NESTINGS OF DIRECTED CYCLE SYSTEMS

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Abstract. We show that for all odd \( m \), there exists a directed \( m \)-cycle system of \( D_n \) that has an \( \lfloor m/2 \rfloor \)-nesting, except possibly when \( n \in \{3m + 1, 6m + 1\} \).

1. Introduction.

Let \( K_n \) be the complete graph on \( n \) vertices. An \( m \)-cycle of a graph \( G \) is an ordered \( m \)-tuple \((v_0, v_1, \ldots, v_{m-1})\) such that \( v_i v_{i+1} \) for \( 0 \leq i \leq m - 1 \) is an edge of \( G \) (where subscripts are reduced modulo \( m \)). An \( m \)-cycle system of \( K_n \) is an ordered pair \((V, C)\) where \( V \) is the vertex set of \( K_n \) (so \( n = |V| \)) and \( C \) is a collection of edge-disjoint \( m \)-cycles of \( K_n \) which induce a partition of \( E(K_n) \).

Let \((v_0, v_1, \ldots, v_{m-1}; w)\) denote the star which joins \( w \) to each of the vertices \( v_0, v_1, \ldots, v_{m-1} \). A nesting of the \( m \)-cycle system \((V, C)\) of \( K_n \) is a function \( \alpha: C \rightarrow V \) such that \( C(\alpha) \) induces a partition of \( E(K_n) \), where \( C(\alpha) \) is the set of stars defined by

\[
C(\alpha) = \{(v_0 v_1, \ldots, v_{m-1}; \alpha(c)) \mid c = (v_0, v_1, \ldots, v_{m-1}) \in C\}.
\]

Whether or not an arbitrary \( m \)-cycle system can be nested is an extremely difficult problem. However, it would seem tractable to consider the problem of finding the set of values of \( n \) for which there exists a nestable \( m \)-cycle system of \( K_n \). A simple counting argument shows that a necessary condition for a nestable \( m \)-cycle system of \( K_n \) to exist is that \( n \equiv 1 \pmod{2m} \). In the case where \( m = 3 \), this problem has been completely settled (this is precisely the nesting problem for Steiner triple systems) [2, 8], the set of possible values being all \( n \equiv 1 \pmod{6} \).

More recently it has been shown that [5] for any odd value of \( m \), with at most 13 possible exceptions the necessary condition is also sufficient, and for the particular case when \( m = 5 \) there are no exceptions. This nesting problem for even length cycles is essentially solved, since for any even \( m \geq 4 \), with at most 13 exceptions

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for each value of \( n \), there exists an \( m \)-cycle system of \( K_n, n \equiv 1 \pmod{2m} \) which has a nesting \([7, 9]\).

In this paper, we introduce analogous problems for directed \( m \)-cycle systems. Let \( D_n \) be the complete directed graph on \( n \) vertices. A directed \( m \)-cycle of a directed graph \( G \) is an ordered \( m \)-tuple \((v_0, v_1, \ldots, v_{m-1})\) such that \((v_i, v_{i+1})\) is an arc of \( G \) for \( 0 \leq i \leq m - 1 \) (reducing sub-scripts modulo \( m \)). A directed \( m \)-cycle system of \( D_n \) is an ordered pair \((V, C)\) where \( V \) is the vertex set of \( D_n \) (so \( n = |V| \)) and \( C \) is a set of arc-disjoint directed \( m \)-cycles of \( D_n \) which induce a partition of \( A(D_n) \) (\( A(D_n) \) is the set of arcs of \( D_n \)). There are clearly several ways to define a nesting of a directed \( m \)-cycle system as the edges in each of the stars can be oriented in different ways. Perhaps the most satisfying problem would require that for some fixed \( x, 0 \leq x \leq m \), each directed star used in the nesting has exactly \( x \) arcs directed in and \( m - x \) arcs directed out of the centre vertex. Therefore, define \((v_0, v_1, \ldots, v_{x-1}; v_x, v_{x+1}, \ldots, v_{m-1}; w)\) to be the directed \((x, m)\)-star in which \((v_i, w)\) is an arc for \( 0 \leq i \leq x - 1 \) and \((w, v_i)\) is an arc for \( x \leq i \leq m - 1 \). Then define an \( x \)-nesting of a directed \( m \)-cycle system \((V, C)\) of \( D_n \) to be an ordered pair \((\alpha, S(\alpha))\) where \( \alpha \) is a function \( \alpha: C \to V \) and \( S(\alpha) \) is a set of directed \((x, m)\)-stars defined by

\[
S(\alpha) = \{ (v_{\pi(c)(0)}, v_{\pi(c)(1)}, \ldots, v_{\pi(c)(x-1)}; v_{\pi(c)(x)}, \ldots, v_{\pi(c)(m-1)}; \alpha(c)) \mid c \in C \}
\]

for some permutations \( \pi_c \) of \( \{0, 1, \ldots, m - 1\} \), \( c \in C \), such that \( S(\alpha) \) induces a partition of \( A(D_n) \).

Example 1.1: Let \( m = 5 \) and \( n = 6 \). Then

\[
C = \{ (5, 0, 1, 3, 2), (5, 1, 2, 4, 3), (5, 2, 3, 0, 4), (5, 3, 4, 1, 0), (5, 4, 0, 2, 1), (0, 3, 1, 4, 2) \}
\]

is a directed 5-cycle system that has a 1-nesting defined by

\[
S(\alpha) = \{ (1; 5, 3, 0, 2; 4), (3; 1, 5, 2, 4; 0), (4; 2, 0, 3, 5; 1), (5; 3, 1, 4, 0; 2), (0; 4, 2, 5, 1; 3), (2; 0, 4, 1, 3; 5) \}
\]

and a 2-nesting defined by

\[
S(\alpha) = \{ (5, 3; 0, 2, 1; 4), (1, 5; 2, 4, 3; 0), (2, 0; 3, 5, 4; 1), (3, 1; 4, 0, 5; 2), (4, 2; 5, 1, 0; 3), (0, 4; 1, 3, 2; 5) \}.
\]

A simple counting argument shows that a necessary condition for the existence of a directed \( m \)-cycle system of \( D_n \) that has an \( x \)-nesting is that \( n \equiv 1 \pmod{m} \).
It is the object of this paper to show that if \( m \) is odd then this is also a sufficient condition, with at most 2 possible exceptions, in the case when \( x = \lfloor m/2 \rfloor \).

It is worth noting that if every arc in an \( x \)-nesting of a directed \( m \)-cycle system is oriented in the opposite direction then a \((m - x)\)-nesting results, so it suffices to consider this problem for \( 1 \leq x \leq \lfloor m/2 \rfloor \).

Finally, notice that if we ignore the directed cycles then what remains is a decomposition of \( D_n \) into directed \((x, m)\)-stars. It is only recently [1] that the problem of finding such decompositions has been found when \( n \equiv 0 \) or \( 1 \pmod{m} \) for all \( x \), the case when \( n \equiv 0 \pmod{m} \) now being possible since the condition of the directed stars arising from a nesting is no longer imposed. Even more recently, this decomposition problem has been completely solved [3].

Throughout the rest of this paper, we assume that \( m \) is odd. Let \( Z_m = \{0, 1, \ldots, m - 1\} \).

2. Directed \( m \)-cycle systems with \( \lfloor m/2 \rfloor \)-nestings.

**Lemma 2.1.** For \( 1 \leq x \leq \lfloor m/2 \rfloor \) there exists a directed \( m \)-cycle system of \( D_{m+1} \) that has an \( x \)-nesting.

Proof: Define a directed \( m \)-cycle on the vertex set \( \{\infty\} \cup Z_m \) by

\[
\alpha = (a_0, a_1, \ldots, a_{m-1}) \quad \text{where}
\]

\[
a_0 = \infty, \quad a_j = (-1)^j \lfloor j/2 \rfloor \quad \text{for} \quad 1 \leq j \leq \lfloor m/2 \rfloor, \quad \text{and}
\]

\[
a_{m-j} = (-1)^{\lfloor m/2 \rfloor} \lfloor m/2 \rfloor + (-1)^j \lfloor j/2 \rfloor \quad \text{for} \quad 1 \leq j \leq \lfloor m/2 \rfloor.
\]

Let \( \alpha + i = (a_0 + i, a_1 + i, \ldots, a_{m-1} + i) \), reducing each component modulo \( m \) and defining \( \infty + i = \infty \). Then we can define a directed \( m \)-cycle system \((\{\infty\} \cup Z_m, C)\) as follows: if \( m \equiv 1 \pmod{4} \) then define

\[
C = \{ \alpha + i | 0 \leq i \leq m - 1 \} \cup \{(0, \lfloor m/2 \rfloor, 2 \lfloor m/2 \rfloor, \ldots, (m - 1) \lfloor m/2 \rfloor)\}
\]

and if \( m \equiv 3 \pmod{4} \) then define

\[
C = \{ \alpha + i | 0 \leq i \leq m - 1 \} \cup \{(0, \lfloor m/2 \rfloor, 2 \lfloor m/2 \rfloor, \ldots, (m - 1) \lfloor m/2 \rfloor)\}.
\]

To nest these directed \( m \)-cycle systems, begin by renaming \( \infty \) with \( m \), so the vertex set is now \( Z_{m+1} \). Of course in this case, for each \( c \in C, \alpha(c) \) is the unique vertex that is not in \( c \). Define

\[
s = (1, -1, \ldots, \lfloor x/2 \rfloor, -\lfloor x/2 \rfloor, (m+1)/2; 1+\lfloor x/2 \rfloor, -(1+\lfloor x/2 \rfloor), \ldots, [m/2], -[m/2]; 0)
\]

if \( x \) is odd, and
\( s = (1, -1, \ldots, x/2, -x/2; 1+x/2, -1-x/2, \ldots, [m/2], -[m/2], (m+1)/2; 0) \)

if \( x \) is even.

Define \( s + i \) to be formed by adding \( i \) (modulo \( m + 1 \)) to each component of \( s \). Then \( S(\alpha) = \{ s + i \mid 0 \leq i \leq m \} \) is an \( x \)-nesting of the directed \( m \)-cycle system \((Z_{m+1}, C)\) of \( D_{m+1} \).

The directed 5-cycle system together with the 1-nesting and 2-nesting in Example 1.1 illustrate the construction in the proof of Lemma 2.1 (with \( \infty \) being replaced by \( m = 5 \) throughout).

**Lemma 2.2.** For \( 1 \leq x \leq \lfloor m/2 \rfloor \) there exists a directed \( m \)-cycle system of \( D_{2m+1} \) that has an \( x \)-nesting.

**Proof:** Let \( m = 2y + 1 \) and so as \( x \leq \lfloor m/2 \rfloor, x \leq y. \) Define

\[
c_1 = (-1, 2, \ldots, (-1)^y y, (-1)^{y+1}(y+1), (-1)^{y+1}(y+2), \ldots, (-1)^{2y}(2y+1))
\]

where each coordinate is reduced modulo \( 2m + 1 \) and define \( c_2 = -c_1 \) (where \( -c_1 \) is formed by multiplying each component of \( c_1 \) by \(-1 \) modulo \( m \)). Also define

\[
s_1 = \begin{cases} 
(-1, 2, \ldots, (-1)^xx; (-1)^{x+1}(x+1), \ldots, (-1)^y y, (-1)^{y+1}(y+1), \ldots, \\
(-1)^{2y}(2y+1); 0) & \text{if } x < y \\
(-1, 2, \ldots, (-1)^xx; (-1)^x(x+1), \ldots, (-1)^{2y}(2y+1); 0) & \text{if } x = y
\end{cases}
\]

and \( s_2 = -s_1 \).

Then \( C = \{ c_1 + i, c_2 + i \mid 0 \leq i \leq 2m \} \) is a directed \( m \)-cycle system and \( S(\alpha) = \{ s_1 + i, s_2 + i \mid 0 \leq i \leq 2m \} \) is an \( x \)-nesting of the directed \( m \)-cycle system \((Z_{2m+1}, C)\) of \( D_{2m+1} \).

For example, Lemma 2.2 produces the directed 5-cycle system \((Z_{11}, C)\) where

\[
C = \{(10+i, 2+i, 3+i, 7+i, 5+i) \mid 0 \leq i \leq 10\}
\]

that has a 1-nesting defined by

\[
S(\alpha) = \{(10+i; 2+i, 3+i, 7+i, 5+i; i) \mid 0 \leq i \leq 10\}
\]

and has a 2-nesting defined by

\[
S(\alpha) = \{(10+i, 2+i; 3+i, 7+i, 5+i; i) \mid 0 \leq i \leq 10\}.
\]
Define an $m$-nesting sequence $d = (d_0, d_1, \ldots, d_{\lfloor m/2 \rfloor})$ by $d_i = (-1)^{i+1} \lfloor (i+1)/2 \rfloor \pmod{m}$. This sequence has two relevant properties. Let $D(i, j) = \min \{i - j \pmod{m}, j - i \pmod{m} \}$. Then this $m$-nesting sequence $d$ satisfies

$$
\{ D(d_i, d_{i-1}) \mid 1 \leq i \leq \lfloor m/2 \rfloor \} = \{ 1, 2, \ldots, \lfloor m/2 \rfloor \}, \quad \text{and} \quad \{ D(d_{\lfloor m/2 \rfloor}, d_i) \mid 0 \leq i < \lfloor m/2 \rfloor \} = \{ 1, 2, \ldots, \lfloor m/2 \rfloor \}.
$$

It will be convenient to denote the directed $m$-cycle

$$(y, d_0) \bullet (z, d_0)$$

$$(y, d_1) \bullet (z, d_1)$$

$$(y, d_2) \bullet (z, d_2)$$

$$(y, d_{\lfloor m/2 \rfloor - 1}) \bullet (z, d_{\lfloor m/2 \rfloor - 1})$$

$$\bullet (r, d_{\lfloor m/2 \rfloor})$$

by $(y, z; d_0, d_1, \ldots, d_{\lfloor m/2 \rfloor})$.

Finally, we need a pair of orthogonal idempotent quasigroups. These exist for all orders except 2, 3, and 6.

**Theorem 2.3.** For all $n \equiv 1 \pmod{m}$ except possibly $n \in \{3m + 1, 6m + 1\}$, there exists a directed $m$-cycle system of $D_n$ that has a $\lfloor m/2 \rfloor$-nesting.

Proof: Let $n = ms + 1$ where $s \notin \{2, 3, 6\}$. Let $(Z_s, o_1)$ and $(Z_s, o_2)$ be a pair of orthogonal idempotent quasigroups of order $s$.

Define a directed $m$-cycle system $(\{\infty\} \cup (Z_s \times Z_m), C)$ of $D_n$ as follows.

1. For each $r \in Z_s$ define a copy of an $\lfloor m/2 \rfloor$-nestable directed $m$-cycle system of $D_{m+1}$ on the set of vertices $\{\infty\} \cup (\{r\} \times Z_m)$ (see Lemma 2.1) and place these directed $m$-cycles into $C$.

2. For $i \in Z_m$, $y \in Z_s$ and $z \in Z_s$, $y \neq z$, place the directed $m$-cycle $(y, z, y o_1 z; d_0 + i, d_1 + i, \ldots, d_{\lfloor m/2 \rfloor} + i)$ into $C$ (reducting all the components $d_j + i \pmod{m}$).

By using property 1 of an $m$-nesting sequence, it is straightforward to check that $(\{\infty\} \cup (Z_s \times Z_m), C)$ is a directed $m$-cycle system. It remains to show that it has an $\lfloor m/2 \rfloor$-nesting.

1. For each $r \in Z_s$ let $(\alpha_r, S_r(\alpha_r))$ be an $\lfloor m/2 \rfloor$-nesting of the directed $m$-cycle system placed on $\{\infty\} \cup (\{r\} \times Z_m)$. 

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(2) For \( i \in \mathbb{Z}_m, y \in \mathbb{Z}_z, z \in \mathbb{Z}_s, y \neq z \) define
\[
\alpha((y, z, y o_1 z; d_0 + i, \ldots, d_{\lfloor m/2 \rfloor} + i)) = (y o_2 z, d_{\lfloor m/2 \rfloor} + i)
\]
and define the corresponding directed \((x, m)\)-star by
\[
s_{(y, z, i)} = ((y, d_0 + i), (y, d_1 + i), \ldots, (y, d_{\lfloor m/2 \rfloor} - 1 + i), (y o_1 z, d_{\lfloor m/2 \rfloor} + i), (z, d_0 + i), \ldots, (z, d_{\lfloor m/2 \rfloor} - 1 + i), (y o_2 z, d_{\lfloor m/2 \rfloor} + i)).
\]

Then the set consisting of the directed stars in the sets \( S_r, r \in \mathbb{Z}_m \) together with \( s_{(y, z, i)} \) for \( y \neq z, y \in \mathbb{Z}_s, z \in \mathbb{Z}_s, i \in \mathbb{Z}_m \) form an \( \lfloor m/2 \rfloor \)-nesting. To see this we should find the directed stars containing the arcs \(((a, j), (b, j)), ((a, j), (a, k))\) and \(((a, j), (b, k))\) for \( a \neq b \) and \( j \neq k \).

Since \((Z_1, o_1)\) and \((Z_2, o_2)\) are orthogonal, for some \( y \) and \( z, y o_1 z = a \) and \( y o_2 z = b \). Also, there is an \( i \) such that \( d_{\lfloor m/2 \rfloor} + i = j \). Then \(((a, j), (b, j))\) is in the directed star \( s_{(y, z, i)} \).

Clearly, \(((a, j), (a, k))\) is in one of the directed stars in \( S_a(\alpha) \).

Finally, by property 2 of \( m \)-nesting sequences, there exist values \( d_r \) and \( i \) such that either \( d_r + i = j \) and \( d_{\lfloor m/2 \rfloor} + i = k \) or \( d_r + i = k \) and \( d_{\lfloor m/2 \rfloor} + i = j \) (but not both). In the first case, let \( a o_2 z = b \), then \(((a, j), (b, k))\) is in \( s_{(a, z, i)} \). In the second case, let \( z o_2 b = a \), then \(((a, j), (b, k))\) is in \( s_{(z, b, i)} \).

The theorem now follows using Lemma 2.1 and Lemma 2.2.

Finally, we remark that several problems remain open.

(1) Find a directed \( m \)-cycle system that has an \( x \)-nesting for \( 1 \leq x \leq \lfloor m/2 \rfloor - 1 \), and for \( x = \lfloor m/2 \rfloor \) when \( m \) is even.

(2) Find a directed \( m \)-cycle system of \( D_n \) that has an \( \lfloor m/2 \rfloor \)-nesting when \( n \in \{3m + 1, 6m + 1\} \).

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