A NOTE ON ONE-FACTORIZATIONS

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ABSTRACT. The graph $G_{n,m}^m$ is defined to have vertex set $\mathbb{Z}_n \times \mathbb{Z}_m$ and $(a,i)(b,j)$ is an edge whenever $a - b \equiv i \pmod{n}$. Parker has shown that these graphs have a one-factorization if and only if $mn$ is even. We construct special one-factorizations of $G_{n,m}^m$ (when $mn$ is even) which are useful in the construction of Room squares with large subsquares.

1. Introduction.

In this note a graph is a graph without multiple edges or loops. The number of edges incident with a vertex $v$ is called the degree of $v$; a graph is $k$-regular if every vertex has degree $k$. The number of vertices is referred to as the order of a graph.

Let $G$ be a graph of even order $2n$. A one-factor of $G$ is a set of $n$ independent edges, that is, a 1-regular subgraph of order $2n$. (A one-factor is sometimes called a perfect matching.) A one-factorization of a $k$-regular graph $G$ is a set $\{F_1,F_2,...,F_k\}$ of one-factors which partitions the edges of $G$. A one-factorization may be thought of as a $k$-colouring of the edges of $G$ so that each vertex is incident with precisely one edge of each colour.

For any graph $G$ and any positive integer $m$ we define $G[K_{m,m}]$ to be the graph formed by replacing each vertex $v$ of $G$ by $m$ vertices $v_1,v_2,...,v_m$. Each edge $vw$ of $G$ is then replaced by all the $m^2$ edges $v_iw_j$, where $1 \leq i, j \leq m$.

Let $G$ be a $k$-regular graph of order $n$ (so $kn$ is even) and let $m$ be a positive integer. We define a $G[K_{m,m}]$-Room rectangle to be a $knm/2$ by $km$ array $A$, each cell of which is either empty or contains an edge of $G[K_{m,m}]$, which satisfies

1. each edge of $G[K_{m,m}]$ occurs in precisely one cell of $A$;
2. the filled cells of any column of $A$ form a one-factor of $G[K_{m,m}]$;
3. the filled cells of any row of $A$ form a one-factor of one of the $K_{m,m}$'s which make up $G[K_{m,m}]$.

Note that condition (2) requires that $G(K_{m,m})$ have a one-factorization, so necessarily $nm$ (the order of $G(K_{m,m})$) must be even.

The study of $G(K_{m,m})$-Room rectangles was motivated by a construction method for Room squares due to the second author [5]. Although the existence question has been solved for Room squares (see Mullin and Wallis [2]), the existence questions for (Room) subsquares of Room squares remain open. It is easily seen that if a Room square of side $u$ contains a subsquare of side $v$, then $u \geq 3v+2$. By using the Room rectangles we construct here, it can be shown that for infinitely many values of $v$, there exists a Room square of side $3v+2$ with a subsquare of side $v$.

The cycle $C_n$ of length $n \geq 3$ is defined to be the 2-regular graph with vertex set $Z_n$ and edges $xy$ for each pair satisfying $x-y \equiv \pm 1 \pmod{n}$.

Parker [3] has shown that the graph $C_{n,m,m}$ has a one-factorization if and only if $nm$ is even. Under the same conditions we show that there exists a $C_{K_{m,m}}$-Room rectangle. We also show the existence of $G(K_{m,m})$-Room rectangles for some more general graphs $G$.

2. The Existence of $C_{n,m,m}$-Room Rectangles.

In this section we show that a $C_{n,m,m}$-Room rectangle exists if and only if $nm$ is even. We begin with some examples and direct constructions.

Example 2.1. A $C_{3}[K_{2,2}]$-Room rectangle.

\[
\begin{array}{ccc}
0_{1} & 0_{1} & 0_{1} \\
1_{2} & 1_{2} & 1_{2} \\
2_{0} & 2_{0} & 2_{0} \\
0_{1} & 0_{1} & 0_{1} \\
1_{2} & 1_{2} & 1_{2} \\
2_{0} & 2_{0} & 2_{0} \\
\end{array}
\]
LEMMA 2.2. A $C_n[2,2]$-Room rectangle exists for all odd $n \geq 3$.

Proof. We give a constructive proof based on the idea of inserting four rows at a time into Example 2.1 and increasing the length of the cycle by 2.

Let $n = 2j+1$ for some $j \geq 1$.

The first two rows of the array are identical to the first two rows of Example 2.1. The next $2(j-1) = n-3$ rows will have the form

$$
\begin{array}{c}
k_1(k+1) \\
k_2(k+1)
\end{array}
$$

$k = 2,3,\ldots,n-2$.

The next row is

$$
\begin{array}{c|c|c}
(2j) & 0 & 2j
\end{array}
$$

replacing the third row of Example 2.1.

The next two rows of the array are identical to rows 4 and 5 of Example 2.1.

The next $2(j-1)$ rows have the form

$$
\begin{array}{c|c|c}
(2k) & (2k+1) \\
(2k+1) & (2k+2)
\end{array}
$$

$k = 1,2,\ldots,j-1$.

The final row is

$$
\begin{array}{c|c|c}
(2j) & 0 & 2j
\end{array}
$$

replacing the final row of Example 2.1. The array thus constructed is a $C_{2j+1}[2,2]$-Room rectangle. \qed

LEMMA 2.3. A $C_n[2,2]$-Room rectangle exists for all even $n \geq 4$.

Proof. The proof is by direct construction. Let $n = 2j$ for some $j \geq 2$.

The first $n = 2j$ rows are

$$
\begin{array}{c|c|c}
k_1(k+1) & k_2(k+1)
\end{array}
$$

$k = 0,1,2,\ldots,n-1$. 

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The final $2j$ rows are

$$
\begin{array}{ccc}
(2k)_1(2k+1)_1 & (2k)_2(2k+1)_2 & \ldots \\
(2k+1)_2(2k+2)_2 & (2k+1)_1(2k+2)_1 & \ldots \\
\end{array}
$$

$k = 0, 1, 2, \ldots, j-1.$

The array thus constructed is a $C_{2j}[K_{2,2}]$-Room rectangle. []

We now give an example of a $C_3[K_{4,4}]$-Room rectangle. It is constructed using Example 2.1 embedded in the top left hand subsquare.

**Example 2.4. A $C_3[K_{4,4}]$-Room rectangle.**

<table>
<thead>
<tr>
<th></th>
<th>0.1</th>
<th>0.1</th>
<th>0.1</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.2</td>
<td>0.4</td>
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<tr>
<td>0.3</td>
<td>0.3</td>
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<td>0.4</td>
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<td>1.2</td>
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<tr>
<td>1.3</td>
<td>1.3</td>
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<td>2.0</td>
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<tr>
<td>2.1</td>
<td>2.1</td>
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<tr>
<td>2.4</td>
<td>2.4</td>
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<td>3.0</td>
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<tr>
<td>3.1</td>
<td>3.1</td>
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<td>4.0</td>
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<tr>
<td>4.1</td>
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</tr>
</tbody>
</table>

**Lemma 2.5.** A $C_n[K_{4,4}]$-Room rectangle exists for all odd $n \geq 3$.

**Proof.** The proof is by a direct construction similar to those of Lemmas 2.2 and 2.3.

Let $n = 2j+1$ for some $j \geq 1$. The first two rows of the array are identical to those of Example 2.4. The next $2(j-1)$ rows have the form

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\[
\begin{array}{cccc}
(2k) & (2k+1) & 4 & (2k) & (2k+1) \\
(2k+1) & (2k+2) & 3 & (2k+1) & (2k+2) \\
\end{array}
\]
\[
\begin{array}{cccc}
(2k) & (2k+1) & (2k+1) & (2k+1) \\
(2k+1) & (2k+2) & (2k+2) & (2k+2) \\
\end{array}
\]
\[k = 1,2,\ldots,j-1.\]

The next row is
\[
\begin{array}{cccc}
(2j) & 0 & 4 & (2i) \\
(2j) & 3 & 0 & (2i) \\
(2j) & 1 & 2 & (2i) \\
(2j) & 2 & 0 & (2i) \\
\end{array}
\]

The next two rows of the array are identical to rows 4 and 5 of Example 2.4. The next \(2(j-1)\) rows have a form obtained from \((*)\) by interchanging the subscripts (1 and 3) and (2 and 4). The next row is obtained from \((†)\) by the same interchange. The final \(2n\) rows are constructed, as in Example 2.4, from the \(n\) subarrays of the form

\[
\begin{array}{cccc}
k_1(k+1) & k_2(k+1) & k_3(k+1) & k_4(k+1) \\
k_4(k+1) & k_3(k+1) & k_1(k+1) & k_2(k+1) \\
\end{array}
\]
\[k = 0,1,2,\ldots,n-1.\]

The array constructed is a \(C_{2j+1}[K_{4,4}]\)-Room rectangle. \(\square\)

We now give a recursive construction for \(G[K_{m,m}]\)-Room rectangles.

We assume that the reader is familiar with the definitions of Latin squares and orthogonal Latin squares (see for example [6]).

**Lemma 2.6.** If a \(G[K_{m,m}]\)-Room rectangle exists and a pair of orthogonal Latin squares of order \(\ell\) exist then a \(G[K_{km,km}]\)-Room rectangle exists.

**Proof.** Notice that \(K_{km,km} = K_{m,m}[K_{k,\ell}]\), so we may think of the vertices of \(G[K_{km,km}]\) as the vertices of \(G[K_{m,m}]\) subscripted by elements of the set \(\{1,2,\ldots,\ell\}\).

Let \(A\) be a \(G[K_{m,m}]\)-Room rectangle and let \(L\) and \(M\) be a pair of orthogonal Latin squares of side \(\ell\). We form a new array from \(A\) by the following process. Each empty cell of \(A\) is replaced by an \(\ell\times\ell\) empty array. Each cell of \(A\) containing an edge \(xy\) of the graph \(G[K_{m,m}]\) is replaced by an \(\ell\times\ell\) array whose \((i,j)\)th entry is \(x_L(i,j) y_M(i,j)\), where \(L(i,j)\) and \(M(i,j)\) are the \((i,j)\)th entries from \(L\) and \(M\) respectively.
The new array is clearly a $G[K_{x_m,x_m}]$-Room rectangle by the orthogonality of $L$ and $M$ and the structure inherited from the array $A$. □

We are now in a position to prove the major result of this section.

**THEOREM 2.7.** A $C_n[K_{m,m}]$-Room rectangle with $n \geq 3$, $m \geq 1$ exists if and only if $mn$ is even.

**Proof.** As noted in the introduction the condition that $mn$ is even is clearly necessary for the existence of a $C_n[K_{m,m}]$-Room rectangle. We shall establish sufficiency in two cases, according to the parity of $n$.

If $n$ is odd, then $m$ is even. When $m = 2$, the result is Lemma 2.2. When $m = 2\ell$ and $\ell \neq 2$ or $6$ we may apply Lemma 2.7 since Bose, Shrikhande, and Parker [1] have established the existence of a pair of orthogonal Latin squares of order $\ell$ provided $\ell \neq 2$ or $6$. When $m = 4$ the result is Lemma 2.4, and when $m = 12$ the result is obtained by applying Lemma 2.7, since $12 = 4 \cdot 3$.

If $n$ is even then a $C_n[K_{1,1}]$-Room rectangle is just a one-factorization of the even cycle $C_n$, which exists. Lemma 2.7 then yields the result for every $m \geq 1$ except $m = 2$ or $6$. When $m = 2$ the result is just Lemma 2.3, and this, together with Lemma 2.7, yields the result for $m = 6 = 2 \cdot 3$. □

3. The Existence of $G[K_{m,m}]$-Room Rectangles.

In this section we shall give necessary and sufficient conditions for the existence of $G[K_{m,m}]$-Room rectangles when $G$ is any 2-regular graph. We also give methods for construction of these rectangles for many cases where $G$ is $k$-regular and $k \geq 3$.

We begin with two simple decomposition lemmas.

**LEMMA 3.1.** Let $G$ be a $k$-regular disconnected graph. A $G[K_{m,m}]$-Room rectangle exists if and only if $G_i[K_{m,m}]$-Room rectangles exist for each component $G_i$ of $G$.

**Proof.** A $G[K_{m,m}]$-Room rectangle may be constructed by "stacking" the component rectangles one above another. The component rectangles may
always be recovered since each row of the $G[K_{m,m}]$-Room rectangle comes from some edge of $G$ which belongs to a unique component. □

**Lemma 3.2.** Let $G$ be a $k$-regular graph with edge-disjoint subgraphs $G_1, G_2, \ldots, G_n$ such that

1. $G = G_1 \cup G_2 \cup \ldots \cup G_n$,
2. each $G_i$ is a spanning regular subgraph, and
3. a $G_i[K_{m,m}]$-Room rectangle exists for each $i$.

Then a $G[K_{m,m}]$-Room rectangle exists.

**Proof.** Let $A_i$ be a $G_i[K_{m,m}]$-Room rectangle for each $i$; then the array

![Diagram of a $G[K_{m,m}]$-Room rectangle]

is a $G[K_{m,m}]$-Room rectangle. □

Note that this lemma was used implicitly in Example 2.4 and Lemma 2.5.

The following is immediate.

**Lemma 3.3.** If $G$ is a 2-regular graph then a $G[K_{m,m}]$-Room rectangle exists if and only if $m$ is even or all the components of $G$ are even cycles.

By analogy with 1-factorization we say that a graph $G$ has a $k$-factorization for some positive integer $k$ if $G$ can be partitioned into spanning $k$-regular subgraphs.

**Theorem 3.4.** If $G$ is a regular graph of even degree and $m$ is even, then a $G[K_{m,m}]$-Room rectangle exists.

**Proof.** Petersen [4] has shown that a regular graph of even degree has a 2-factorization. Apply Lemmas 3.2 and 3.3. □
REFERENCES


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Received December 23, 1980.

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