MORE SKEW ROOM SQUARES

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ABSTRACT. The authors construct skew Room squares for four new sides, reducing the number of unsettled cases to seven.

1. *Introduction.*

The purpose of this note is to point out that the theory of frames (in the generalized sense of [1] and [2]) can be combined with the generalized direct singular (indirect) product of [4] to produce skew Room squares of previously undecided orders.

A *skew Room square* of side s is an $s \times s$ array A of cells and an $(s+1)$-set S of objects called symbols which satisfy the following properties:

(i) every cell of A is either empty or contains an unordered pair of elements;
(ii) every symbol occurs precisely once in each row and column of A;
(iii) every unordered pair of distinct symbols occurs in precisely one cell of A;
(iv) there is a distinguished element ∞, which occurs in each cell of the main diagonal of A and hence every other element occurs precisely once on the main diagonal of A;
(v) for each pair of non-diagonal cells (i,j) and (j,i) of A, precisely one of these is empty.

A *subsquare* of side t of a skew Room square of side s is a $t \times t$ subarray B of A, situated symmetrically with respect to the main diagonal of A, which is a skew Room square in its own right (based on the symbols occurring on the diagonal of B). Every skew Room square contains a skew subsquare of side 1, and by convention, of side 0 as well. Clearly the side s of a skew Room square must be odd, and it is known that no skew Room square of side 3 or 5 exists. However, combining results from a series of papers by various authors, it is shown in [4] that skew Room squares exist for all positive odd integral sides $s \geq 7$ with the possible exception of $s \in \{69,75,87,93,95,115,123,129,$

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159, 213, 215, 237}. Moreover, Stinson [7], using frames, has shown that there is a skew Room square of side 129. It is our purpose here to employ frame techniques to show that there are also skew Room squares of side s for $s \in \{75,115,213,215\}$. For the definition of a *skew* t-*frame* of order u (often hereafter referred to simply as a skew (t,u)-frame), see [2]. For the definition of an incomplete array $IA(n,s)$ see [4].

The following theorem is a straightforward generalization of Theorem 3.2 of [4].

THEOREM 1.1. *Suppose that there exists a skew* (t,v_1)-*frame, and a skew Room square of order* v_2 *with a skew subsquare of side* v_3. *Let* a *be an integer such that* $0 \le a < v_3$ *such that there exists an incomplete array* $IA((v_2-a)/t,(v_3-a)/t)$ *and that there is a skew Room square of side* $v_1(v_3-a)+a$. *Then there exists a skew Room square of side* $v_1(v_2-a)+a$ *which contains a skew subsquare of side* $v_1(v_3-a)+a$.

Proof. The proof is that of Theorem 3.2 of [4], mutatis mutandis. □

Closely associated with this is the following.

THEOREM 1.2. *If there exists a skew* (t,v_1)-*frame and if there exists a skew Room square of side* v_2 *with a skew subsquare of side* v_3, *and if there exists a pair of orthogonal latin squares of side* $(v_2-v_3)/t$, *then there exists a skew Room square of side* $v_1(v_2-v_3)+v_3$ *which contains skew subsquares of side* v_2 *and* v_3.

Proof. The proof is that of Theorem 3.1 of [5], mutatis mutandis. □

Since a skew Room square of side s can itself be viewed as a skew 1-frame of order s, the above constructions can be applied where the frames in question are in fact skew Room squares.

2. *The Construction of Skew Room Squares.*

For convenience we let $SS = \{s: \exists \text{ a skew Room square of side } s\}$.

To show the existence of the skew Room squares discussed herein, we give the source of the frame used and then list the associated equations, justifying the existence of the required intermediate objects when their existence is not immediately apparent.

A. In [11], a skew 3-frame of order 5 is presented. Since $75 = 5(15-0) + 0$, $75 \in SS$ by Theorem 1.2.

B. In [7], a skew 4-frame of order 4 is given.

(i) Since $115 = 4(31-3) + 3$ (and since an IA(7,1) is a pair of orthogonal latin squares of side 7), if we set $v_3 = 7$, all that is required is the existence of a skew Room square of order 31 with a subsquare of side 7. But as displayed in [1,p.145], there exists a skew 2-frame of order 5, and $31 = 5(7-1) + 1$, so by Theorem 1.2, such a square of order 31 exists. Thus $115 \in SS$.

(ii) Since $213 = 4(57-5) + 5$ (and since an IA(13,1) is a pair of orthogonal latin squares of side 13), if we take $v_3 = 9$ all that is required for the existence of a skew Room square of order 213 is a skew Room square of side 57 with a skew subsquare of order 9. But $57 = 7(9-1) + 1$, and using a skew Room square as a (7,1) frame produces the required square. Hence $213 \in SS$.

C. For a definition of *starter* and *skew adder* see [6].

The following skew adder generates a skew Room square of side 35. It has the further property that the subgroup of Z_{35} isomorphic to Z_7 gives rise to a skew Room square of side 7 in the skew Room square of side 35. Thus by deleting the elements of this subgroup from the given starter and adder, one obtains a starter and adder which can be used to generate a skew 7-frame of order 5.

Starter	1,28	2,13	3,17	4,21	6,22
Adder	1	31	9	32	17
7,8	9,32	11,33	12,16	14,23	18,24
14	2	8	16	29	24
19,26	27,29	31,34	5,30	10,25	15,20
12	22	28	10	20	5

Now $215 = 7(35-5) + 5$, and using a skew Room square as a (1,7)-frame and taking $v_3 = 7$, we have a skew Room square of order 215 provided that there exists an IA(30,2). But in [3] it is noted that an $IA(n,2)$ exists for all $n \geq 6$. Hence $215 \in SS$. Thus the only odd sides greater than or equal to 7 for which the existence of a skew Room square is in doubt are 69, 87, 93, 95, 123, 159, and 237.

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