THE NON-EXISTENCE OF A (2,4)-FRAME

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Abstract

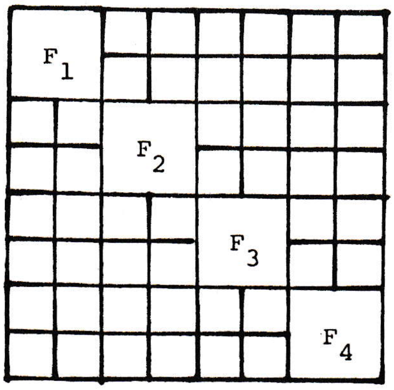
It is shown that a (2,4)-frame does not exist.

1. *Introduction.*

A (2,4)-frame, if it were to exist, would be an eight-by-eight array F of cells with the following properties:

(1) A cell either is empty or contains an unordered pair of elements chosen from the set $S = \{1,2,3,4\} \times \{1,2\}$.

(2) There are four empty two-by-two blocks F_1, F_2, F_3, F_4, down the diagonal of F:



(3) A row or column which meets F_i contains precisely the elements of $S \backslash (\{i\} \times \{1,2\})$.

(4) The unordered pairs occurring in F are precisely those $\{(k,m),(\ell,n)\}$ where $k \neq \ell$.

For any ordered pair of positive integers (t,u), a (t,u)-frame is defined analogously. Frames have been of considerable use in the construction of Howell designs and Room squares (see [1] and [4]). The following existence results have been shown.

THEOREM 1.1 (Dinitz and Stinson [2])

(1) If $u \geq 6$, then a (t,u)-frame exists if and only if $t(u-1)$ is even,

(2) If $\gcd(t,210) \neq 1$, then a $(t,5)$-frame exists,

(3) If t is a multiple of four, then a $(t,4)$-frame exists, whereas if t is odd, then no $(t,4)$-frame exists,

ARS COMBINATORIA, Vol. 11 (1981), pp. 99-106.

(4) *No $(t,3)$- or $(t,2)$-frame exists, whereas all $(t,1)$-frames exist.*

The non-existence results of the above theorem are trivial. Other than those exceptions there are precisely two frames which have been shown not to exist: a $(1,5)$-frame, which is equivalent to a Room square of side 5 (see [3]); and a $(2,4)$-frame. In this note we demonstrate the non-existence of a $(2,4)$-frame.

The author conjectures that all $(t,5)$-frames exist for $t > 1$, and that all $(t,4)$-frames exist for t even, $t > 2$.

2. *The non-existence proof.*

Let the *two-cell* F_{ij} be the two-by-two subarray of F determined by the rows meeting F_i and the columns meeting F_j:

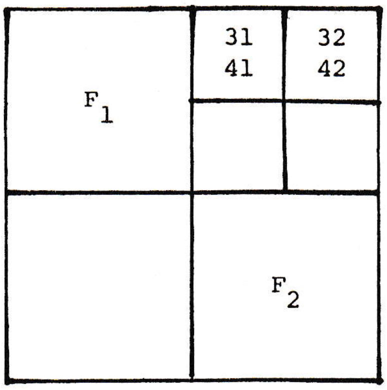
F_1	F_{12}	F_{13}	F_{14}
F_{21}	F_2	F_{23}	F_{24}
F_{31}	F_{32}	F_3	F_{34}
F_{41}	F_{42}	F_{43}	F_4

LEMMA 2.1 *A cell in the two-cell F_{ij} can only contain a pair of the type $\{(k,m),(\ell,n)\}$, where $\{i,j,k,\ell\} = \{1,2,3,4\}$.*

LEMMA 2.2 *A two-cell contains exactly two filled cells, not both in the same row or column.*

Proof. The twocells F_{ij} and F_{ji} together have four filled cells. Thus, we assume (without loss of generality) that F_{12} contains two filled cells in the first row, and obtain a contradiction.

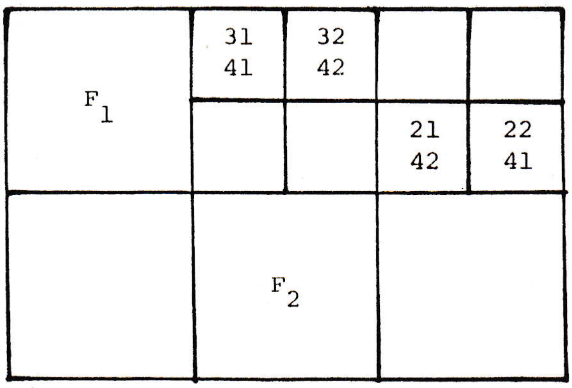
Without loss of generality, we may fill in:



Note that the other two cells of F_{12} must be left empty, since pairs cannot be repeated.

Now, (4,1) and (4,2) must occur in the second row. They cannot occur in F_{12} or F_{14}, so they must occur in F_{13}.

Again, without loss of generality, we may fill in:



As before, the other two cells of F_{13} must be left empty. Now, the symbol (2,1) has not occurred in the first row. It can only appear in F_{14}. Thus (2,1) must occur in the first row with some (3,j). However, (3,1) and (3,2) have both already occurred in the first row, so we have a contradiction.

LEMMA 2.3 *If an element (k,m) occurs twice in some two-cell, then $(k,1)$ and $(k,2)$ both occur exactly zero times or twice in every two-cell.*

Proof. Suppose, without loss of generality, that (3,1) occurs twice in F_{12}. Then (3,2) occurs twice in F_{42}, (3,1) occurs twice in F_{41}, (3,2) occurs twice in F_{21}, (3,1) occurs twice in F_{24}, and (3,2) occurs twice in F_{14}.

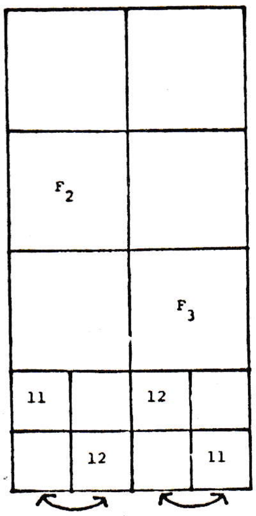
LEMMA 2.4 *If an element $x = (k,m)$ occurs twice in a two-cell, and an element $y = (\ell,n)$ occurs twice in a two-cell, then $k = \ell$.*

Proof. Suppose, without loss of generality, $x = (1,1)$, $y = (2,1)$, and obtain a contradiction. As in Lemma 2.3 we see that, in F_{34}, (1,m) occurs twice, and (2,n) occurs twice. This yields a repeated pair.

In view of Lemma 2.4, we may assume that (1,1),(1,2),(2,1),(2,2), (4,1) and (4,2) never occur twice in a two cell.

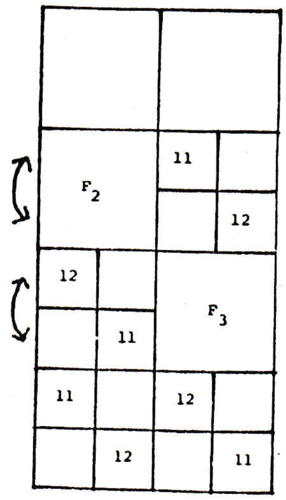
We now try to construct F. Each step is without loss of generality, modulo a permissible permuting of columns, rows and symbols. That is, we allow interchanging the two symbols (i,1) and (i,2), and we allow switching two rows or columns meeting a (given) F_i. Arrows indicate such a switching of rows or columns can be performed, if needed.

(1)



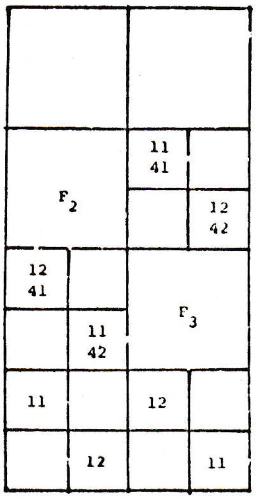
Place (1,1), (1,2) in F_{42} and F_{43}

(2)



Place (1,1), (1,2) in F_{23} and F_{32}

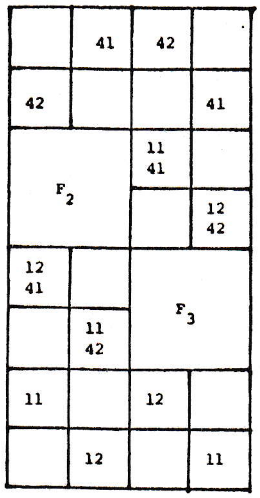
(1)



Place (4,1), (4,2)

in F_{23} and F_{32}

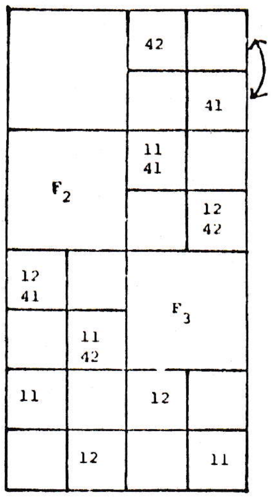
(5)



Place in (4,1), (4,2)

in F_{12}

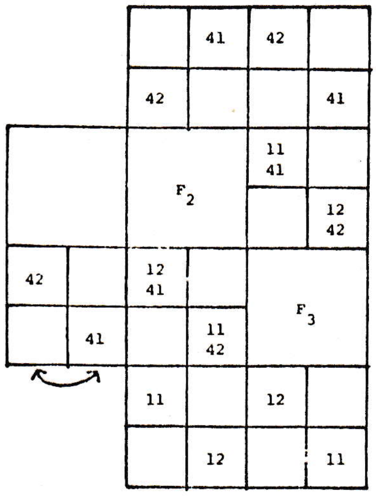
(4)



Place (4,1), (4,2)

in F_{13}

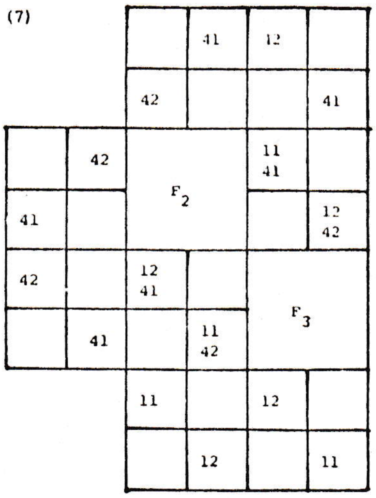
(6)



Place (4,1), (4,2)

in F_{31}

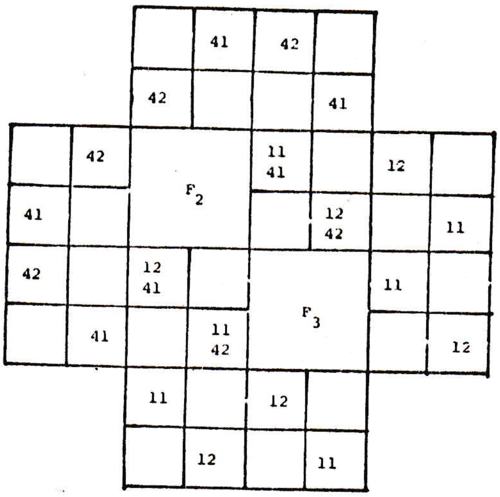
(7)



Place (4,1), (4,2)

in F_{21}

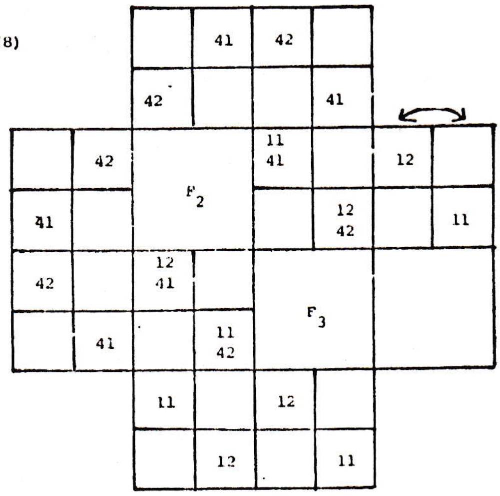
(9)



Place (1,1), (1,2)

in F_{34}

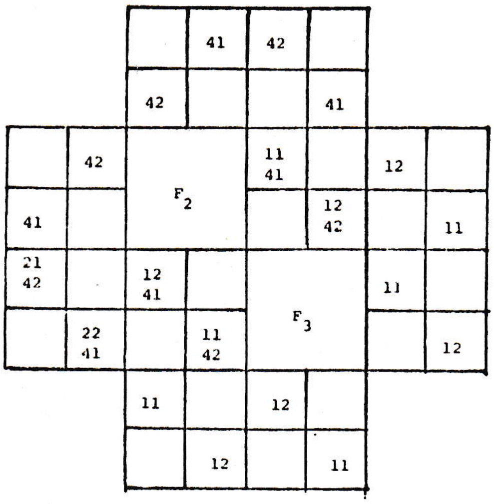
(8)



Place (1,1), (1,2)

in F_{24}

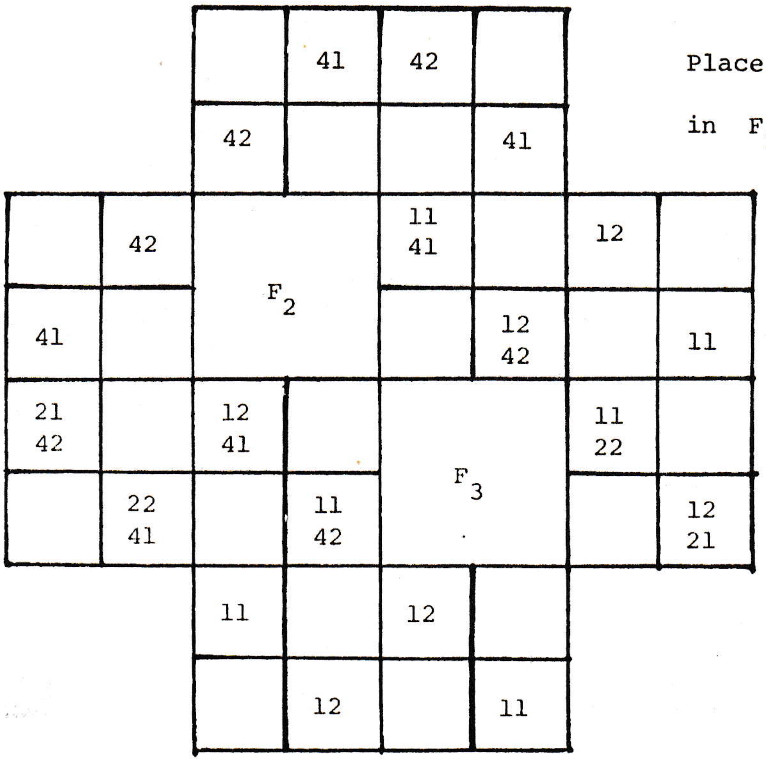
(10)



Place (2,1), (2,2)

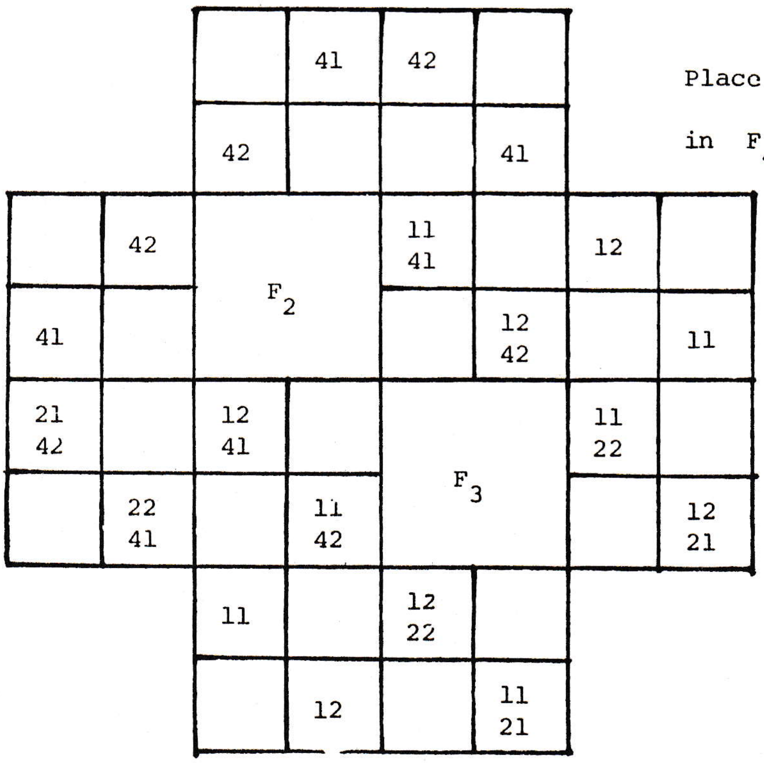
in F_{31}

(11)



Place (2,1), (2,2) in F_{34}

(12)



Place (2,1), (2,2) in F_{43}

Now if we try to place the elements (2,2),(2,1) into F_{13}, we obtain a contradiction. (2,2) has already appeared with (4,1) so it must now appear with (4,2). But then (2,2) occurs twice in a column. Thus we have shown:

THEOREM 2.5 *A (2,4)-frame does not exist.*

Acknowledgement

I would like to thank P.J. Schellenberg for carefully reading this note.

References

[1] J.H. Dinitz and D.R. Stinson, *The construction and uses of frames*, Ars Combinatoria 10(1980), 31-53.

[2] J.H. Dinitz and D.R. Stinson, *Further results on frames*, Ars Combinatoria 11 (1981), to appear.

[3] R.C. Mullin and W.D. Wallis, *The existence of Room squares*, Aequationes Math. 13 (1975), 1-7.

[4] D.R. Stinson, *Some results concerning frames, Room squares and subsquares*, Journal of the Australian Mathematical Society (series A), to appear.

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