PERFECT PAIR-COVERINGS AND AN ALGORITHM FOR CERTAIN (1-2) FACTORIZATIONS OF THE COMPLETE GRAPH K_{2s+1}

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1. *Introduction.*

Let X be a finite set. A family $\tilde{B}$ of subsets of X is said to be a perfect covering (or simply a covering) provided that each pair of elements of X occurs in a unique set in the family $\tilde{B}$ (a perfect covering is also called a finite linear space or a pairwise balanced design). Elements of X are called points, and elements of B are called blocks.

In [1] the following problem was introduced. For $2 \leq k \leq v$, k and v integers, define $g^{(k)}(v)$ to be the least integer b such that there exists a perfect covering of a v-set which possesses b blocks and in which the largest block has length k. For $k \geq v/2$, $g^{(k)}(v)$ was determined in [1]; we quote the result as

THEOREM 1.1. *If* $k \geq v/2$, *then*

$$g^{(k)}(v) = 1 + (v-k)(3k-v+1)/2 .$$

Also, if $k = (v-1)/2$ is odd, then [1] gave the result for $g^{(k)}(v)$ as

THEOREM 1.2. *If* k *is odd, then*

$$g^{(k)}(2k+1) = 1 + k(k+1)/2 .$$

For k even, the bound given in Theorem 1.2. can not be attained (cf. [1], Lemma 5.2). In this paper, we prove that, for k even and $k > 2$, then

$$g^{(k)}(2k+1) = 1 + k(k+1)/2 + \lceil k/4 \rceil ,$$

where $\lceil \ \rceil$ denotes the usual ceiling function.

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2. *A Lower Bound.*

Define

$$SK(k,v) = 1 + k^2(v-k)/(v-1) .$$

In [2], it is shown that

$$g^{(k)}(v) \geq \lceil SK(k,v) \rceil .$$

We now improve this bound slightly in the case when $(v-1)/k = t$, where t is an integer.

THEOREM 2.1. *Suppose* $v-1 = kt$ *and* $k-1 = ut + w$, *where* $t,u,$ *and* w *are integers and* $0 \leq w < t$. *Then*

$$g^{(k)}(v) \geq 1 + \left\lceil \frac{k^2(v-k)}{v-1} + \frac{wk^3}{(v-1)^2} \right\rceil .$$

Proof. Let $\tilde{B}$ be a covering of a v-set X which contains b blocks and includes a block B_0 of length k. Let $\tilde{B}_1$ be the family formed by deleting B_0 and all points on B_0. Let the blocks of $\tilde{B}_1$ be

$$\tilde{B}_1 = \{B_1,\ldots,B_{b-1}\},$$

and let the length of B_i be k_i $(1 \leq i \leq b-1)$.

The following summations are from 1 to b-1. Clearly,

$$\Sigma 1 = b - 1,$$

and

$$\Sigma k_i(k_i-1) = (v-k)(v-k-1).$$

Also, since every point of $X\backslash B_0$ occurs (in $\tilde{B}$) on a block with every point of B_0, we have

$$\Sigma k_i \geq k(v-k).$$

Actually, it is easy to deduce from Theorem 1 of [3] that $\Sigma k_i = k(v-k) + \Sigma' k_i$, where $\Sigma' k_i$ denotes summation over those blocks of $\tilde{B}_1$ which are also blocks of $\tilde{B}$ (that is, which are disjoint to B_0).

Now we calculate

$$(1) \quad \Sigma(k_i-t)^2 \leq (v-k)(v-k-1) - (2t-1)k(v-k) + t^2(b-1)$$
$$= (v-k)(v-1-2tk) + t^2(b-1) .$$

Since $\tilde{B}_1$ was constructed from $\tilde{B}$, we have k subsets P_j of disjoint blocks in $\tilde{B}_1$ $(1 \le j \le k)$ such that each point of $X \backslash B_0$ occurs in precisely one block of each P_j. (P_j is made up of those blocks of $\tilde{B}_1$ which derive from the blocks of $\tilde{B}$ that pass through point j on B_0). We thus have

$$\sum_{B_i \in P_j} k_i = v - k .$$

Now

$$v - k = kt - (k-1)$$

$$= kt - (ut+w) .$$

Hence the remainder, when $v - k$ is divided by t, is equal to w, and so

(2) $$\sum_{B_i \in P_j} (k_i - t)^2 \ge w .$$

Combination of equations (1) and (2) produces

$$kw \le (v-k)(v-1-2tk) + t^2(b-1);$$

this simplifies to

$$b \ge 1 + \{kw + (v-k)(2tk-v+1)\}/t^2 .$$

Now, substitute $t = (v-1)/k$, and use the fact that b is an integer to obtain

$$b \ge 1 + \left\lceil \frac{k^2(v-k)}{v-1} + \frac{wk^3}{(v-1)^2} \right\rceil .$$

This is our desired result.

Corollary 2.1. $g^{(2s)}(4s+1) \ge 2s^2 + s + 1 + \lceil s/2 \rceil$.

Proof. We set $v = 4s + 1$, $k = 2s$; then $t = 2$, $w = 1$, and the result follows.

3. *Construction of the Coverings.*

First, we note that $g^{(2)}(5) = 10$, since all blocks must have length 2. Henceforth, we assume that $s > 1$, and construct coverings to show that

$$g^{(2s)}(4s+1) = 2s^2 + s + 1 + \lceil s/2 \rceil .$$

There are two cases, depending on the parity of s, although basically both cases are identical.

Case 1 (s even). We have $2s$ points on B_0 and $2s+1$ other points. We need to take the complete graph K_{2s+1} on these latter points and obtain a complete factorization of it into $2s$ factors (one for each point of B_0). The normal factorization of K_{2s+1} produces $2s+1$ factors each of which is made up of a singleton and various pairs (the easiest way to proceed is as in [1]; put the $2s+1$ points on a circle and, with each point i, use the set of chords perpendicular to Oi, O being the centre of the circle).

Here, we allow ourselves singletons, pairs, and triangles. Let the $2s+1$ points be denoted by $0, 1, 2, \ldots, 2s$, and let the $2s$ points $1, 2, \ldots, 2s$ be equally spaced around a circle. Let $P(i, j)$ denote the line (i,j) and all chords parallel to (i,j). Let $Q(i,j)$ denote two singletons i and j situated at opposite ends of a diameter, together with all chords perpendicular to the diameter (i,j).

We define $2s$ factors, or classes of blocks, on the $2s+1$ points $0, 1, \ldots, 2s$.

First take the s classes of blocks $P(i,i+1)$, where $1 \le i \le s$; take also the s classes $Q(i,i+s)$ where $1 \le i \le s$. Clearly, every edge from $\{1,2,\ldots,2s\}$ occurs precisely once in these classes. Now, extend these $2s$ classes to be classes on $\{1,2,\ldots,2s\}$ by the following algorithm:

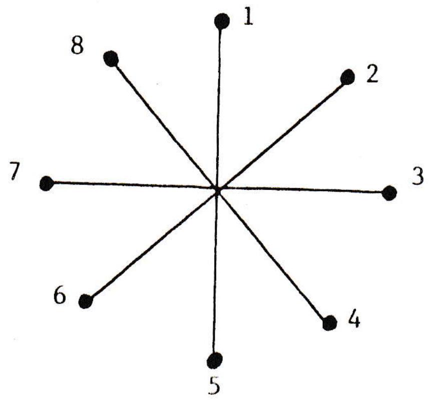
(1) if i is odd, change the pair $(i, i+1)$ of $P(i, i+1)$ to the triangle $(0, i, i+1)$;

(2) if i is even, add the singleton 0 to the set $P(i, i+1)$;

(3) adjoin 0 to the singleton $i+s$ of $Q(i, i+s)$ for $1 \le i \le s$.

Denote the sets, with 0 adjoined, by $P^*(i, i+1)$, $Q^*(i, i+s)$. We illustrate the procedure for $2s = 8$, $2s + 1 = 9$, in

Example 3.1.



P*(1,2)	P*(2,3)	P*(3,4)	P*(4,5)
012	0	034	0
38	23	25	45
47	14	16	36
57	58	78	27
	67		18

Q*(1,5)	Q*(2,6)	Q*(3,7)	Q*(4,8)
1	2	3	4
50	60	70	80
28	13	24	17
37	48	15	26
46	57	68	35

Clearly, we can now show that $g^{(8)}(8+9) = 39$ by taking a block $A_1 A_2 \ldots A_8$ and combining each A_i with one of the 8 resolutions of $\{0,1,\ldots,8\}$ just displayed.

The method of the illustration works in general. We use a long block $A_1 A_2 \ldots A_{2s}$ and associate each A_i with one of the 2s classes P*(i,i+1) and Q*(i,i+s). The covering thus formed contains the following blocks:

(1) one block of length 2s;

(2) s/2 blocks of length 4 formed from the s/2 triangles of P*(i,i+1), i odd;

(3) 3s/2 blocks of length 2;

(4) $2s^2-s/2$ blocks of length 3.

The total number of blocks is $2s^2 + 3s/2 + 1$, and this can be written as

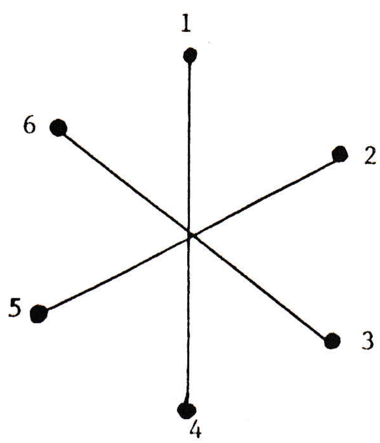
$$2s^2 + s + 1 + \lceil s/2 \rceil .$$

<u>Case 2 (s odd)</u>. We use a procedure similar to that in Case 1. We create s classes P(i, i+1) for $1 \le i \le s$ and s classes Q(i,i+s) for $1 \le i \le s$. The adjunction of the extra point 0 proceeds as follows. P*(i,i+1) is formed by using the triangle (0,i,i+1) when i is odd; P*(i,i+1) contains a singleton 0 when i is even. As before, Q*(i,i+s) is formed by replacing the singleton i+s by the pair (0,i+s), but there is one variation. We do this only for $i \ge 2$.

The set Q(1,1+s) is extended to Q*(1,1+s) by adjoining a singleton 0; this leaves 3 singletons in Q*(1,1+s).

We illustrate the procedure for 2s=6, 2s+1=7, in

Example 3.2.



P*(1,2)	P*(2,3)	P*(3,4)
012	0	034
36	23	35
45	14	16
	56	

Q*(1,4)	Q*(2,5)	Q*(3,6)
0	2	3
1	50	60
4	13	15
26	46	24
35		

We thus have a construction showing that $g^{(6)}(6+7) = 24$ by taking a block $A_1A_2 \ldots A_6$ and combining each A_i with one of the 6 resolutions of $\{0,1,\ldots,6\}$ just displayed.

The method of the illustration works in general. We use a long block $A_1A_2 \ldots A_{2s}$ and associate each A_i with one of the 2s classes P*(i,i+1) and Q*(i,i+s). The covering thus formed contains the following blocks:

(1) one block of length 2s;
(2) $(s+1)/2$ blocks of length 4;
(3) $3(s+1)/2$ blocks of length 2;
(4) $2s^2-(s+3)/2$ blocks of length 3.

The total number of blocks is $2s^2 + (3s+1)/2 + 1$, and this can be written as

$$2s^2 + s + 1 + \lceil s/2 \rceil .$$

4. *Conclusion.*

In Section 2, we showed (Corollary 2.1) that

$$g^{(2s)}(4s+1) \geq 2s^2 + s + 1 + \lceil s/2 \rceil .$$

In Section 3, we gave a construction which displayed a covering that achieved this lower bound. Hence, we may state our results as

THEOREM 4.1. *The perfect covering number* $g^{(2s)}(4s+1)$ *is equal to* $2s^2 + s + 1 + \lceil s/2 \rceil$, for $s > 1$.

REFERENCES

1 R. G. Stanton, J. L. Allston, and D. D. Cowan, *Pair-coverings with Restricted Largest Block Length,* to appear, Ars Combinatoria 11(1981).

2 R. G. Stanton and J. G. Kalbfleisch, *The* $\lambda - \mu$ *Problem:* $\lambda = 1$ *and* $\mu = 3$, Proc. 2nd Chapel Hill Conf. on Combinatorics, Univ. of North Carolina (1972), 451-462.

3 R. G. Stanton, P. D. Eades, G. H. J. van Rees, and D. D. Cowan, *Computation of Some Exact g-Coverings,* Utilitas Math. 18(1980), 269-282.

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