A NOTE ON THE EXISTENCE OF 7 AND 8 MUTUALLY ORTHOGONAL LATIN SQUARES

D.R. Stinson

Abstract

If \( N(v) \) denotes the maximum number of mutually orthogonal Latin squares of order \( v \), then \( N(v) > 8 \) if \( v \) is odd and \( v > 2343 \), and \( N(v) > 7 \) if \( v > 2862 \).

1. Introduction.

For the definition of Latin square, and mutually orthogonal Latin squares (henceforth MOLS), see [1].

The purpose of this report is to update the results of [1], where it is shown that \( N(v) \geq 7 \) if \( v > 4146 \) and \( N(v) \geq 8 \) if \( v > 9402 \).

2. A Corollary to Wilson's Construction for MOLS.

In [2], Wilson describes a recursive construction for MOLS. Various corollaries of this construction have been instrumental in establishing bounds for the existence of MOLS. In [3], Wotjas obtains the following corollary to Wilson's construction. We state it as Theorem 2.1.

**Theorem 2.1.** If \( 0 \leq \omega \leq t \) then

\[
N(mt+\omega) \geq \min \{N(m), N(m+1), N(m+\omega)+1, N(t)-\omega\}.
\]

We will use Theorem 2.1 to obtain 8 MOLS of some odd orders, and 7 MOLS of a few orders.

3. 8 MOLS of Odd Orders.

In [1], a list of all orders for which 8 MOLS are not known is given.

We are able to eliminate many of the possible exceptions of odd orders using Theorem 2.1. Below we list equations of the form

\[ v = mt + w, \text{ with } m, t \text{ and } w \text{ as in Theorem 2.1.} \]

Thus there exist 8 MOLS of order \( v \), for such \( v \). The existence of the required number of MOLS of orders \( m, m+1, m+w, \) and \( t \) may be found in [1].

\[
\begin{align*}
2737 &= 16 \cdot 167 + 65 & 1267 &= 16 \cdot 79 + 3 & 721 &= 16 \cdot 43 + 33 \\
2577 &= 16 \cdot 157 + 65 & 1105 &= 16 \cdot 67 + 33 & 695 &= 16 \cdot 43 + 7 \\
2471 &= 16 \cdot 151 + 55 & 1045 &= 16 \cdot 64 + 21 & 663 &= 16 \cdot 41 + 7 \\
1981 &= 16 \cdot 121 + 45 & 965 &= 16 \cdot 59 + 21 & 595 &= 31 \cdot 19 + 6 \\
1941 &= 31 \cdot 61 + 50 & 959 &= 16 \cdot 59 + 15 & 511 &= 16 \cdot 31 + 15 \\
1939 &= 16 \cdot 121 + 3 & 951 &= 16 \cdot 59 + 7 & 471 &= 16 \cdot 29 + 7 \\
1799 &= 16 \cdot 109 + 55 & 917 &= 31 \cdot 29 + 18 & 447 &= 16 \cdot 27 + 15 \\
1655 &= 16 \cdot 103 + 7 & 905 &= 31 \cdot 29 + 6 & 435 &= 16 \cdot 27 + 3 \\
1603 &= 16 \cdot 97 + 51 & 855 &= 16 \cdot 53 + 7 & 415 &= 16 \cdot 25 + 15 \\
1547 &= 31 \cdot 49 + 28 & 847 &= 31 \cdot 27 + 10 & 375 &= 16 \cdot 23 + 7 \\
1479 &= 31 \cdot 47 + 22 & 843 &= 31 \cdot 27 + 6 & 371 &= 16 \cdot 23 + 3 \\
1463 &= 31 \cdot 47 + 6 & 805 &= 16 \cdot 49 + 21 & 259 &= 16 \cdot 16 + 3 \\
1445 &= 16 \cdot 89 + 21 & 759 &= 16 \cdot 47 + 7 \\
1385 &= 16 \cdot 83 + 57 & 755 &= 16 \cdot 47 + 3 \\
1285 &= 16 \cdot 79 + 21 & 723 &= 31 \cdot 23 + 10
\end{align*}
\]

The above equations, and the results of [1], prove the following.

THEOREM 3.1. If \( v \) is odd and \( v > 2343 \), then \( N(v) \geq 8 \).

Remark: If \( v \) is odd, \( v > 1935 \) and \( v \nmid 2343, 2181, 2005 \), then \( N(v) \geq 8 \).

4. 7 MOLS.

The following is shown in [1].
THEOREM 4.1. If \( v > 2862, v \neq 3620 \) or 4146, then \( N(v) \geq 7 \).

Using Theorem 2.1, the two exceptions above can be eliminated. As in Section 3, we have \( 4146 = 8 \cdot 509 + 74 \) and \( 3620 = 88 \cdot 41 + 12 \). Since \( N(8), N(9), N(89) \geq 7, N(82), N(100) \geq 8, N(509) \geq 81 \), and \( N(41) \geq 19 \);

Theorem 2.1 shows \( N(4146), N(3620) \geq 7 \). Thus the above discussion and Theorem 4.1 imply

THEOREM 4.2. If \( v > 2862 \), \( N(v) \geq 7 \).

5. Summary.

Thus we have established the new bounds \( N(v) \geq 8 \) if \( v \) odd and \( v > 2343 \) and \( N(v) \geq 7 \) if \( v > 2862 \).

REFERENCES


Ohio State University, Columbus, Ohio.