NESTINGS OF DIRECTED CYCLE SYSTEMS

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Abstract. We show that for all odd m, there exists a directed m-cycle system of D_n that has an $\lfloor m/2 \rfloor$-nesting, except possibly when $n \in \{3m+1, 6m+1\}$.

1. Introduction.

Let K_n be the complete graph on n vertices. An *m-cycle* of a graph G is an ordered m-tuple $(v_0, v_1, \ldots, v_{m-1})$ such that $v_i v_{i+1}$ for $0 \le i \le m-1$ is an edge of G (where subscripts are reduced modulo m). An *m-cycle system* of K_n is an ordered pair (V, C) where V is the vertex set of K_n (so $n = |V|$) and C is a collection of edge-disjoint m-cycles of K_n which induce a partition of $E(K_n)$ ($E(K_n)$ is the edge set of K_n).

Let $(v_0, v_1, \ldots, v_{m-1}; w)$ denote the *star* which joins w to each of the vertices $v_0, v_1, \ldots, v_{m-1}$. A *nesting* of the m-cycle system (V, C) of K_n is a function $\alpha: C \to V$ such that $C(\alpha)$ induces a partition of $E(K_n)$, where $C(\alpha)$ is the set of stars defined by

$$C(\alpha) = \{(v_0 v_1, \ldots, v_{m-1};\ \alpha(c)) \mid c = (v_0, v_1, \ldots, v_{m-1}) \in C\}.$$

Whether or not an arbitrary m-cycle system can be nested is an extremely difficult problem. However, it would seem tractable to consider the problem of finding the set of values of n for which there exists a nestable m-cycle system of K_n. A simple counting argument shows that a necessary condition for a nestable m-cycle system of K_n to exist is that $n \equiv 1 \pmod{2m}$. In the case where $m = 3$, this problem has been completely settled (this is precisely the nesting problem for Steiner triple systems) [2, 8], the set of possible values being all $n \equiv 1 \pmod 6$. More recently it has been shown that [5] for any odd value of m, with at most 13 possible exceptions the necessary condition is also sufficient, and for the particular case when $m = 5$ there are no exceptions. This nesting problem for even length cycles is essentially solved, since for any even $m \ge 4$, with at most 13 exceptions for each value of n, there exists an m-cycle system of K_n, $n \equiv 1 \pmod{2m}$ which has a nesting [7, 9].

In this paper, we introduce analogous problems for directed m-cycle systems. Let D_n be the complete directed graph on n vertices. A *directed m-cycle* of a directed graph G is an ordered m-tuple $(v_0, v_1, \ldots, v_{m-1})$ such that (v_i, v_{i+1}) is an arc of G for $0 \leq i \leq m-1$ (reducing sub-scripts modulo m). A *directed m-cycle system* of D_n is an ordered pair (V, C) where V is the vertex set of D_n (so $n = |V|$) and C is a set of arc-disjoint directed m-cycles of D_n which induce a partition of $A(D_n)$ ($A(D_n)$ is the set of arcs of D_n). There are clearly several ways to define a nesting of a directed m-cycle system as the edges in each of the stars can be oriented in different ways. Perhaps the most satisfying problem would require that for some fixed x, $0 \leq x \leq m$, each directed star used in the nesting has exactly x arcs directed in and $m - x$ arcs directed out of the centre vertex. Therefore, define $(v_0, v_1, \ldots, v_{x-1};\ v_x, v_{x+1}, \ldots, v_{m-1};\ w)$ to be the *directed (x, m)-star* in which (v_i, w) is an arc for $0 \leq i \leq x-1$ and (w, v_i) is an arc for $x \leq i \leq m-1$. Then define an *x-nesting* of a directed m-cycle system (V, C) of D_n to be an ordered pair $(\alpha, S(\alpha))$ where α is a function $\alpha\colon C \to V$ and $S(\alpha)$ is a set of directed (x, m)-stars defined by

$$S(\alpha) = \{(v_{\pi_c(0)}, v_{\pi_c(1)}, \ldots, v_{\pi_c(x-1)};\ v_{\pi_c(x)}, \ldots, v_{\pi_c(m-1)};\ \alpha(c)) \mid c = (v_0, \ldots, v_{m-1}) \in C\}$$

for some permutations π_c of $\{0, 1, \ldots, m-1\}$, $c \in C$, such that $S(\alpha)$ induces a partition of $A(D_n)$.

Example 1.1: Let $m = 5$ and $n = 6$. Then

$$C = \{(5,0,1,3,2), (5,1,2,4,3), (5,2,3,0,4), (5,3,4,1,0), (5,4,0,2,1), (0,3,1,4,2)\}$$

is a directed 5-cycle system that has a 1-nesting defined by

$$S(\alpha) = \{(1; 5,3,0,2; 4), (3; 1,5,2,4; 0), (4; 2,0,3,5; 1), (5; 3,1,4,0; 2), (0; 4,2,5,1; 3), (2; 0,4,1,3; 5)\}$$

and a 2-nesting defined by

$$S(\alpha) = \{(5,3; 0,2,1; 4), (1,5; 2,4,3; 0), (2,0; 3,5,4; 1), (3,1; 4,0,5; 2), (4,2; 5,1,0; 3), (0,4; 1,3,2; 5)\}.$$

A simple counting argument shows that a necessary condition for the existence of a directed m-cycle system of D_n that has an x-nesting is that $n \equiv 1 \pmod{m}$.

It is the object of this paper to show that if m is odd then this is also a sufficient condition, with at most 2 possible exceptions, in the case when $x = \lfloor m/2 \rfloor$.

It is worth noting that if every arc in an x-nesting of a directed m-cycle system is oriented in the opposite direction then a $(m - x)$-nesting results, so it suffices to consider this problem for $1 \leq x \leq \lfloor m/2 \rfloor$.

Finally, notice that if we ignore the directed cycles then what remains is a decomposition of D_n into directed (x, m)-stars. It is only recently [1] that the problem of finding such decompositions has been found when $n \equiv 0$ or $1 \pmod{m}$ for all x, the case when $n \equiv 0 \pmod{m}$ now being possible since the condition of the directed stars arising from a nesting is no longer imposed. Even more recently, this decomposition problem has been completely solved [3].

Throughout the rest of this paper, we assume that m is odd. Let $Z_m = \{0, 1, \ldots, m-1\}$.

2. Directed m-cycle systems with $\lfloor m/2 \rfloor$-nestings.

Lemma 2.1. *For $1 \leq x \leq \lfloor m/2 \rfloor$ there exists a directed m-cycle system of D_{m+1} that has an x-nesting.*

Proof: Define a directed m-cycle on the vertex set $\{\infty\} \cup Z_m$ by

$$\begin{aligned} a &= (a_0, a_1, \ldots, a_{m-1}) \quad \text{where} \\ a_0 &= \infty, \\ a_j &= (-1)^j \lfloor j/2 \rfloor \quad \text{for} \quad 1 \leq j \leq \lfloor m/2 \rfloor, \quad \text{and} \\ a_{m-j} &= (-1)^{\lfloor m/2 \rfloor} \lfloor m/2 \rfloor + (-1)^j \lfloor j/2 \rfloor \quad \text{for} \quad 1 \leq j \leq \lfloor m/2 \rfloor. \end{aligned}$$

Let $a + i = (a_0 + i, a_1 + i, \ldots, a_{m-1} + i)$, reducing each component modulo m and defining $\infty + i = \infty$. Then we can define a directed m-cycle system $(\{\infty\} \cup Z_m, C)$ as follows: if $m \equiv 1 \pmod{4}$ then define

$$C = \{a + i \mid 0 \leq i \leq m-1\} \cup \{(0, \lceil m/2 \rceil, 2\lceil m/2 \rceil, \ldots, (m-1)\lceil m/2 \rceil)\}$$

and if $m \equiv 3 \pmod{4}$ then define

$$C = \{a + i \mid 0 \leq i \leq m-1\} \cup \{(0, \lfloor m/2 \rfloor, 2\lfloor m/2 \rfloor, \ldots, (m-1)\lfloor m/2 \rfloor)\}.$$

To nest these directed m-cycle systems, begin by renaming ∞ with m, so the vertex set is now Z_{m+1}. Of course in this case, for each $c \in C$, $\alpha(c)$ is the unique vertex that is not in c. Define

$$\begin{aligned} s = (&1, -1, \ldots, \lfloor x/2 \rfloor, -\lfloor x/2 \rfloor, (m+1)/2;\ 1 + \lfloor x/2 \rfloor, -(1 + \lfloor x/2 \rfloor), \ldots, \\ &\lfloor m/2 \rfloor, -\lfloor m/2 \rfloor; 0) \end{aligned}$$

if x is odd, and

$$s=(1,-1,\dots,x/2,-x/2;1+x/2,-1-x/2,\dots,\lfloor m/2\rfloor,-\lfloor m/2\rfloor,(m+1)/2;0)$$

if x is even.

Define $s+i$ to be formed by adding i (modulo $m+1$) to each component of s. Then $S(\alpha)=\{s+i \mid 0\le i\le m\}$ is an x-nesting of the directed m-cycle system (Z_{m+1},C) of D_{m+1}. ∎

The directed 5-cycle system together with the 1-nesting and 2-nesting in Example 1.1 illustrate the construction in the proof of Lemma 2.1 (with ∞ being replaced by $m=5$ throughout).

Lemma 2.2. *For $1\le x\le \lfloor m/2\rfloor$ there exists a directed m-cycle system of D_{2m+1} that has an x-nesting.*

Proof: Let $m=2y+1$ and so as $x\le\lfloor m/2\rfloor$, $x\le y$. Define

$$c_1=(-1,2,\dots,(-1)^y y,(-1)^y(y+1),(-1)^{y+1}(y+2),\dots,(-1)^{2y}(2y+1))$$

where each coordinate is reduced modulo $2m+1$ and define $c_2=-c_1$ (where $-c_1$ is formed by multiplying each component of c_1 by -1 modulo m). Also define

$$s_1=\begin{cases}(-1,2,\dots(-1)^x x;(-1)^{x+1}(x+1),\dots,(-1)^y y,(-1)^y(y+1),\dots,\\ (-1)^{2y}(2y+1);0) & \text{if } x<y\\ (-1,2,\dots(-1)^x x;(-1)^x(x+1),\dots,(-1)^{2y}(2y+1);0) & \\ & \text{if } x=y\end{cases}$$

and $s_2=-s_1$.

Then $C=\{c_1+i,c_2+i \mid 0\le i\le 2m\}$ is a directed m-cycle system and $S(\alpha)=\{s_1+i,s_2+i \mid 0\le i\le 2m\}$ is an x-nesting of the directed m-cycle system (Z_{2m+1},C) of D_{2m+1}. ∎

For example, Lemma 2.2 produces the directed 5-cycle system (Z_{11},C) where

$$C=\{(10+i,2+i,3+i,7+i,5+i)\mid 0\le i\le 10\}$$

that has a 1-nesting defined by

$$S(\alpha)=\{(10+i;2+i,3+i,7+i,5+i;\ i)\mid 0\le i\le 10\}$$

and has a 2-nesting defined by

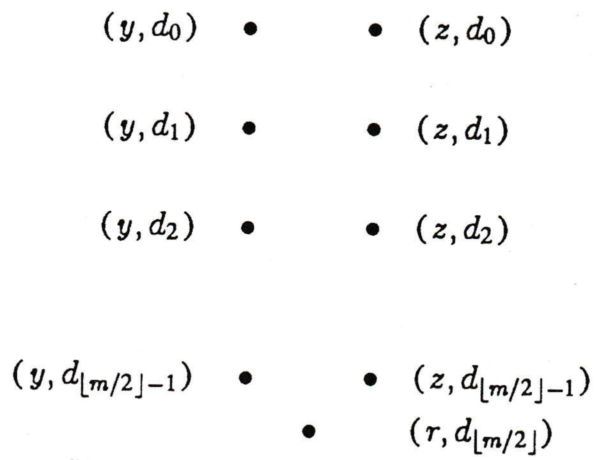
$$S(\alpha)=\{(10+i,2+i;3+i,7+i,5+i;\ i)\mid 0\le i\le 10\}.$$

Define an m-*nesting sequence* $d = (d_0, d_1, \ldots, d_{\lfloor m/2 \rfloor})$ by $d_i = (-1)^{i+1} \lfloor (i+1)/2 \rfloor \pmod{m}$. This sequence has two relevant properties. Let $D(i,j) = \min\{i-j \pmod{m}, j-i \pmod{m}\}$. Then this m-nesting sequence d satisfies

$$\{D(d_i, d_{i-1}) \mid 1 \le i \le \lfloor m/2 \rfloor\} = \{1, 2, \ldots, \lfloor m/2 \rfloor\}, \quad \text{and} \tag{1}$$

$$\{D(d_{\lfloor m/2 \rfloor}, d_i) \mid 0 \le i < \lfloor m/2 \rfloor\} = \{1, 2, \ldots, \lfloor m/2 \rfloor\}. \tag{2}$$

It will be convenient to denote the directed m-cycle



by $(y, z, r;\ d_0, d_1, \ldots, d_{\lfloor m/2 \rfloor})$.

Finally, we need a pair of orthogonal idempotent quasigroups. These exist for all orders except $2, 3$, and 6.

Theorem 2.3. *For all $n \equiv 1 \pmod{m}$ except possibly $n \in \{3m+1, 6m+1\}$, there exists a directed m-cycle system of D_n that has a $\lfloor m/2 \rfloor$-nesting.*

Proof: Let $n = ms + 1$ where $s \notin \{2, 3, 6\}$. Let $(Z_s, \circ_1)$ and $(Z_s, \circ_2)$ be a pair of orthogonal idempotent quasigroups of order s.

Define a directed m-cycle system $(\{\infty\} \cup (Z_s \times Z_m), C)$ of D_n as follows.

(1) For each $r \in Z_s$ define a copy of an $\lfloor m/2 \rfloor$-nestable directed m-cycle system of D_{m+1} on the set of vertices $\{\infty\} \cup (\{r\} \times Z_m)$ (see Lemma 2.1) and place these directed m-cycles into C.

(2) For $i \in Z_m$, $y \in Z_s$ and $z \in Z_s$, $y \ne z$, place the directed m-cycle $(y, z, y \circ_1 z;\ d_0 + i, d_1 + i \ldots, d_{\lfloor m/2 \rfloor} + i)$ into C (reducing all the components $d_j + i$ modulo m).

By using property 1 of an m-nesting sequence, it is straightforward to check that $(\{\infty\} \cup (Z_s \times Z_m), C)$ is a directed m-cycle system. It remains to show that it has an $\lfloor m/2 \rfloor$-nesting.

(1) For each $r \in Z_s$ let $(\alpha_r, S_r(\alpha_r))$ be an $\lfloor m/2 \rfloor$-nesting of the directed m-cycle system placed on $\{\infty\} \cup (\{r\} \times Z_m)$.

(2) For $i \in Z_m, y \in Z_s, z \in Z_s, y \neq z$ define

$$\alpha\left((y, z, y \circ_1 z;\ d_0 + i, \dots, d_{\lfloor m/2 \rfloor} + i)\right) = (y \circ_2 z, d_{\lfloor m/2 \rfloor} + i)$$

and define the corresponding directed (x, m)-star by

$$s_{(y,z,i)} = \big((y, d_0 + i), (y, d_1 + i), \dots, (y, d_{\lfloor m/2 \rfloor - 1} + i), (y \circ_1 z, d_{\lfloor m/2 \rfloor} + i);\\ (z, d_0 + i), \dots, (z, d_{\lfloor m/2 \rfloor - 1} + i);\ (y \circ_2 z, d_{\lfloor m/2 \rfloor} + i)\big).$$

Then the set consisting of the directed stars in the sets S_r, $r \in Z_s$ together with $s_{(y,z,i)}$ for $y \neq z, y \in Z_s, z \in Z_s, i \in Z_m$ form an $\lfloor m/2 \rfloor$-nesting. To see this we should find the directed stars containing the arcs $((a,j),(b,j)), ((a,j),(a,k))$ and $((a,j),(b,k))$ for $a \neq b$ and $j \neq k$.

Since $(Z_1, \circ_1)$ and $(Z_2, \circ_2)$ are orthogonal, for some y and z, $y \circ_1 z = a$ and $y \circ_2 z = b$. Also, there is an i such that $d_{\lfloor m/2 \rfloor} + i = j$. Then $((a,j),(b,j))$ is in the directed star $s_{(y,z,i)}$.

Clearly, $((a,j),(a,k))$ is in one of the directed stars in $S_a(\alpha_a)$.

Finally, by property 2 of m-nesting sequences, there exist values d_r and i such that either $d_r + i = j$ and $d_{\lfloor m/2 \rfloor} + i = k$ or $d_r + i = k$ and $d_{\lfloor m/2 \rfloor} + i = j$ (but not both). In the first case, let $a \circ_2 z = b$, then $((a,j),(b,k))$ is in $s_{(a,z,i)}$. In the second case, let $z \circ_2 b = a$, then $((a,j),(b,k))$ is in $s_{(z,b,i)}$.

The theorem now follows using Lemma 2.1 and Lemma 2.2. ∎

Finally, we remark that several problems remain open.

(1) Find a directed m-cycle system that has an x-nesting for $1 \leq x \leq \lfloor m/2 \rfloor - 1$, and for $x = \lfloor m/2 \rfloor$ when m is even.

(2) Find a directed m-cycle system of D_n that has an $\lfloor m/2 \rfloor$-nesting when $n \in \{3m+1, 6m+1\}$.

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