* $(X,\mathcal{B})$ *of index* λ *has* v *points and* b *blocks, and every point occurs in at least* r *blocks. Then* $b \geq \frac{r^2 v}{\lambda v + r - \lambda}$. *Furthermore, equality occurs if and only if* $(X,\mathcal{B})$ *is a BIBD.*

Proof. Let k_i, for $1 \leq i \leq b$, denote the size of the ith block. The following summations are over the block set $\mathcal{B}$:

$$\Sigma 1 = b,$$

$$\Sigma k_i \geq vr,$$

and

$$\Sigma \binom{k_i}{2} = \lambda \binom{v}{2};$$

thus

$$\Sigma k_i^2 \leq v(\lambda v + r - \lambda).$$

If $\bar{k}$ denotes the mean of the k_i's, then the variance is

$$\Sigma (k_i - \bar{k})^2 = \Sigma k_i^2 - \frac{1}{b}(\Sigma k_i)^2 \geq 0$$

Thus

$$b \geq \frac{(vr)^2}{v(\lambda v + r - \lambda)} = \frac{r^2 v}{\lambda v + r - \lambda}$$

Also, equality occurs if and only if

$$k_1 = \ldots = k_b = \bar{k} = \frac{vr}{b}. \quad \square$$

2. *Consequences.*

In this section we derive several inequalities as corollaries to the inequality proved in the introduction. For the remainder of this section, let $f(r,\lambda,v) = \frac{r^2 v}{\lambda v + r - \lambda}$.

First, we prove Fisher's inequality [3].

COROLLARY 2.1. *In a BIBD* $b \geq v$. *Further,* $b = v$ *if and only if any two blocks have* λ *common points.*

Proof. Let B_0 be any block. Then $(B_0, \{B \cap B_0 : B \in \mathcal{B}, B \neq B_0\})$ is an $(r-1, \lambda-1)$-design with k points and $b-1$ blocks. Thus $b-1 \geq f(r-1, \lambda-1, k)$ or $k(b-1)(\lambda-(k-1)+r-k)-k^2(r-1)^2 \geq 0$. Using the basic relations, the left side equals $(r-k)(v-k)(r-\lambda)$. We have $v > k$ and $r > \lambda$; thus $r \geq k$. Since $vr = bk$, we obtain $b \geq v$. In the case of equality, every block meets B_0 in $\frac{k(r-1)}{b-1} = \frac{k(k-1)}{v-1} = \lambda$ points. Since B_0 was an arbitrary block, the result follows. □

Our next inequality is due to Stanton and Sprott [13]. (See also Bose [1].)

A parallel class in a PBD is a set of blocks which forms a partition of the point set. An (r,λ)-design is resolvable if its blocks may be partitioned into parallel classes. A resolvable BIBD is called affine resolvable if any two blocks contain either zero or k^2/v common points.

COROLLARY 2.2. *If a BIBD* $(X,\mathcal{B})$ *contains a parallel class* $\mathcal{P}$ *then* $b \geq v+r-1$. *Further, a resolvable BIBD is affine resolvable if and only if* $b = v+r-1$.

Proof. Let $\mathcal{P} \subseteq \mathcal{B}$ be a parallel class, and let B_0 be any block in $\mathcal{P}$. Then $(B_0, \{B \cap B_0 : B \in \mathcal{B} \setminus \mathcal{P}\})$ is an $(r-1, \lambda-1)$-design with k points and $b - v/k$ blocks. Thus $b-v/k \geq f(r-1, \lambda-1, k)$. This reduces to $b \geq v+r-1$ (see [13]). For equality, B_0 meets every block of $\mathcal{B}\backslash\mathcal{P}$ in $\frac{k(r-1)}{b-v/k} = \frac{k^2}{v}$ points. □

Next, we consider the situation of PBDs of index 1. Let us denote by $g(k,v)$ the minimum number of blocks in a PBD (of index 1) which has v points and contains a block of size k. The following was proved in [12].

COROLLARY 2.3. $$g(k,v) \geq 1 + \frac{k^2(v-k)}{v-1}.$$

Proof. Let $(X,\mathcal{B})$ be a PBD of index 1 on v points, with $B_0 \in \mathcal{B}$ having size k. The PBD $(X\backslash B_0, \{B\backslash B_0 : B \in \mathcal{B}, B \neq B_0\})$ has $v-k$ points, $b-1$ blocks and is of index 1. Further every point occurs in at least k blocks. Thus $b-1 \geq f(k,1,v-k)$, and the result follows. □

A well-known Theorem of de Bruijn and Erdős [2] states that, in a PBD of index 1, $b \geq v$ with equality if and only if the PBD is a projective plane or a near-pencil ($v-1$ blocks of size 2 and one block of size $v-1$). Using Corollary 2.3, Stanton and Kalbfleisch [12] give a very concise proof of this result (see also [11]).

We now turn to a problem which has received considerable attention recently. Let S be a set of size kn. Define a (k,n)-round to be a partition of S into k blocks of size n. Denote the minimum number of pairs common to two (k,n)-rounds by $\sigma(k,n)$. The Cordes problem is to determine the maximum number $R(k,n)$ of (k,n)-rounds, any two of which contain $\sigma(k,n)$ pairs in common. McCarthy and van Rees [5] obtain an upper bound for $R(k,n)$. More recently Mullin et al [7] investigated a more general form of the

Cordes problem, and proved the result of Corollary 2.4. (The bound of McCarthy and van Rees is an immediate consequence of this result.)

COROLLARY 2.4. *Let* G *be a finite graph having* E *edges. Suppose* $H_1,\dots,H_\delta$ *are subgraphs of* G, *each having* e *edges, such that* H_i *and* H_j *have* θ *common edges for all* $i \neq j$. *Then* $\delta \le \frac{E(e-\theta)}{e^2-E\theta}$.

Proof. Let X denote the set of edges of G; for $1 \le i \le \delta$, let $B_i = \{x \in H_i : x \in X\}$; and let $\mathcal{B} = \{B_i; 1 \le i \le \delta\}$. $\mathcal{B}$ is a set of δ e-subsets of the E-set X, any two of which have θ points (i.e. edges of X) in common. $(X,\mathcal{B})$ is the dual of an (e,θ)-design with $v = \delta$, $b = E$. Thus $E \le \frac{e^2\delta}{\theta\delta+e-\theta}$. Solving for δ, the result follows. □

3. *Extensions.*

Let us consider the equations used to prove Theorem 1.1 : $\Sigma 1 = b$, $\Sigma k_i \ge rv$, and $\Sigma(k_i^2-k_i) = \lambda v(v-1)$. Let ℓ be an integer; then

$$\begin{aligned} 0 &\le \Sigma(k_i-\ell)(k_i-\ell-1) \\ &= \Sigma(k_i^2-k_i) - 2\ell\Sigma k_i + (\ell^2+\ell)b \\ &\le \lambda v(v-1) - 2\ell rv + (\ell^2+\ell)b . \end{aligned}$$

Thus we have

THEOREM 3.1. *Suppose a PBD of index* λ *has* v *points,* b *blocks, and every point occurs in at least* r *blocks. If* ℓ *is any integer then* $b \ge v\,\frac{2\ell r-\lambda(v-1)}{\ell^2+\ell}$. *Further, equality occurs if and only if every block has size* ℓ *or* $\ell+1$, *and every point occurs* *in precisely* r *blocks.*

Let us define $h(r,\lambda,v,\ell) = v\left(\frac{2\ell r-\lambda(v-1)}{\ell^2+\ell}\right)$. We have $b \ge h(r,\lambda,v,\ell)$; for fixed r,λ, and v, we wish to maximize $h(r,\lambda,v,\ell)$. We compute $h(r,\lambda,v,\ell) - h(r,\lambda,v,\ell-1) = \frac{2\ell v}{\ell^2(\ell^2-1)}[\lambda(v-1)-(\ell-1)r]$. Thus h is maximized, for fixed r,λ, and v, by letting $\ell = \left\lfloor\frac{\lambda(v-1)+r}{r}\right\rfloor$.

THEOREM 3.2. *If* $\ell = \frac{\lambda(v-1)+r}{r}$, *then* $h(r,\lambda,v,\ell) \ge f(r,\lambda,v)$.

Proof. Let us consider the inequality

$$v\left(\frac{2\ell r-\lambda(v-1)}{\ell^2+\ell}\right) \ge \frac{r^2 v}{\lambda(v-1)+r} .$$

This reduces to

$$r^2\ell^2 - (2\lambda r(v-1)+r^2)\ell + \lambda^2(v-1)^2 + r\lambda(v-1) \le 0 .$$

The left side is a quadratic in ℓ, having roots

$$\frac{2\lambda r(v-1)+r^2 \pm \sqrt{(2\lambda r(v-1)+r^2)^2-4r^2(\lambda^2(v-1)^2+r\lambda(v-1))}}{2r^2}$$

$$= \frac{2\lambda r(v-1) + r^2 \pm r^2}{2r^2}$$

$$= \lambda\left(\frac{v-1}{r}\right) \quad \text{or} \quad \frac{\lambda(v-1)+r}{r},$$

However $\frac{\lambda(v-1)}{r} \le \left\lfloor\frac{\lambda(v-1)+r}{r}\right\rfloor \le \frac{\lambda(v-1)+r}{r}$, so the result is obtained. □

Thus Theorem 3.1 is an improvement of Theorem 1.1. As before, we can obtain a lower bound for $g(k,v)$. The proof is the same as that of Corollary 2.3.

COROLLARY 3.3. *For any integer* ℓ, $g(k,v) \geq 1 + (v-k)\left(\frac{2\ell k-(v-k-1)}{\ell^2+\ell}\right)$. *The optimum* ℓ *is*

$$\left\lfloor \frac{1(v-k-1)+k}{k} \right\rfloor = \left\lfloor \frac{v-1}{k} \right\rfloor .$$

Woodall [14] first proved the following inequality, which is obtained by setting $\ell=1$ in Corollary 3.3.

COROLLARY 3.4. $g(k,v) \geq 1+\frac{(v-k)(3k-v+1)}{2}$.

In [10], Stanton et al establish that, for $v+1 \leq k \leq 2v$, $g(k,v) = 1 + \frac{(v-k)(3k-v+1)}{2}$. (These are precisely the values for which $\left\lfloor \frac{v-1}{k} \right\rfloor = 1$.) In the same paper, it is shown that, for k odd, $g(k,2k+1) = 1 + \frac{k(k+1)}{2}$. In [8], it is proven that, for k even ($k > 2$), $g^{(k)}(v) = 1 + \frac{k(k+1)}{2} + \left\lceil \frac{k}{4} \right\rceil$. The next cases are $2k+2 \leq k \leq 3k$. Here $\ell = 2$ gives the optimum value, and Corollary 3.3 yields $g(k,v) \geq 1 + \frac{(v-k)(5k-v+1)}{6}$. Preliminary investigation suggests that this bound can usually be attained. We hope to report further in a future paper.

We finish this section by proving a result using Theorem 3.1. It is well-known (see [4]) that a BIBD$(X,\mathcal{B})$ with parameters $\left(\frac{k^2+k}{2}, \frac{k^2+3k+2}{2}, k+2, k, 2\right)$ can be embedded into a BIBD with parameters $\left(\frac{k^2+3k+4}{2}, \frac{k^2+3k+4}{2}, k+2, k+2, 2\right)$. (We say that such a BIBD is quasi-residual, with $\lambda = 2$.) The first step in the proof is to show that any two blocks have one or two points in common. We can obtain this result as a corollary to Theorem 3.1.

COROLLARY 3.5. *In a quasi-residual BIBD with* $\lambda = 2$, *any two blocks meet in either one or two points.*

Proof. Let B_0 be any block. Then $(B_0, \{B \cap B_0 : B \in \mathcal{B}, B \neq B_0\})$ is a (k+1,1)-design with k points and $\frac{k^2+3k+2}{2}$ blocks. On setting $\ell = 1$, we have $\frac{k^2+3k+2}{2} \geq k\left(\frac{2(k+1)-(k-1)}{2}\right) = \frac{k^2+3k+2}{2}$. Thus B_0 meets every other block in one or two points. Since B_0 was arbitrary, the result follows. □

Finally, we consider t-wise balanced designs ($t \geq 2$). A t-wise balanced design (of index 1) is a pair $(X,\mathcal{B})$, where $\mathcal{B}$ is a family of proper subsets of X, such that each t-subset of X is contained in a unique block. As before, v and b will (respectively) denote the number of points and blocks. The term t-wise balanced design will be abbreviated to tBD. We will denote by $g(t,k,v)$ the minimum number of blocks in any tBD on v points, where one block has size k.

Let B_0 be a block of size k in a tBD $(X,\mathcal{B})$. Let $\mathcal{B}_1 \subseteq \mathcal{B}$ denote the blocks which have t-1 points in common with B_0; denote $\alpha = |\mathcal{B}_1|$. The following summations are over $\mathcal{B}_1$ (k_i denotes the length of B_i, where $\mathcal{B}_1 = \{B_i : 1 \leq i \leq \alpha\}$):

$$\Sigma 1 = \alpha$$

$$\Sigma(k_i-t+1) = \binom{k}{t-1}(v-k)$$

and

$$\Sigma\binom{k_i-t+1}{2}(t-1) \leq \binom{k_i}{t-2}\binom{v-k}{2}.$$

The second equation is obtained by counting t-subsets of X which meet B_0 in t-1 points. The third equation is obtained by counting t-subsets which contain t-2 points of B_0 (not all such t-subsets need be contained in the blocks of $\mathcal{B}_1$; hence the inequality). Let us denote $\ell_i = k_i-t+1$, $1 \leq i \leq \alpha$. If we calculate the variance of the ℓ_i's, we obtain the following inequality due to Stanton and Kalbfleisch [12].

THEOREM 3.6. $$g(t,k,v) \geq 1 + \frac{k-t+2}{v-t+1}\binom{k}{t-1}(v-k).$$

If, however, we observe that $0 \leq \Sigma(\ell_i-\ell)(\ell_i-\ell-1)$, for any integer ℓ, we obtain

THEOREM 3.7. *For any integer* ℓ,

$$g(t,k,v) \geq 1 + \frac{v-k}{\ell^2+\ell}\binom{k}{t-1}\left(2\ell+1 - \frac{v-t+1}{k-t+2}\right).$$

It is easy to show that the optimum ℓ is given by $\left\lfloor \frac{v-t+1}{k-t+2} \right\rfloor$. As before we have

THEOREM 3.8. *For* $\ell = \left\lfloor \frac{v-t+1}{k-t+2} \right\rfloor$, *the bound of Theorem 3.7 is greater than or equal to the bound of Theorem 3.6.*

We also observe that Woodall's bound [14] can be obtained from Theorem 3.7 by setting $\ell=1$.

THEOREM 3.9. $$g(t,k,v) \geq 1 + (v-k)\binom{k}{t-1}\left(1 - \frac{v-k-1}{2(k-t+2)}\right).$$

Finally, it is not difficult to see when equality occurs in Theorem 3.8.

4. *Summary*.

We have attempted to unify some well-known inequalities for combinatorial designs. This can be accomplished since the proofs use a common technique, the calculation of a variance. This idea can be generalized by investigating other quadratic functions. Improved bounds are obtained in this way.

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